CUL-H06389-22-P17558.

P17558

CALCUTTA UNIVERSITY LIBRARY.
Acc. No.
Book No.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

510.5
.003/5

EDITED BY

EARLE RAYMOND HEDRICK
ARNOLD DRESDEN
JOHN WESLEY YOUNG

WITH THE ASSISTANCE OF

ROBERT W. BURGESS
TULLIO LEVI-CIVITA
DUNHAM JACKSON
WILLIAM R. LONGLEY
DERRICK N. LEHMER
HENRY L. RIETZ
CAROLINE E. SEELY

22

VOLUME XXXI

JANUARY TO DECEMBER, 1925

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK
1925

PRESS OF
LÜTCKE & WULFF
HAMBURG

BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

THE INTERNATIONAL CONGRESS AT TORONTO*

The seventh International Mathematical Congress was opened on Monday morning, August 11, 1924, in the Convocation Hall of the University of Toronto by Professor Ch. de la Vallée Poussin, President of the International Mathematical Union. Addresses of welcome were delivered by the Honorable H. S. Beland on behalf of the Government of the Dominion, and by Sir Robert Falconer, President of the University of Toronto. At this session, Professor J. C. Fields was elected President of the Congress; Vice-Presidents were appointed for the sessions of the Congress, which were to be held in eight sections as follows:

SECTION I. Algebra, Theory of Numbers, Analysis.
SECTION II. Geometry.
SECTION III. (a) Mechanics, Mathematical Physics.
SECTION III. (b) Astronomy, Geophysics.
SECTION IV. (a) Electrical, Mechanical, Civil and Mining Engineering.
SECTION IV. (b) Aeronautics, Naval Architecture, Ballistics, Radiotelegraphy.
SECTION V. Statistics, Actuarial Science, Economics.
SECTION VI. History, Philosophy, Didactics.

---

* A report prepared for this BULLETIN by Arnold Dresden.

These Sections began their meetings on Monday afternoon, and continued every morning throughout the week, with the exception of Thursday which was set aside for an all day excursion to Niagara Falls.

The afternoons and evenings were devoted to general lectures, and to social functions. The meetings of the British Association for the Advancement of Science which had started during the preceding week overlapped with the earlier sessions of the Congress, thus furnishing an opportunity for hearing many of the most eminent scientists in many fields.

Not enough praise can be given to the Committee on Entertainment, to the officials of the Province of Ontario, of the City of Toronto, and of the University, and to the citizens of Toronto, who provided lavish opportunity for social intercourse in the form of garden parties, a conversazione in Hart House, an excursion around the City of Toronto, etc.

On Saturday afternoon the Congress paid homage to the students of the University of Toronto who laid down their lives in the war. A wreath was placed in the newly erected Memorial Tower of the University; short addresses were made by Professor de la Vallée Poussin and by President Sir Robert Falconer; and it was announced that arrangements had been made for inserting in the wall of the Tower a medallion bearing the following inscription: "To the heroes of the University of Toronto, the members of the International Mathematical Congress, Toronto, 1924."

A special convocation of the University of Toronto was held on Wednesday afternoon. Honorary degrees were conferred upon Sir David Bruce, Sir Charles Parsons, Sir John Russell, Sir Ernest Rutherford of the British Association for the advancement of Science, and upon Professors F. Severi, Ch. de la Vallée Poussin, V. Stekloff and G. Koenigs of the International Mathematical Congress.

The general assembly of the International Mathematical Union took place on Friday morning. Of the six delegates elected by the American Section of the Union, Professors

Coble, Richardson, and Snyder were present.* Professor S. Pincherle, of the University of Bologna, was elected President of the Union for the ensuing quadrennium; Professors G. Koenigs and A. Demoulin were reelected as Secretary and Treasurer respectively. Professors G. A. Bliss, E. Phragmen, and H. Fehr were elected as Vice-Presidents in place of Professors Dickson, Larmor, and Volterra, whose terms expired, while Professors P. Appell and W. H. Young continue as Vice-Presidents. As Honorary Presidents of the Union were elected Professors L. E. Dickson, J. C. Fields, H. Lamb, G. Mittag Leffler, Emile Picard, Ch. de la Vallée Poussin, and V. Volterra.

A Committee on International Bibliography was appointed, consisting of Professors Archibald, Bortolotti, Fréchet, van der Woude, and W. H. Young.

The following announcement was made by the American Section of the Union and received the endorsement of the delegations from Italy, Holland, Sweden, Denmark, Norway, and England.

"The American Section of the International Mathematical Union has unanimously passed the following resolution and requests that it be transmitted to the International Research Council by the Executive Committee of the International Mathematical Union:

"Resolved that the American Section of the International Mathematical Union requests the International Research Council to consider whether the time is ripe for the removal of the restrictions on membership now imposed by the rules of the Council.

"The American Section also wishes to announce that it intends to present to the National Research Council of the United States resolutions requesting action in this direction."

The closing session of the Congress took place on Saturday afternoon.

Professor Synge, Secretary of the Congress, announced that 387 persons, representing 25 countries, had registered as members of the Congress, the distribution being as follows: Argentina 2, Belgium 7, Canada 67, Cuba 1, Czecho Slovakia 3, Denmark 3, France 26, Georgia 1, Great Britain 39, Holland 4, India 2, Ireland 3, Italy 11, Mexico 1, Norway 5, Poland 3, Portugal 2, Rumania 1, Russia 5, Spain 3, Sweden 3, Switzerland 4, Ukrania 1, United States 189, Yugo Slavia 1.

---

* See this BULLETIN, vol. 30 (1924), p. 377.

A resolution was unanimously adopted stating that the Congress hears with pleasure that the Royal Irish Academy contemplates the publication of a collected edition of the works of Sir William Rowan Hamilton.

Resolutions were also adopted expressing to the government of the Dominion, of the Province of Ontario, and of the City of Toronto, and to the many persons and organizations who had contributed to the success of the meeting the recognition and thanks of the Congress.

The number of papers read at the meetings of the various Sections reached the enormous total of 235. In addition to these, the following 7 lectures were given before general sessions of the Congress:

C. Störmer, *Modern Norwegian Researches on the Aurora Borealis.*

F. Severi, *Géométrie algébrique.*

E. Cartan, *La théorie des groupes et les recherches récentes de géométrie différentielle.*

W. H. Young, *Some characteristic features of twentieth century pure mathematical research.*

L. E. Dickson, *Outline of the theory to date of the arithmetics of algebras.*

S. Pincherle, *Opérations fonctionnelles.*

M. Le Roux, *Sur l'intégration des équations aux dérivées partielles par des intégrales définies.*

The following is a list, as nearly accurate as it has been possible to secure, of the papers presented at the sectional sessions.

Section I. W. Kapteyn, *Expansion of functions in terms of Bernoullian functions*; G. A. Bliss, *The transformation of Clebsch in the calculus of variations*; J. Ouspensky and B. Venkov, *On some new class number relations*; M. Plancherel, *Sur les séries de fonctions orthogonales*; P. A. MacMahon, *Some recent developments in determinant theory*; L. E. Dickson, *Further development of the theory of arithmetics of algebras*; G. Prasad, *On the numerical solution of integral equations*; O. E. Glenn, *Differential combinants and associated parameters*; J. A. Shohat (Jacques Chokhate), *On the asymptotic properties of a certain class of Tchebycheff polynomials*; B. A. Bernstein, *Modular representations of finite algebras*; N. Gunter, *Sur la résolution des systèmes des équations* Grad $X = A$ Rot $X = A$; F. H. Murray, *The asymptotic distribution of the characteristic numbers for the self-adjoint linear partial differential equations of the second order*; J. F. Ritt, *Elementary functions and their inverses*; Th. Varopoulos, *Sur les valeurs exceptionnelles des fonctions multiformes*; L. Pomey, *Les équations intégro-différentielles à plusieurs variables*; O. Öre, *A new method in the theory of algebraic numbers*; H. S. Vandiver, *On the first case of Fermat's last theorem*; R. Birkeland, *On the solution of quintic equations*; W. L. G. Williams, *Formal modular invariants of forms in several variables*; J. Wolff, *On the sufficient conditions for analyticity*; E. T. Bell, *General class number relations whose degenerates involve indefinite forms*; O. E. Glenn, *The method of invariant elements in a modular field*; N. Kryloff, *Sur quelques recherches dans le domaine de la théorie de l'interpolation et des quadratures dites mécaniques*; N. Kryloff and T. Tamarkine, *Sur une formule d'interpolation*; J. Touchard, *Sur la théorie des différences*; J. Touchard, *Sur certaines équations fonctionelles*; J. I. Del Corrail, *Meditations on trigonometry*; A. Lévy, *Sur le calcul des idéaux d'un corps du second degré*; M. Petrovitch, *Correspondance entre la fonction et la fraction décimale*; L. E. Dickson, *A new theory of linear transformations and pairs of bilinear forms*; L. Pomey, *La théorie des nombres*; D. R. Curtiss, *Rational processes for separating the real branches of a plane curve at a multiple point*; T. R. Rosebrugh, *A theorem on the moduli of substitutions involving products of variables themselves given by linear substitutions*; N. Gunter, *Sur un problème fondamental de la hydrodynamique*; E. Narishkina, *On the analogs of Bernoullian numbers in quadratic fields*; J. Drach, *Sur "l'intégration logique" des équations différentielles de la géométrie et de la mécanique*; L. Crelier, *Sur les équations intégrales simples*; E. Hille, *On the zeros of functions defined by linear differential equations*; R. Fueter, *Some applications of the theory of functions to the theory of numbers*; G. C. Evans, *The Dirichlet problem for the general finitely connected open region*; A. Razmadze, *Sur quelques formules de la moyenne*; G. A. Miller, *Commutative conjugate cycles in subgroups of* *the holomorph of an Abelian group*; W. Sierpinski, *Les ensembles bien définis non mesurables*; V. Stekloff, *Sur les problèmes de représentation approchée des fonctions à l'aide des polynomes, du calcul approché des intégrales définies, du développement des fonctions en séries de polynomes et de l'interpolation, considérés au point de vue des idées de Tchebycheff*; J. C. Fields, *A basis for the theory of ideals*; B. Delaunay, *Sur le nombre de représentations d'un nombre par une forme binaire cubique d'un discriminant négatif*; R. L. Wilder, *On a certain type of connected set which cuts the plane*; J. I. Hutchinson, *On the roots of the Riemann zeta function*; W. B. Ford, *On determining the asymptotic developments of a given function*; L. Tonelli, *Sul calcolo delle variazioni*; A. Razmadze, *Sur les extrémales discontinues dans le calcul des variations*; L. G. Du Pasquier, *L'évolution du concept du nombre hypercomplexe entier*; T. Andrade, *Problème proposé sur les équations fonctionnelles*; O. C. Hazlett, *On the arithmetic of a general associative algebra*; M. Kossler, *A generalization of the theorems of Fabry and Szasz*; M. Kravtschok, *Note sur l'interpolation généralisée.*

Section II. C. Servais, *Sur les lignes asymptotiques*; E. Cartan, *Sur la déformation projective des surfaces*; A. B. Coble, *The behavior of the rational plane sextic and its related Cayley symmetroid under regular Cremona transformation*; T. Bonnesen, *Recherches géométriques sur le problème isopérimétrique*; J. H. Weaver, *On a system of triangles related to a poristic system*; J. A. Barrau, *Conditions of intersection of flat spaces situated in a quadratic variety*; F. Morley, *The condition that the curves of a net have a common point*; M. Fréchet, *Expression la plus générale de la "distance" sur une droite*; G. Fubini, *Principles of projective differential geometry*; A. R. Da Costa, *L'enseignement des mathématiques doit être orienté pour l'étude de la relativité*; A. Naess, *On a method of generalizing the vector products to S_m*; A. Naess, *Three theorems of analysis derived by the vector method as corollaries from a single proposition*; F. D. Murnaghan, *The generalized Kronecker symbol*; N. Delaunay, *Sur les bases nouvelles de la théorie des systèmes articulés*; G. Tzitzeica, *Un nouveau problème sur les suites de Laplace*; M. Fréchet, *Sur une représentation paramétrique intrinsèque de la courbe continue la plus générale*; C. H. Sisam, *On surfaces whose asymptotic curves are cubics*; E. A. L. Merlin, *Sur les réseaux asymptotiques*; B. Delaunay, *Sur la sphère vide*; M. Janet, *Sur les systèmes linéaires d'hypersurfaces*; G. Koenigs, *Sur les mouvements à deux paramètres doublement décomposables*; C. Servais, *Sur la géométrie du tétraèdre*; N. B. MacLean, *On certain surfaces related covariantly to a given ruled surface*; L. A. Godeaux, *Sur les involutions régulières d'ordre deux appartenant à une surface irrégulière*; C. T. Sullivan, *The determination of all surfaces characterized by a reducible directrix quadric*; A. Errera, *Quelques remarques sur le problème* *des quatre couleurs*; J. A. Eiesland, *Quadratic flat-complexes in odd N-space and their singular spreads. Flat sphere transformation*; J. L. Synge, *Normals and curvatures of a curve in a Riemannian manifold*; B. Bydzovsky, *Contribution à la théorie de la sextique à huit points doubles*; A. Torroje, *Sur la représentation des espaces pluri-dimensionnels*; M. W. Haskell, *Curves autopolar with respect to a finite number of conics*; D'A. W. Thompson, *The repeating patterns of the regular polygons and their relation to the Archimedean bodies*; A. H. Wheeler, *New methods for constructing geometry models*; L. D. Cummings, *Cyclic systems of six points in a binary correspondence*; W. van der Woude, *On the finiteness of a system of invariants*; M. Fréchet, *Number of dimensions of an abstract set*; A. Demoulin, *La théorie des équations M et quelques unes de ses applications à la géométrie*; C. Ricci, *Contributo alla teoria delle varietà Riemanniane*; E. Bortolotti. *L'algebra geometrica ed i prodromi della geometria analytica in un manuscritto inedito de R. Bombelli*; J. Pierpont, *Non-euclidean geometry from a non-projective standpoint.*

Section III. S. Zaremba, *Sur un groupe de transformations qui se présente en électrodynamique*; W. F. G. Swann, *A new deduction of the electromagnetic equations*; P. Haag, *Sur un problème général de probabilités et ses diverses applications*; C. V. Raman, *Theory of the structure of liquid surfaces*; E. W. Brown, *The orbit of the eighth satellite of Jupiter*; L. Silberstein, *A finite world radius and some of its cosmological implications*; S. Chapman, *Steps towards a new theory of magnetic storms*; R. H. Fowler, *The equilibrium properties of gases at high (stellar) temperatures*; C. Snow, *Alternating current distribution in cylindrical conductors*; A. W. Conway, *On the quantisation of certain orbits*; P. Haag, *Sur l'application des méthodes du calcul tensoriel à la théorie des moindres carrés*; G. Greenhill, *The spinning top in two moves*; G. Giorgi, *The functional dependence of physical variables*; A. C. Dixon, *A rectangular plate, clamped at edges*; J. Giancheschi, *The perturbations in electrons' orbits*; H. M. Dadourian, *On the fundamental principle of dynamics*; S. J. Jacobsohn, *Note on the force equation of electrodynamics*; W. J. Humphreys, *The effect of surface drag on surface winds*; G. F. McEwen, *Calculation of the velocity of vertical ocean currents in the San Diego region from the accompanying temperature reduction below "normal" values*; A. H. S. Gillson, *The dynamical theory of tides in an ocean of varying depth*; N. H. Heck, *Velocity of sound in sea water*; D. Buchanan, *Asymptotic solutions in the problem of three bodies*; J. Patterson, *The theory of the anemometer*; H. C. Plummer, *Note on the reduction of parallax plates*; P. Haag, *Sur le problème des séquences*; L. V. King, *On the direct numerical calculation of elliptic functions and integrals*; D. Wrinch, *A subject in hydrodynamics*; S. Chapman, *The electrostatic* *potential energy of crystals of the calcite type*; A. W. Conway, *The mathematical work of Sir W. R. Hamilton*; G. A. Campbell, *A system of "definitive units" proposed for universal use*; A. Bezikovitch, *Über relative Maxima des Newtonschen Potentials*; C. de Losada Y Puga, *A short contribution to the kinetic theory of gases*; R. Risser, *Note au sujet des ondes d'émersion*; V. Kostitzin, *Sur une application des équations intégrales au problème d'hystérésis magnétique*; J. G. Gray, *Gyroscopic tops*; A. S. Eddington, *Absolute rotation*; E. Cartan, *La stabilité ordinaire des ellipsoides de Jacobi*; V. Stekloff, *Sur les recherches posthumes de Liapounoff sur les figures d'équilibre d'une liquide hétérogène en rotation*; C. E. St. John, *The red shift of the solar lines and relativity*; H. C. Levinson, *The gravitational field of n moving particles in the theory of relativity*; L. Silberstein, *Modern photographic theory*; J. Andrade, *Chronométrie: Problème actuel des horloges élastiques*; J. Andrade, *Balances spirales: Frictions hydrostatiques et viscosités*; Da Costa Lobo, *Nouvelles théories physiques: application à l'astronomie*; V. Bjerknes, *Solved and unsolved problems in dynamical meteorology*; N. Shaw, *The convective energy of saturated air in natural environment*; W. F. G. Swann, *A generalization of electro-dynamics consistent with restricted relativity and affording an explanation of the earth's magnetic and gravitational fields, and the maintenance of the earth's charge*; H. L. Vanderlinden, *The gravitational field of an electrical sphere with a variable density of matter*; L. A. Bauer, *The mathematical analysis of the earth's magnetic field*; J. Drach, *Sur le mouvement d'un solide pesant qui a un point fixe*; J. Chazy, *Sur l'arrivée dans le système solaire d'un astre étranger*; J. B. Pomey, *Sur la nature des grandeurs électriques considérées en électrostatique*; Barre, *Sur la propagation des ondes planes dans les milieux élastiques anisotropes*; H. Castro-Borel, *Sur quelques méthodes graphiques pour la détermination de la position géographique d'un dirigeable.*

Section IV. S. D. Carothers, *On the application of the principle of the elastic equivalence of statically equipollent loads to engineering problems*; A. P. M. Fleming and R. W. Bailey, *Mathematics in industrial research*; G. A. Campbell, *Mathematics in industrial research*; J. R. Carson, *A generalization of Rayleigh's reciprocal theorem*; E. Schou, *Sur quelques recherches aérodynamiques faites en Danemark avant 1900*; Marchis, *Development of aeronautics in France*; J. B. Henderson, *The oscillations of a gyroscopic compass comprising two gyroscopes*; A. Ferrier, *The duration and length of run required by a seaplane in taking off*; C. F. Jenkin, *What the engineer expects of the mathematician*; J. B. Henderson, *Mathematics for students of engineering*; J. B. Pomey, *Sur les nouveaux appareils multiplex de télégraphie*; U. Puppini, *Azioni sismiche sussultorie tra montanti verticali incastrati alla base con carichi e vincoli elastici all'estremo* *superiore*; E. R. Hedrick, *Effects of variations in Hooke's law on impact, the theory of beams, and elasticity*; H. B. Dwight, *A new formula for use in calculating repulsion of coaxial coils*; T. R. Rosebrugh, *Calculation of long transmission systems*; J. A. L. Waddell, *Mathematics from a consulting engineer's viewpoint*; Charbonnier, *Sur la balistique extérieure*; V. Bjerknes, *On the forces which lift aeroplanes*; W. H. Roever, *Derivation of the differential equations of motion of a projectile regarded as a particle*; F. R. W. Hunt, *The choice of independent variable in the calculation of trajectories by small arcs*, T. R. Wilkins, *A method of computation for sound ranging data*; H. C. Plummer, *Design in gun construction*; W. F. Gerhardt, *New aerodynamical conceptions and formulae*; G. W. O. Howe, *A new theory of long distance radio communication*; N. Yamaga, *On the equilibrium of gases in the reaction of explosives*; G. Puppini, *Principe de réciprocité dans les sciences appliquées*; S. D. Carothers, *Test loads on foundations as affected by scale of tested area*; T. R. Rosebrugh, *The binary linear substitution of modulus unity in problems of general dynamics, acoustics and electricity*; E. G. Coker, *The teaching of the elementary theory of elasticity to engineering students*; R. W. Angus, *Arithmetic solution of engineering problems*; A. Boyajian, *Physical interpretation of complex angles and of their trigonometric functions*; A. E. Kennelly, *Hyperbolic-function series of integral numbers and the occasions for their presentation in electrical engineering*; R. M. Foster, *Two-mesh electric circuits realizing any specified driving-point impedance*; A. F. Samsioe, *Berechnung der Airyschen Spannungsfunktion fur rechteckige Scheiben*; C. Parsons, *Physics and engineering*; B. P. Haigh and A. Beale, *Resonant vibration in steel bridges*, L. J. Briggs, *Research in mechanics and sound at the Bureau of Standards*; P. Cormack, *The use of exponentials in the analysis of machine motions*; L. Breguet, *Sur l'aviation*; M. Montoriel, *Sur les récents perfectionnements apportés à l'appareil Baudot*; Lesaffre, *Appareils Baudot présentés à l'exposition de physique et de T. S. F.*; J. G. Gray, *Gyroscopic stabilizers*; T. C. Fry, *The use of mechanical integration in the practical solution of differential equations by Picard's method of successive approximation*; W. J. Berry, *The influence of mathematics on the development of naval architecture*, L. Woollard, *The teaching of mathematics to students of naval architecture*; A. Planiol, *Sur les pertes par frottements dans les moteurs à explosion*, J. Larmor, *On the cones of steady compression for flying bullets*.

Section V. M. Fréchet, *On a general formula for the computation of net premiums*; R. Henderson, *Some points in the general theory of graduation*; G. U. Yule, *Some life-table approximations*; G. F. McEwen, *A method of estimating the significance of the difference between two* *averages by means of Bayes' theorem on the probability of proportions*; W. P. Elderton, *Mathematical law of mortality; a suggestion*; L. E. Phragmen, *Sur une méthode d'évaluer les intégrales de probabilité*; A. W. Whitney, *Actuarial science in the field of workmen's compensation insurance; particularly the mathematics of schedule-rating and experience-rating*; W. F. Willcox, *Estimates of population in the United States*; J. W. Glover, *Quadrature formulas when ordinates are not equidistant*; W. F. Sheppard, *Interpolation with least square of error*; H. L. Rietz, *On a certain law of probability of Laplace*; J. F. Steffensen, *On a class of quadrature formulas*; P. R. Rider, *A generalized law of error*; A. L. Bowley, *Use of mathematics in economic, social and public statistics*; A. Fisher, *Application of frequency curves to the construction of mortality tables*; L. March, *De l'erreur probable dans le calcul des moyennes*; C. Gini, *Alcune richerche sulla "fecondabilità" della donna*; H. H. Wolfenden, *On the development of formulae for graduation by linear compounding with special reference to the work of Erastus L. de Forest*; L. J. Reed, *Correlations between climatic factors and death rates*; R. H. Coats and M. C. Maclean, *Jottings from the Canadian census*; R. A. Fisher, *On a distribution yielding the error functions of several well known statistics*; E. C. Molina, *A formula for the solution of some problems in sampling*; G. F. McEwen, *Note on a short method of computing terms and sums of terms of the asymmetrical binomial.*

Section VI. E. Bortolotti, *La memoria "De infinitis hyperbolis" di Torricelli*, F. Cajori, *Past struggles between symbolists and rhetoricians in mathematics*; L. C. Karpinski, *The colonial American arithmetic*; C. J. Keyser, *The doctrinal function: Its rôle in mathematics and general thought*; A. Korzybski, *Time-binding: The general theory*; F. Cajori, *Uniformity of mathematical notations —a retrospect and prospect*; G. A. Miller, *History of several fundamental mathematical concepts*, J. H. Rogers, *Vilfredo Pareto*; L. G. Du Pasquier, *Unification of arithmetical terminology*; J. Andrade, *Modèles de mouvements pour l'éducation géométrique*; L. Crelier, *Observations pratiques de méthodologie*; J. Vasconcellos, *Sur quelques points de l'histoire des mathématiques des Egyptiens et aussi sur les Siddhautas des Indiens*; H. Fehr, *L'université et la préparation des professeurs de mathématiques.*

ARNOLD DRESDEN

A THEOREM ON SIMPLE ALGEBRAS*

BY J. H. M. WEDDERBURN

In a previous paper† I showed that every simple algebra A can be expressed as the direct product of a division algebra D and a simple matric algebra $M = (e_{pq})$; the object of this note is to show that this expression is unique, that is, *if* $A = D_1 \times M_1 = D_2 \times M_2$, *where* D_1 *and* D_2 *are division algebras and* M_1 *and* M_2 *are simple matric algebras, then* D_1 *and* M_1 *are simply isomorphic*‡ *with* D_2 *and* M_2 *respectively.*

Let δ_1 and δ_2 be the orders of D_1 and D_2, and let e_1 and e_2 be primitive idempotent elements of M_1 and M_2 respectively. If e_1 and e_2 are supplementary or equal, then§

$$D_1 \cong e_1Ae_1 \cong e_2Ae_2 \cong D_2;$$

M_1 and M_2 are then of the same order and are therefore simply isomorphic. We shall therefore suppose that $e_1 \neq e_2$ and, say, $e_1e_2 \neq 0$.

Assume in the first place that $x = e_1e_2$ is not nilpotent; there then exists a rational polynomial

$$y = \alpha_1x + \alpha_2x^2 + \cdots + \alpha_rx^r,$$

which is an idempotent element of the algebra X generated by x. Now $e_1x = x$; therefore, since every element of X has the form $xf(x)$, $f(x)$ a polynomial in x, it follows that $e_1y = y$ and

$$(ye_1)^2 = ye_1ye_1 = y^2e_1 = ye_1 = e_1ye_1,$$

so that $ye_1 \leqq e_1Ae_1$; also $ye_1 \neq 0$ since $ye_1y = y^2 = y \neq 0$; hence ye_1, being idempotent, equals e_1. In the same way it follows that $e_2y = e_2$ and $ye_2 = y$. Also, if we set $y_{10} = y - e_1$, $y_{02} = y - e_2$, then

---

* Presented to the Society, May 3, 1924.

† Proceedings of the London Society, (2), vol. 6 (1907), p. 99.

‡ *Simple isomorphism* will be denoted by $\cong$.

§ See L. E. Dickson, *Algebras and Their Arithmetics*, Chicago, 1923, pp. 74, 77.

$$e_1y_{10} = y_{10},\ y_{10}e_1 = 0,\ e_2y_{02} = 0,\ y_{02}e_2 = y_{02},$$

and, since $e_1 \neq e_2$, one of y_{10}, y_{02} is not zero, say y_{10}.

We shall now show that y is primitive in A. Since $e_1y = y$, every element of yAy has the form[*]

$$w = d_{11}e_{11} + d_{12}e_{12} + \cdots, \qquad (d_{ij} < D_1;\ e_{11} = e_1,\ e_{ij} < M_1).$$

If w is idempotent, this gives $d_{11} = a$, the modulus of A, so that, if w is primitive, $y - w$ lacks the term in e_{11} and, being therefore nilpotent, must equal 0 since $(y-w)^2 = y - w$. Hence y is primitive and we may set[†] $A = D \times M$ where M is a simple matric algebra which contains y and D is a division algebra simply isomorphic with yAy.

Now, since $ye_1 = e_1$,

$$De_1 = Dye_1 = yAye_1 \geqq ye_1Ae_1ye_1 = e_1Ae_1 = D_1e_1$$

also

$$D_1e_1 = e_1Ae_1 \geqq e_1yAye_1 = yAye_1 = Dye_1 = De_1;$$

hence $De_1 = D_1e_1$ and therefore $D \cong D_1$. Similarly $D \cong D_2$; hence $D_1 \cong D_2$ and, as before, also $M_1 \cong M_2$.

Suppose in the second place that e_1e_2 is nilpotent; then $(e_1e_2e_1)^r = (e_1e_2)^re_1$ is also nilpotent and, as e_1Ae_1 is a division algebra, it follows that $e_1e_2e_1 = 0$; hence $(e_2e_1)^2 = 0$ and so, by a repetition of the same argument, $e_2e_1e_2 = 0$. If now we set $y = e_1 - e_1e_2 - e_2e_1$, then

$$e_1y = e_1 - e_1e_2 \neq e_1,\ ye_1 = e_1 - e_2e_1,\ e_1ye_1 = e_1,$$
$$e_2y = 0 = ye_2,\ ye_1y = y,\ y^2 = y.$$

As before we must show that y is primitive. If $y = y_1 + y_2$ where y_1 and y_2 are supplementary idempotent elements, then

$$0 = e_2y = e_2y_1 + e_2y_2,$$

so that

$$0 = (e_2y_1 + e_2y_2)y_1 = e_2y_1,$$

and similarly

$$y_1e_2 = e_2y_2 = y_2e_2 = 0.$$

---

[*] See Dickson, loc. cit.

[†] See Dickson, loc. cit.

From $y_1 + y_2 = y = e_1 - e_1e_2 - e_2e_1$ we have therefore

$$y_1 = yy_1 = e_1y_1 - e_2e_1y_1,$$

so that $y_1e_1y_1 = y_1$, and similarly

$$y_2e_1y_2 = y_2,\ y_2e_1y_1 = y_2y_1 = 0 = y_1e_1y_2.$$

Now, if $z_1 = e_1y_1e_1$, $z_2 = e_1y_2e_1$, then

$$e_1 = e_1ye_1 = e_1y_1e_1 + e_1y_2e_1 = z_1 + z_2,$$

$$z_1^2 = e_1y_1e_1 \cdot e_1y_1e_1 = e_1 \cdot y_1e_1y_1 \cdot e_1 = e_1y_1e_1 = z_1,$$

$$z_1z_2 = e_1y_1e_1 \cdot e_1y_2e_1 = e_1y_1e_1y_2e_1 = 0,$$

and similarly $z_2^2 = z_2$, $z_2z_1 = 0$. Also $z_1 \neq 0$ unless $y_1 = 0$ since

$$y_1z_1y_1 = y_1e_1y_1e_1y_1 = y_1e_1y_1 = y_1;$$

and similarly $z_2 \neq 0$ unless $y_2 = 0$. But z_1 and z_2, if not zero, are supplementary idempotent elements in e_1Ae_1 whereas e_1 is primitive; hence one of them is zero, that is, y is primitive.

We may now, as before, set $A = D \times M$ where $D \cong yAy$ is a division algebra and M is a simple matric algebra containing y. Remembering that $e_1ye_1 = e_1$, we then have

$$e_1Dye_1 = e_1yAye_1 \geqq e_1ye_1Ae_1ye_1 = e_1Ae_1 = D_1e_1,$$

also

$$D_1e_1 = e_1Ae_1 \geqq e_1yAye_1 = e_1Dye_1;$$

hence $D_1e_1 = e_1Dye_1$. If d and d' are any elements of D, we therefore have

$$e_1dye_1 = d_1e_1,\ e_1d'ye_1 = d_1'e_1, \qquad (d_1,\ d_1' < D_1)$$

and hence

$$e_1(d + d')ye_1 = (d_1 + d_1')e_1,$$

$$d_1d_1'e_1 = e_1dye_1d'ye_1 = e_1dye_1yd'e_1 = e_1dyd'e_1 = e_1dd'ye_1,$$

and, since $e_1 = e_1ye_1$, it follows that $D \cong D_1$ and therefore $M \cong M_1$. Finally, since $e_2y = 0 = ye_2$, e_2 and y are supplementary and hence $D_2 \cong e_2Ae_2 \cong yAy \cong D \cong D_1$, from which, as before, $M_1 \cong M_2$. The proof of the theorem is therefore complete.

PRINCETON UNIVERSITY

NOTE ON A CLASS OF HARMONIC FUNCTIONS

BY G. C. EVANS

1. *Introduction.* Let S be a multiply-connected open region, bounded by an exterior circle s_0 and n interior circles $s_1, \cdots, s_n$, not touching, and let $u(P)$ be harmonic, not necessarily bounded, in S. The author has shown that a necessary and sufficient condition that $u(P)$ be given as the sum of logarithmic terms and Poisson-Stieltjes integrals around the several boundaries is that the integrals $\int_0^{2\pi}|u|d\theta_i$, extended around circles concentric with each boundary circle, remain bounded in the neighborhood of each boundary circle. He has shown also that a necessary and sufficient condition that $u(P)$ be given by the corresponding Stieltjes integral generalization of the formula in terms of the normal derivative of the Green's function is that $\int|u|dh$ remain bounded, where the integration is extended over the whole of a variable curve $g = \text{const.}$, with g the Green's function and h its conjugate, referred to some fixed pole Q. Finally the author has shown that a necessary and sufficient condition for the latter formula is that $u(P)$ be the difference of two functions harmonic and not-negative in S.* It is desired to show that this condition is also necessary and sufficient for the former mode of representation, so that the two forms of representation apply to the same class of functions. That the condition is necessary follows at once from the mode of representation. It is only the sufficiency that requires proof.

2. *Lemma* I. Let S_0 be a simply-connected open region, bounded internally or externally by a circle s_0, of radius

---

* *The Dirichlet problem for the general open finitely-connected region,* PROCEEDINGS OF THE INTERNATIONAL CONGRESS AT TORONTO (1924).

R_0, and let $u(P)$ be the difference of two not-negative harmonic functions $u'(P)$, $u''(P)$ in an annular neighborhood in S_0 of the boundary s_0. Then $\int_0^{2\pi}|u|d\theta$ is bounded for the circles of that neighborhood concentric with s_0.

For convenience let S_0 be an interior region. Let s', of radius r', be a fixed circle in the annular neighborhood and let s be a circle of radius r, $r' \leqq r < R_0$, all these circles having the same center 0. The lemma follows immediately from Bôcher's device, evaluating the surface integral of $(1/r)\partial u/\partial r$. In fact, by integrating this expression in u' over the annular region between s' and s, and equating its two evaluations as an iterated integral, we have the equation

$$\left(\int_0^{2\pi} u'd\theta\right)_r = \left(\int_0^{2\pi} u'd\theta\right)_{r'} + \alpha_1 \log r - \alpha_1 \log r',$$

where α_1 is a constant. Hence

$$\int u'd\theta = \int |u'|d\theta$$

is bounded. Similarly $\int |u''|d\theta$ is bounded, and therefore $\int |u|d\theta$ is bounded.

3. *Lemma* II. In the open region S of § 1, let $u(P)$ be the difference of two functions $u'(P)$, $u''(P)$, harmonic and not-negative in S. Then $u(P)$ is the sum of $n+1$ functions $u_i(P)$, each harmonic except for possible logarithmic singularities at infinity, in the simply-connected open region S_i bounded by s_i, and each satisfying in its region S_i the conditions of Lemma 1.

The first part of this lemma is proved by Osgood.* The last part follows immediately by writing, in the neighborhood of s_i, $u_i(P)$ as the difference $u(P) - \overline{u}(P)$ where

$$\overline{u}(P) = \sum\nolimits_j' u_j(P),$$

the summation not including $j = i$.

---

* Osgood, *Lehrbuch der Funktionentheorie*, Leipzig and Berlin, 1912, pp. 642–644.

4. *Proof of the Theorem.* The statement § 1 can now be proved. In fact, if the hypothesis of Lemma 2 holds, $\int|u_i|\,d\theta_i$ is bounded for circles concentric with s_i in the annular neighborhood in S of s_i, by Lemma 1. The same is true for $\int|u_j|\,d\theta_i, j \neq i$, since $u_j(P)$ is harmonic on s_i itself if j is unequal to i. But

$$\int|u|\,d\theta_i \leqq \sum_{j=0}^{n}\int|u_j|\,d\theta_i.$$

Hence the condition of the first cited theorem of § 1 is satisfied. This is what was to be proved.

4. *Conclusion.* If we make a conformal transformation of S into a general open region T of the same connectivity, without isolated point boundaries, the Toronto memoir already cited shows that the condition that $u(P)$ be the difference of two not-negative harmonic functions is necessary and sufficient that $u(P)$ be given by a generalized Stieltjes integral in terms of the conjugate to the Green's function for T on the ordered boundary points of T. But if we make conformal transformations of each S_i into the corresponding simply-connected region T_i, we obtain a representation of u as the sum of terms u_i where each u_i is the sum of a possible logarithmic term and a generalized Stieltjes integral on the frontier points of T_i, expressed in terms of the conjugate to the Green's function for T_i. And from what we have proved in § 3, it follows that this representation applies to precisely the same class of harmonic functions as the other. The Dirichlet problem is a special case of the problem here discussed.

The connection between the two forms of representation of $u(P)$ is given by integral equations which are worth study on their own account.

THE RICE INSTITUTE

THREE THEOREMS ON NORMAL ORTHOGONAL SETS

BY M. H. STONE

The three theorems which we desire to record here may be of some interest, in spite of their simplicity. So far as the author knows, they have not appeared in the literature. On the other hand, the corollaries appended are to be found in the journals cited in the footnotes, but were there obtained in different fashion. Throughout our work we shall assume on the part of the reader some familiarity with the theorem of Riesz-Fischer and with convergence in the mean.* All functions considered are understood to be real functions defined on the interval (a, b). They are to be summable with summable square in the Lebesgue sense on this interval. The first result is embodied in the following theorem.

THEOREM I. *A necessary and sufficient condition that every pair of functions f_1, f_2 having identical Fourier coefficients with respect to a normal orthogonal set $\{u_k\}$ should have identical coefficients with respect to a second set $\{v_k\}$ is that the formal series for each function of the set $\{v_k\}$ in terms of $\{u_k\}$ should converge in the mean to that function.*

The condition is necessary. We write

$$\int_a^b v_i u_k = c_{ik},$$

so that the formal series for v_i is $\sum_{k=1}^{\infty} c_{ik} u_k$. Suppose that for some fixed i the series does not converge in the mean to v_i; that is, that

---

* Plancherel, RENDICONTI DI PALERMO, vol. 30 (1910), pp. 289–335, especially pp. 290–297. See also Riesz, GÖTTINGER NACHRICHTEN, 1907, p. 117.

$$\sum_{k=1}^{\infty} c_{ik}^2 < \int_a^b v_i^2.$$

By the Riesz-Fischer theorem, the series converges in the mean to a second function v_i^* for which we have

$$\int_a^b v_i^* u_k = c_{ik}$$

for all k. Now

$$\lim_{K\to\infty} \int^b \left(v_i^* - \sum_{k=1}^{K} c_{ik} u_k\right) v_i = 0$$

as we see by the use of Schwarz's inequality. In other words

$$\int_a^b v_i^* v_i = \sum_{k=1}^{\infty} c_{ik}^2 \neq \int_a^b v_i v_i.$$

The functions v_i and v_i^* have identical coefficients with respect to $\{u_k\}$ but different coefficients with respect to the set $\{v_k\}$. This contradicts the hypothesis, and therefore shows the necessity of the condition.

The condition is sufficient. We write $\int_a^b f_1 u_k = \int_a^b f_2 u_k = a_k$. We use the Schwarz inequality to prove that

$$\lim_{K\to\infty} \int_a^b f_1 \left(v_i - \sum_{k=1}^{K} c_{ik} u_k\right) = 0,$$

$$\lim_{K\to\infty} \int_a^b f_2 \left(v_i - \sum_{k=1}^{K} c_{ik} u_k\right) = 0.$$

It follows that $\int_a^b f_1 v_i = \int_a^b f_2 v_i = \sum_{k=1}^{\infty} c_{ik} a_k$. This completes the proof of Theorem I.

We shall next prove the following theorem.

THEOREM II. *A necessary and sufficient condition that the formal Fourier series* $\sum_{k=1}^{\infty} a_k u_k$, $\sum_{k=1}^{\infty} b_k v_k$ *associated with an arbitrary function* f *should converge in the mean to the same function* f^* *is that the formal series for each function of the set* $\{v_k\}$ *in terms of the set* $\{u_k\}$ *should converge in the mean to that function and vice versa.*

The condition is necessary. This fact appears as obvious as soon as we set $f = u_k$, $f = v_k$, $(k = 1, 2, \cdots, \infty)$.

The condition is sufficient. The series $\sum_{k=1}^{\infty} a_k u_k$ converges in the mean to a function f_1^*, the series $\sum_{k=1}^{\infty} b_k v_k$ to a function f_2^*, by the theorem of Riesz-Fischer. We now appeal to Theorem I to show that

$$\int_a^b f u_k = \int_a^b f_1^* u_k = \int_a^b f_2^* u_k, \quad (k = 1, 2, \cdots, \infty).$$

Since f, f_2^* have identical coefficients with respect to $\{v_k\}$, they also have identical coefficients with respect to $\{u_k\}$ by Theorem I. The equalities above then result. In the same manner we show that

$$\int_a^b f v_k = \int_a^b f_1^* v_k = \int_a^b f_2^* v_k, \quad (k = 1, 2, \cdots, \infty).$$

Next, from Bessel's inequality, we obtain

$$\sum_{k=1}^{\infty} a_k^2 = \int_a^b f_1^{*2} \geqq \sum_{k=1}^{\infty} b_k^2, \quad \sum_{k=1}^{\infty} b_k^2 = \int_a^b f_2^{*2} \geqq \sum_{k=1}^{\infty} a_k^2.$$

The inequalities are therefore impossible. We see that either of the two series $\sum_{k=1}^{\infty} a_k u_k$, $\sum_{k=1}^{\infty} b_k v_k$ converges in the mean to each of the two functions f_1^*, f_2^*. The two functions are therefore the same, in the sense that they can differ at most on a set of zero measure. This completes the proof of Theorem II.

The two corollaries which we shall now discuss are to be found in the literature.

COROLLARY I.* *A necessary and sufficient condition that a normal orthogonal set $\{u_k\}$ be closed is that the formal series for each function of an arbitrary closed normal orthogonal set $\{v_k\}$ in terms of $\{u_k\}$ converge in the mean to that function.*

In order that $\{u_k\}$ be a closed set, it is necessary and sufficient that every function represented by the convergence in the mean of its formal series in terms of the closed set $\{v_k\}$ be so represented in terms of $\{u_k\}$. Theorem II then gives us the last link in the proof.

---

* Lauricella, RENDICONTI DEI LINCEI, (5), vol. 21 (1912), pp. 675–85.

COROLLARY II.* *Let* $\{\Phi_k\}$ *be a set of functions such that the system of equations* $\int_a^b \theta \Phi_k = 0$, $k = 1, 2, \cdots, \infty$, *implies that the function* θ *is identically zero except possibly on a set of zero measure. Then a necessary and sufficient condition that a normal orthogonal set* $\{u_k\}$ *be closed is that the formal series for each function of the set* $\{\Phi_k\}$ *in terms of* $\{u_k\}$ *converge in the mean to that function.*

By Schmidt's process of normalization and orthogonalization we form from the set $\{\Phi_k\}$ a closed normal orthogonal set $\{v_k\}$. Each function of $\{v_k\}$ is a linear combination with constant coefficients of a finite set of functions of $\{\Phi_k\}$, and conversely. We recall that series which can be summed in the sense that they converge in the mean can be added term by term. By using Corollary I in conjunction with this fact we complete our demonstration.

Of particular interest under the second corollary is the special case obtained by setting $\Phi_k = x^{k-1}$, $k = 1, 2, \cdots, \infty$. For further remarks on these two corollaries we refer to the papers already mentioned.

A theorem very similar to Theorem II follows.

THEOREM III. *A necessary and sufficient condition that the difference of the two formal series* $\sum_{k=1}^{\infty} a_k u_k$, $\sum_{k=1}^{\infty} b_k v_k$ *associated with an arbitrary function* f *should converge in the mean to zero, is that the formal series for each function of* $\{v_k\}$ *in terms of* $\{u_k\}$ *converge in the mean to that function, and vice versa.*

The condition is necessary. We can see this fact readily by putting $f = u_k$, $f = v_k$, $(k = 1, 2, \cdots, \infty)$.

The condition is sufficient. We know by Theorem II that the two series converge in the mean to the same function f^*. Hence their difference converges in the mean to zero.

---

* Severini, RENDICONTI DI PALERMO, vol. 36 (1913), pp. 177–202, especially pp. 194–95.

This last theorem may be looked upon as giving a necessary and sufficient condition that two normal orthogonal sets $\{u_k\}$, $\{v_k\}$ be equivalent in the sense that the formal series for an arbitrary function in terms of them always converge in the mean to the same function.

NEW YORK CITY

ON THE COMPLETE INDEPENDENCE OF THE FUNCTIONAL EQUATIONS OF INVOLUTION*

BY C. C. MAC DUFFEE

1. *Introduction.* The three fundamental theorems or "laws" of involution are commonly written

$$\text{I}\ a^m \cdot a^n = a^{m+n}, \qquad \text{II}\ (a^m)^n = a^{mn}, \qquad \text{III}\ a^n \cdot b^n = (ab)^n.$$

In § 2 of this paper these equations are abstractly formulated as functional equations. In § 3 it is proved that any function satisfying Equations I and II also satisfies III, provided certain underlying conditions A and B are fulfilled.

In § 4 a number system $\mathfrak{R}$ is introduced whose elements are the numbers $[\xi, r]$ where ξ and r are real numbers. Two operations, addition and multiplication, are introduced, and it is shown that these operations obey all the ordinary laws of algebra except the associative law of addition. This number system $\mathfrak{R}$ is then used in the discussion of the complete independence† of Equations I, II and III.

---

* Presented to the Society, April 28, 1923.

† Consider n conditions. There are conceptually 2^n distinct cases to be considered according as Condition 1 holds or does not hold, Condition 2 holds or does not hold, $\cdots$, Condition n holds or does not hold. If none of these 2^n cases is empty, the n conditions are said to be completely independent. Cf. E. H. Moore, THE NEW HAVEN MATHEMATICAL COLLOQUIUM, (Yale University Press, 1910), pp. 81, 82.

P 17558

Subject to the conditions of § 2 the functional equations are not completely independent, for I and II imply III. However, the remaining 7 cases are non-empty, for in § 5 of this paper 7 functions are defined which obey the conditions of § 2, each fulfilling one of these 7 cases.

The writer wishes to acknowledge his indebtedness to Professor E. H. Moore of the University of Chicago and to Professor E. V. Huntington of Harvard University for suggestions and criticisms.

2. *A General Formulation of the Functional Equations.* Consider a system Σ composed of the following terms:

(1) Two classes C, C' whose elements will be denoted respectively by c, c'.

(2) Three single-valued functions* or operations:

$$\times \text{ on } CC \text{ to } C, \quad \times' \text{ on } C'C' \text{ to } C', \quad +' \text{ on } C'C' \text{ to } C'.$$

(3) A single-valued function φ on CC' to C.

We assume that the following underlying conditions are satisfied:

A. The right-hand distributive law holds for C', namely

$$(c_1' +' c_2') \times' c_3' = (c_1' \times' c_3') +' (c_2' \times' c_3')$$

for every c_1', c_2', c_3'.

B. There is an element c_0 of C such that for every c there is a c' such that $c = \varphi(c_0, c')$.

We now consider the following postulates in connection with system Σ:

$$\text{I.} \qquad \varphi(c, c_1') \times \varphi(c, c_2') = \varphi(c, c_1' +' c_2'),$$

$$\text{II.} \qquad \varphi(\varphi(c, c_1'), c_2') = \varphi(c, c_1' \times' c_2'),$$

$$\text{III.} \qquad \varphi(c_1, c') \times \varphi(c_2, c') = \varphi(c_1 \times c_2, c').$$

It is understood that these relations hold for all values of the arguments.

As an example of a system Σ in which Conditions *A*

---

[*] Thus for example, if c_1, c_2 are two elements of C, $\times(c_1, c_2)$ or $c_1 \times c_2$ denotes a definite element c_3 of C.

and B and all three postulates are satisfied, we may identify the class C with the class of all real numbers $\geqq 1$, C' with the class of all numbers $\geqq 0$, $\times$ and $\times'$ with the ordinary operation of multiplication, $+'$ with ordinary addition, and $\varphi(\ ,c')$ with the power function $c^{c'}$.

3. THEOREM. *If system Σ satisfies Conditions A and B, and φ satisfies Postulates* I *and* II, *then φ also satisfies Postulate* III.

Suppose that c_0 is the element c_0 of Condition B, and that c_1 and c_2 are any two elements of C. Then by Condition B elements c_1' and c_2' exist so that

$$c_1 = \varphi(c_0, c_1'), \qquad c_2 = \varphi(c_0, c_2'). \tag{1}$$

Therefore we have

$$\begin{aligned}
\varphi(c_1 \times c_2, c') &= \varphi(\varphi(c_0, c_1') \times \varphi(c_0, c_2'), c'), && (1),\\
&= \varphi(\varphi(c_0, c_1' +' c_2'), c'), && (\mathrm{I}),\\
&= \varphi(c_0, (c_1' +' c_2') \times' c'), && (\mathrm{II}),\\
&= \varphi(c_0, c_1' \times' c' +' c_2' \times' c'), && (A),\\
&= \varphi(c_0, c_1' \times' c') \times \varphi(c_0, c_2' \times' c'), && (\mathrm{I}),\\
&= \varphi(\varphi(c_0, c_1'), c') \times \varphi(\varphi(c_0, c_2'), c'), && (\mathrm{II}),\\
&= \varphi(c_1, c') \times \varphi(c_2, c'), && (1),
\end{aligned}$$

which is precisely Postulate III.

4. *An Algebra Whose Elements are Pairs of Real Numbers.* Consider the class of all "numbers" or number pairs of the type $[\xi, r]$ where ξ and r are real numbers. The equality

$$[\xi_1, r_1] = [\xi_2, r_2]$$

implies that $r_1 = r_2$, and unless $r_1 = r_2 = 0$ it also implies that $\xi_1 = \xi_2$. The number $[\xi, 0]$ is supposed to be independent of ξ and will be called *zero*.

When $r_1 r_2 \neq 0$, addition of two such number pairs is defined by the identity

$$[\xi_1, r_1] + [\xi_2, r_2] = [\xi_3, r_3]$$

where r_3 and ξ_3 are defined by the relations

$$r_3 \sin \xi_3 = r_1 \sin \xi_1 + r_2 \sin \xi_2,$$
$$r_3 \cos \xi_3 = r_1 \cos \xi_1 + r_2 \cos \xi_2,$$
$$\frac{\xi_1+\xi_2}{2} - \frac{\pi}{2} < \xi_3 \leqq \frac{\xi_1+\xi_2}{2} + \frac{\pi}{2}.$$

When either r_1 or r_2 or both are zero, addition is defined by the relations

$$[\xi_1, 0] + [\xi_2, r_2] = [\xi_2, r_2], \qquad [\xi_1, r_1] + [\xi_2, 0] = [\xi_1, r_1].$$

The sum always exists uniquely.

The operation of multiplication is defined by the identity

$$[\xi_1, r_1] \times [\xi_2, r_2] = [\xi_1 + \xi_2, r_1 r_2].$$

The product always exists uniquely.

The algebra of such number pairs subject to the operations of addition and multiplication as defined above we shall call the algebra $\mathfrak{N}$. It is evident that both addition and multiplication in $\mathfrak{N}$ are commutative, for the identities which define these operations are symmetric in ξ_1, r_1 and ξ_2, r_2. It may be shown directly that multiplication is associative and distributive with respect to addition. It is worthy of note however that addition is not always associative. This fact is sufficient to show that $\mathfrak{N}$ is not isomorphic with any linear algebra.

5. *Concerning the Complete Independence of the Postulates.* In connection with three conditions 1, 2, 3 there are conceptually eight distinct cases to be considered, each case being defined by the holding or non-holding of each condition. If none of these cases is empty, the three conditions are completely independent. We wish to consider the complete independence of Postulates I, II, III of § 2 in a system Σ with Conditions A and B holding. To settle this question completely we must consider eight cases, and the existence of eight functions φ_{ijk} where i is 1 or 0 according as the function φ_{ijk} satisfies or does not satisfy I, j is 1 or 0 according as it satisfies or does not satisfy II, and k is 1 or 0 according as it satisfies or does not satis-

fy III. In view of the theorem of § 3 no function φ_{110} can exist, for every function φ in Σ satisfying A, B, I and II must necessarily satisfy III. We show that no further dependence theorems are possible by exhibiting examples of functions in each of the remaining seven cases.

In each of the following examples we shall use $\mathfrak{N}$ as the class C', and $\mathfrak{N}$ or a subclass of $\mathfrak{N}$ which is closed under the $\times$-operation as the class C. We shall denote by C_1 the subclass of $\mathfrak{N}$ composed of all numbers $[\xi, r]$ for which $r > 0$; by C_2 the subclass of $\mathfrak{N}$ for which $\xi = 0$ and $r > 0$; by C_3 the subclass for which $r = 1$; and by C_4 the subclass composed of zero and all the other numbers of $\mathfrak{N}$ for which $\xi > 0$. Evidently all four of these subclasses are closed under the $\times$-operation.

By considering in connection with C and C' a single-valued function φ_{ijk}, we have a system Σ. Moreover, since the distributive law holds for $\mathfrak{N}$, Condition A is satisfied. It will be necessary to verify Condition B for each case, to show that $[\zeta, p] = \varphi_{ijk}([\xi, r], [\eta, s])$ is independent of η if $s = 0$, and then investigate whether each of the three postulates I, II, III is or is not satisfied by φ_{ijk} for all values of its arguments.

Case 111. In addition to the example given in § 2, we may mention the function

$$\varphi_{111}([\xi, r], [\eta, s]) \equiv [\zeta, p]$$

on $C_1 C'$ to C_1 where

$$\zeta = s\xi \cos\eta + s \sin\eta \log r, \quad \log p = s \cos\eta \log r - s\xi \sin\eta.$$

If we choose $[0, e]$ to be the element c_0 of Condition B, we find that we may choose as c' any element $[\eta, s]$ defined by the relations $s \sin\eta = \zeta$, $s \cos\eta = \log p$. Hence c' always exists in C' for every element $[\zeta, p]$ in C_1.

When $s = 0$, $[\zeta, p] = [0, 1]$ is independent of η. It may be shown by a direct calculation that φ_{111} satisfies Postulates I, II and III. In fact, this function corresponds to the power function of complex variable theory. The latter however has a denumerable infinity of values, whereas the function φ_{111} on C_1C' to C_1 is single-valued.

Case 011. The function φ_{011} on C_2C' to C_2 is defined by the identity

$$\varphi_{011}([0, r], [\eta, s]) \equiv [\zeta, p] \equiv [0, r^s].$$

If we take $c_0 = [0, e]$, then $c' = [\eta, \log p]$ where η is arbitrary.

Case 101. The function φ_{101} on C_1C' to C_1 is defined by the identity

$$\varphi_{101}([\xi, r], [\eta, s]) \equiv [\zeta, p] \equiv [s\xi \cos \eta + s \sin \eta \log r, r^{s \cos \eta}].$$

If $c_0 = [0, e]$, then c' is any element $[\eta, s]$ such that

$$s \sin \eta = \zeta, \qquad s \cos \eta = \log p.$$

Case 110 is empty in view of the theorem of § 3.

Case 001. The function φ_{001} on C_3C' to C_3 is defined as

$$\varphi_{001}([\xi, 1], [\eta, s]) \equiv [\zeta, 1] \equiv [\xi s \eta, 1].$$

If $c_0 = [1, 1]$, then c' is any element $[\eta, s]$ such that $\eta s = \zeta$.

Case 010. The function φ_{010} on C_4C' to C_4 is defined as

$$\varphi_{010}([\xi, r], [\eta, s]) \equiv [\zeta, p] \equiv [\xi e^{\eta}, rs].$$

If $c_0 = [e, 1]$, then $c' = [\log \zeta - 1, p]$.

Case 100. The function φ_{100} on C_1C' to C_1 is defined as

$$\varphi_{100}([\xi, r], [\eta, s]) \equiv [\zeta, p] \equiv [s \sin \eta, r^{\xi s \cos \eta}].$$

If $c_0 = [1, e]$, then c' is any element $[\eta, s]$ defined by

$$s \sin \eta = \zeta, \qquad s \cos \eta = \log p.$$

Case 000. For the function φ_{000} on $\mathfrak{R}C$ to $\mathfrak{R}$, we may take

$$\varphi_{000}([\xi, r], [\eta, s]) \equiv [\zeta, p] \equiv [\eta, s].$$

Let c_0 be any convenient element of $\mathfrak{R}$. Then $c' = [\zeta, p]$.

Thus the problem of the complete independence of these postulates is completely solved, subject to the mild restrictions of Conditions A and B. It is found that I *and* II *jointly imply* III, *and no other dependence relations exist.*

OHIO STATE UNIVERSITY

THE FREQUENCY LAW OF A FUNCTION OF ONE VARIABLE[*]

BY E. L. DODD

1. *Introduction.* If the probalility that a variable X will take on a value not greater than x is

$$\Phi(x) = \int_{-\infty}^{x} \varphi(x)\,dx,$$

then $\Phi(x)$ is the cumulative frequency law, "Verteilung", for X; whereas the frequency law is $\Phi'(x)$, which is equal to $\varphi(x)$ at a point of continuity of $\varphi(x)$. Two closely related problems will be treated in this paper:

(1) Given the frequency law $\varphi(x)$ for a variable X, to find the frequency law $\psi(y)$ for a function Y of X;

(2) Given $\varphi(x)$ and $\psi(y)$, to find Y.

Under (1), where $Y = f(X)$ is given as a continuous increasing function, no special difficulty arises.[†] When, however, $f(X)$ has an infinitely multiple-valued inverse $g(Y)$, the expression naturally assignable to $\psi(y)$ will not be valid without restriction.

Two real functions $\varphi(x)$ and $f(x)$ will be introduced defined for all real values of x. In case a given $\varphi(x)$ or $f(x)$ is undefined outside a finite interval, the value zero may be assigned to it outside this interval.

As the foregoing problems belong essentially to general analysis, the two theorems to be stated will avoid the language of probability.

---

[*] Presented to the Society, December 30, 1924.

[†] Mayr, *Wahrscheinlichkeitsfunktionen und ihre Anwendungen*, Monatshefte für Mathematik und Physik, vol. 30 (1920), pp. 17–43. Rietz, *Frequency distributions obtained by certain transformations of normally distributed variates*, Annals of Mathematics, (2), vol. 23 (1922), pp. 292–300.

2. *The Determination of the Frequency Law for a Function.*

THEOREM I. *Let*

$$z = \varphi(x) \tag{1}$$

be a single-valued real function of x, integrable over the real continuum. Let

$$y = f(x) \tag{2}$$

be a continuous single-valued real function, defined for all real values of x, with an inverse function,

$$x = g(y), \tag{3}$$

in general multiple-valued. In a formal manner, set

$$\psi(y) = \sum \varphi[g(y)]\ |g'(y)|, \tag{4}$$

where a term is to appear in the sum for each x corresponding to the given y, and where $\psi(y) = 0$ if there is no such x; and set

$$\Psi(\eta) = \int_{f \leqq \eta} \varphi(x)\, dx, \tag{5}$$

this definite integral to be taken over those portions of the x-axis for which $f(x) \leqq \eta$, and to be zero if $f(x) > \eta$ identically. Given a real number η for which $f(x) - \eta$ has at least one zero, suppose that $\varphi[g(\eta)]$ is continuous for each corresponding x. Postulate, further, either Condition A or Condition B, below. Then we shall have

$$\Psi'(\eta) = \psi(\eta). \tag{6}$$

CONDITION A. $f(x) - \eta$ vanishes for but a finite number of values x_i of x; and each $f'(x_i)$ exists and is not zero.

CONDITION B. (1) There is a constant M such that for small enough $\Delta\eta$

$$|g'(\eta + \theta\Delta\eta)| < M, \qquad |\theta| \leqq 1. \tag{7}$$

(2) The lengths of intervals between consecutive zeros of $f(x) - \eta$ have a positive lower bound b.

(3) $\varphi(x)$ has but a finite number of discontinuities.

(4) When $|x|$ is sufficiently large, $\varphi(x)$ does not change sign, and $|\varphi(x)|$ does not increase with $|x|$.

PROOF. Under Condition A and also under Condition B, because of 2^0, the intersections x_i of $y=f(x)$ with $y=\eta$ are isolated. The curve—see 1^0—cuts $y=\eta$, and is not merely tangent to it. We may suppose, then, that for x_i with even subscripts, $g'(\eta)\leqq 0$, and for odd subscripts $g'(\eta)\geqq 0$. With i even, take v_i so that $x_i<v_i<x_{i+1}$,—this becomes simply $x_i<v_i$ in case x_i is the greatest abscissa of intersection. Set

$$(8)\qquad u_i=\int_{x_i}^{v_i}\varphi(x)dx;\qquad u_{i+1}=\int_{v_i}^{x_{i+1}}\varphi(x)dx.$$

Then from (5), using all values of i involved, odd and even,

$$(9)\qquad \psi(\eta)=\sum u_i.$$

But, since $\varphi(x)$ is continuous at x_i,

$$(10)\qquad u_i'=\frac{du_i}{d\eta}=\varphi(x_i)\left|\frac{dx_i}{d\eta}\right|.$$

Under Condition A, then, (6) follows from (3), (4), (9), and (10).

Now, under Condition B, we may take $\eta\Delta$ so small that, on account of 1^0, 3^0, $\varphi(x)$ is continuous when $\eta-\Delta\eta\leqq y\leqq\eta+\Delta\eta$. In (8) and (9) we may now think of η as replaced by any y in $(\eta-\Delta\eta,\ \eta+\Delta\eta)$ to form terms u_i with derivatives u_i'. Suppose that 4^0 is applicable for $x\geqq x_n-b$. Then, by (7), (10), and 2^0,

$$(11)\qquad \sum_n^\infty|u_i'|<M\sum_n^\infty|\varphi(x_i)|\leqq\frac{M}{b}\left|\int_{x_n-b}^\infty\varphi(x)dx\right|.$$

And, since a similar inequality can be set up for terms with $i\leqq -n$, and $\varphi(x)$ is integrable, $\sum u_i'$ converges; and also $\sum u_i$ converges. Thus (6) is also valid under Condition B, by virtue of a theorem* of function theory.

---

* Dini, *Grundlagen für eine Theorie der Functionen einer veränderlichen reellen Grösse*, Leipzig, 1892, p. 154.

Porter, *On the differentiation of an infinite series term by term*, ANNALS OF MATHEMATICS, (2), vol. 3 (1901), pp. 19–20.

If (6) fails for not more than a finite number of values of η

$$(12)\qquad \Psi(\eta) = \int_{-\infty}^{\eta} \sum \varphi[g(y)]\,|g'(y)|\,dy;$$

and, indeed, less restrictive conditions may be given.

To show that the conditions of the theorem are not altogether superfluous, let us form a block diagram, and then round off slightly the corners to make it the graph of a single-valued function,—a graph very roughly representing the wave curve of damped vibration. Suppose, then, that rectangles of unit base are placed alternately above and below the X-axis, resting upon this axis, and each with a vertex upon $x^2y^2 = 1$, except near the origin. And suppose that $\varphi(x) = |x|^{-3/2}$ except near the origin. Then $\Psi'(0) = +\infty$; but $\psi(0) = 0$, since at each intersection, $g'(0) = 0$. The "curve" just described may be modified so as to oscillate about an infinity of lines parallel to the X-axis; and thus for an unlimited number of values in a finite interval $\Psi'(y) \neq \psi(y)$.

The application of the theorem to probability is obvious. The $\Psi(\eta)$ and $\Psi'(\eta)$ in (5) and (6) are respectively the cumulative frequency law and the frequency law or "frequency density" at η for the function $Y = f(X)$, when X is subject to $\varphi(x)$. E. g., if $Y = \cos X$, and $\varphi(x) = 1/2e^{-|x|}$,

$$(13)\qquad \psi(y) = \frac{e^{-x_0} + e^{-x_1}}{1 - e^{-2\pi}}\,\frac{1}{\sqrt{1-y^2}};$$

$x_0 = \text{arc}\cos y$, $0 \leqq x_0 < \pi$, $x_1 = \text{arc}\cos y$, $\pi \leqq x_1 < 2\pi$, for $|y| < 1$; and $\psi(y) = 0$ for $|y| > 1$.

3. *The Determination of a Function connecting two Frequency Laws.*

THEOREM II. *For $a \leqq x \leqq b$, $\alpha \leqq y \leqq \beta$, where a, b, α, β, are constants, finite or infinite, let $\varphi(x)$ and $\psi(y)$ be positive except possibly for isolated values of x or y; and let these functions have finite integrals. Set*

$$(14)\qquad \Phi(\xi) = \int_a^{\xi} \varphi(x)dx; \qquad \Psi(\eta) = \int_\alpha^{\eta} \psi(y)dy.$$

Suppose that for some $c \leqq b$,

$$\Psi(\beta) = \Phi(c). \tag{15}$$

Then there exists a single-valued continuous increasing function,

$$Y = F(X), \tag{16}$$

defined in the interval (a, c) *such that for any* ξ *in* (a, c),

$$\Psi(\eta) = \Phi(\xi); \qquad \eta = F(\xi). \tag{17}$$

PROOF. Set

$$u = \Psi(\eta). \tag{18}$$

Then u is a continuous increasing function of η in (α, β). Hence

$$\eta = \Psi^{-1}(u) \tag{19}$$

is a continuous increasing function of u in $[\Psi(\alpha), \Psi(\beta)]$, that is, by (15), in $[\Phi(a), \Phi(c)]$. Hence

$$\eta = \Psi^{-1}[\Phi(\xi)] = F(\xi) \tag{20}$$

is a continuous increasing function of ξ in (a, c).

This theorem suggests that although a cause X may be subject to some generally recognized frequency law $\varphi(x)$, its effect Y may be subject to almost any imaginable law $\psi(y)$, in the absence of rather definite knowledge of the functional relation between X and Y. In this connection, not only would $F(X)$ be available, as a possible expression of this unknown relation, but an infinite number of functions $f(X)$ to which Theorem I applies.

4. *Conclusion.* If the conditions of both theorems are satisfied everywhere, then there exists one and only one continuous increasing function $F(X)$ whose frequency law is identical with that of $f(X)$,—the cumulative law $\Psi(\eta)$, indeed, being the same for $F(X)$ and for $f(X)$ if only a finite number of points need to be excepted. Thus $F(X)$ may be looked upon as the chief representative of a whole class of functions $f(X)$ associated with each other through $\varphi(x)$.

THE UNIVERSITY OF TEXAS

ON THE ACCESSIBILITY OF AN ARC FROM ITS COMPLEMENT IN SPACE OF THREE DIMENSIONS

(Extract from a letter to R. L. Moore, dated Jan. 29, 1924)

BY CASIMIR KURATOWSKI

In a paper on continuous curves printed in the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, July, 1923, you have raised the question, *whether an arc lying in a three-dimensional space is accessible at each of its ends from every point which does not lie on it.*

I shall answer *positively* this question.

Let AB be an arc lying in the space S, X a point of $S—AB$, P a point of AB (not necessarily an end-point). Let H be a plane containing the points X and P.

The point-set G, composed of those points of the arc AB which lie on H, is a closed plane and bounded set. Since the arc AB passes through G, the set G does obviously satisfy a condition (which you gave with Professor Kline in a paper printed in ANNALS OF MATHEMATICS)* which is necessary and sufficient in order that it should be possible to pass an arc through it. Let T be an arc lying on H, containing G but not X. Hence T is accessible at each of its points on the plane H (see, e. g., Schoenflies). Let XP be an arc lying on H and having in common with T only the point P. Hence XP has in common with AB only the point P and thus the theorem is proved.

Obviously the proof holds true in n-dimensional space, $n \geqq 3$.

THE UNIVERSITY OF WARSAW

---

* The paper here referred to is *On the most general plane closed point-set through which it is possible to pass a simple continuous arc*, ANNALS OF MATHEMATICS, (2), vol. 20 (1919), pp. 218–223. R. L. M.

ON SETS OF THREE CONSECUTIVE INTEGERS WHICH ARE QUADRATIC OR CUBIC RESIDUES OF PRIMES*

BY H. S. VANDIVER

1. *Introduction.* The problem of finding sets of two consecutive integers which are quadratic residues of a prime has been considered by a number of writers† from the point of view of finding integers x and y such that

$$x^2 \equiv y^2 + 1 \pmod{p}$$

p being a prime. I know of no references however, on the problem of finding three consecutive integers which are squares.

As to consecutive integers which are cubic residues, the congruence

$$x^3 \equiv y^3 + 1 \pmod{p}$$

has been studied,‡ but the problem of determining sets of three consecutive integers or sets in arithmetic progression which are cubic residues has apparently not been considered.

In the present note special results on such distribution of quadratic and cubic residues will be obtained.

2. *Three Consecutive Quadratic Residues.* It is known that

$$\frac{x^5-1}{x-1} \equiv 0 \pmod{p}$$

has solutions prime to 5 only when the prime $p \equiv 1 \pmod{5}$.

---

* Presented to the Society, December 29, 1923. The author was enabled to carry out this investigation through a grant from the Heckscher Foundation for the Advancement of Research, established by August Heckscher at Cornell University.

† See the references in Dickson, *History of the Theory of Numbers*, vol. 2, pp. 282–303 (These are included incidentally in the literature on representation of a number as the sum of four squares.)

‡ See Libri[24], Pellet[128–244], Dickson[199], Cornacchia[217], Mantel[277], Hurwitz[213], Schur[288], of Chapter 26 of vol. 2 of Dickson's *History*.

Under this assumption put $v=x+1/x$; then $(x^5-1)/(x-1)=0$ may be written $v^2+v-1=0$ and $v\equiv(-1\pm a)/2 \pmod{p}$ where $a^2\equiv 5 \pmod{p}$. From $v=x+1/x$ it follows that the congruence*

$$\frac{-1\pm a}{2}\equiv x+\frac{1}{x} \pmod{p}$$

has solutions in x. Hence the discriminant is a square modulo p, or there exists an integer u such that

$$(1)\qquad \begin{aligned} 2u^2 &\equiv -5\pm a && \pmod{p},\\ 2u^2 &\equiv -a^2\pm a && \pmod{p},\\ -2au^2 &\equiv a^2(a\pm 1) && \pmod{p} \end{aligned}$$

and, using Legendre's quadratic residue symbol we have

$$(2)\qquad \left(\frac{a\pm 1}{p}\right)=\left(\frac{-2a}{p}\right).$$

If p is of the form $5n-1$, then the congruence $x^2\equiv 5 \pmod{p}$ has a solution, but $(x^5-1)/(x-1)\equiv 0 \pmod{p}$ has no solution. Hence

$$(3)\qquad \left(\frac{a\pm 1}{p}\right)=-\left(\frac{-2a}{p}\right);$$

(2) and (3) prove a proposition proposed as a problem in the AMERICAN MATHEMATICAL MONTHLY.†

From (1) we have $4u^2\equiv -10\pm 2a \pmod{p}$ for $p=5n+1$, but no such congruence holds for $p=5n-1$. Hence we have the following theorem.

THEOREM I. *If* $a^2\equiv 5 \pmod{p}$, *then*

$$\left(\frac{10+2a}{p}\right)=\left(\frac{10-2a}{p}\right)=\pm 1$$

according as p *is a prime of the form* $5n\pm 1$.

In (2), let p be a prime of the form $40k+11$. Then $(-2/p)=1$, and therefore

---

* The fraction m/n in a congruence stands for the integer z, where $zn\equiv m \pmod{p}$.

† Problem 152, vol. 15, 1908, p. 235.

$$\left(\frac{a+1}{p}\right) = \left(\frac{a}{p}\right) = \left(\frac{a-1}{p}\right)$$

and $a+1$, a and $a-1$ are successive integers which are all quadratic residues of p, or all quadratic non-residues. There are two roots of $x^2 \equiv 5 \pmod{p}$, so if $(a/p) = -1$, then $(p-a)/p = 1$, and we have therefore determined three successive integers which are all quadratic residues of p. If $p = 40k+39$, we have by (3)

$$\left(\frac{a+1}{p}\right) = \left(\frac{a-1}{p}\right) = -\left(\frac{-2}{p}\right)\left(\frac{a}{p}\right) = \left(\frac{a}{p}\right)$$

and as before we may select a to be a quadratic residue of p. If $p = 40n+1$, then

$$\left(\frac{a+1}{p}\right) = \left(\frac{a-1}{p}\right) = \left(\frac{a}{p}\right)$$

and these are not quadratic residues of p unless a is a quadratic residue of p, or in other words 5 is a biquadratic residue of p. Since p is of the form $40n+1$, it is not always possible to select a so that it is a quadratic residue. Similar remarks apply to the case $p = 40n+29$. Hence we have the following theorem.

THEOREM II. *If p is of one of the forms $40k+1$, $40k+11$, $40k+29$, $40k+39$, then $a+1$, a, $a-1$ are all quadratic residues of p, where $a^2 \equiv 5 \pmod{p}$ and 5 is a biquadratic residue of p. If p is of one of the two forms $40k+11$, $40k+39$, it is always possible to select a so that $(a/p) = 1$. If p is one of the two forms $40k+1$, $40k+29$, then if one set $a+1$, a, $a-1$, exists then at least one other exists, namely $(p-a)+1$, $p-a$, $(p-a)-1$, having the desired property.*

In case we are testing the existence of sets in the case $p = 40k+1$ or $40k+29$, and p is large, we may avoid the determination of a by the use of the law of biquadratic reciprocity,* which enables us to find the value of $(5/p)_4$.

---

* Smith, *Report on the theory of numbers*, COLLECTED WORKS, vol. 2, p. 77.

EXAMPLES: The three consecutive integers 3, 4, and 5 are quadratic residues of 11; also 19, 20, and 21 are quadratic residues of 79.

It is evident that we can also set up, by similar methods, theorems regarding integers a, $a+1$, $a+2$, modulo p, where each of these integers have prescribed quadratic characters.

3. *Cubic Residues.* We shall now discuss cubic residues. We shall first prove the following lemma.

LEMMA. *If p is an odd prime >3, and*

$$y^3+ay+b\equiv 0 \pmod{p} \tag{4}$$

where a, b, and y are integers, and also k is an integer not congruent to 0 (mod p) *such that*

$$R=\frac{b^2}{4}+\frac{a^3}{27}\equiv k^2 \pmod{p}$$

then

$$\left(-\frac{b}{2}+k\right) \text{ and } \left(-\frac{b}{2}-k\right)$$

are cubic residues modulo p.

To show this we follow a procedure analogous to that employed in Cardan's solution of the cubic. Let $y=u+v$; then (4) gives $3uv=-a$, $u^3+v^3=-b$. Assuming that y is an integer, then $u^2+2uv+v^2$ is an integer. Assuming $p>3$, then $3uv=-a$ shows that uv is congruent to an integer modulo p, hence u^2+v^2 has the same property, as well as u^2+uv+v^2. Also since $(u^3-v^3)^2=4R$ the same is true of u^3-v^3. Now $(u-v)(u^2+uv+v^2)=u^3-v^3$ and $k\not\equiv 0 \pmod{p}$, whence $u-v$ is congruent to an integer modulo p. Since $u+v$ is also, we have both u and v congruent to integers modulo p. Since $u^3=(-b/2+k)$ and $v^3=(-b/2-k)$ the lemma is proved.

We now apply the above result to the function

$$f(x)=x^3-3\lambda x-m\lambda \tag{5}$$

where λ is a prime,

$$4\lambda = m^2+27n^2, \quad m \equiv 1 \pmod{3}.$$

We also have

$$f(x) = (x-\eta_1)(x-\eta_2)(x-\eta_3)$$

where

$$\eta_1 = \alpha \;\; + \alpha^{r^3} + \alpha^{r^6} + \cdots + \alpha^{r^{\lambda-4}},$$
$$\eta_2 = \alpha^{r} \; + \alpha^{r^4} + \alpha^{r^7} + \cdots + \alpha^{r^{\lambda-3}},$$
$$\eta_3 = \alpha^{r^2} + \alpha^{r^5} + \alpha^{r^8} + \cdots + \alpha^{r^{\lambda-2}},$$

$\alpha^{\lambda} = 1$, $\alpha \neq 1$, r a primitive root of λ. By a theorem of Kummer* we know that the congruence $f(x) \equiv 0 \pmod{p}$ always has solutions if $p^{(\lambda-1)/3} \equiv 1 \pmod{\lambda}$. Identifying (5) with (4) we have, since $a = -3\lambda$, $b = -m\lambda$,

$$R = \frac{-27n^2\lambda^2}{4}.$$

Now if p is a prime congruent to 1 or 7, modulo 12, then $(-3/p) = 1$, so that an integer k exists so that $k^2 \equiv R \pmod{p}$. Put $j^2 \equiv -3 \pmod{p}$; then we may write

$$k \equiv \frac{j^3 n}{2} \pmod{p},$$

so that, from the lemma,

$$\frac{\lambda m}{2} + \frac{n\lambda}{2}j^3, \qquad \frac{\lambda m}{2} - \frac{n\lambda}{2}j^3$$

are cubic residues of p.

Hence, if both $4\lambda m$ and $4\lambda n$ are cubic residues, so that $4\lambda m \equiv g^3$ and $4\lambda n j^3 \equiv h^3 \pmod{p}$, then

$$\left(\frac{g}{2}\right)^3 + \left(\frac{h}{2}\right)^3, \quad \left(\frac{g}{2}\right)^3, \quad \left(\frac{g}{2}\right)^3 - \left(\frac{h}{2}\right)^3$$

are all cubic residues and if $(h/2)f \equiv 1 \pmod{p}$ then, modulo p,

---

* CRELLE, vol. 30 (1846), pp. 107–116.

$$\left(\frac{fg}{2}\right)^3+1,\quad \left(\frac{fg}{2}\right)^3,\quad \left(\frac{fg}{2}\right)^3-1$$

give a set of consecutive integers which are cubic residues. If either $4\lambda m$ or $4\lambda n$ is a cubic residue then we may find three cubic residues in arithmetic progression. Noting that if s is a cubic residue of p then $(-s)$ is also, and

$$\frac{n\lambda}{2}j^3-\frac{\lambda m}{2}$$

is a cubic residue, the result follows easily. We may then state the following theorem.

THEOREM. *If* λ *is a prime,*

$$\lambda=\frac{m^2+27n^2}{4},\ m\equiv 1\pmod 3,\quad p\equiv 1 \text{ or } 7\pmod{12},$$

$p^{(\lambda-1)/3}\equiv 1 \pmod\lambda$ *and if* $4\lambda m$ *or* $4\lambda n$ *is a cubic residue of* p *then it is possible to find three cubic residues of* p *in arithmetic progression. If both* $4\lambda m$ *and* $4\lambda n$ *are cubic residues, it is possible to find three consecutive integers which are cubic residues of* p.

EXAMPLE. Let $p=37$, $\lambda=19$, whence $m=7$, $n=1$, $j=16$. Here we find $4\lambda m=14$ is a cubic residue of 37 and 36, 11, 23 are cubic residues of 37 in arithmetic progression, with common difference 12.

The methods employed above apply to other similar problems, but I have not been able to obtain any theorems which are very general. In particular owing to the use of the equations $\varphi(z)=(z-\eta_1)(z-\eta_2)\cdots(z-\eta_e)$, where $\eta_1,\eta_2,\ldots,\eta_e$ are the cyclotomic periods, $ef=n-1$, of the equation $x^n=1$, in the algebraic solution of this equation, and the results of Kummer already cited on the integral divisors of $\varphi(z)$, it is possible to obtain a variety of results, but they all seem to be rather special in character, as are the theorems of this paper.

CORNELL UNIVERSITY

NORMAL CONGRUENCES OF CURVES IN RIEMANN SPACE*

BY HARRY LEVY

The purpose of this note is to determine necessary and sufficient conditions that a congruence of curves in a Riemann space of n dimensions be normal to a family of k—dimensional hypersurfaces. We proceed first to find certain necessary conditions.

If the hypersurfaces, which we denote for brevity by V_k, are defined by the equations

$$(1)\qquad f_i(x^1 x^2 \cdots x^n) = c_i, \quad (i = 1, 2, \cdots, n-k),$$

where the c's are constants, any one of the $n-k$ vectors, $\partial f_i / \partial x^n$, $(i = 1, 2, \cdots, n-k)$ will be orthogonal to the family. Hence, if a congruence of curves is normal to a family V_k, there are $n-k$ linearly independent congruences normal to the same family. Define these congruences by the $n-k$ systems of equations

$$(2)\qquad \frac{dx^1}{\lambda_h|^1} = \frac{dx^2}{\lambda_h|^2} = \cdots = \frac{dx^n}{\lambda_h|^n}, \quad (h = 1, 2, 3, \cdots, n-k).$$

There will be no loss of generality in assuming that the congruences are mutually orthogonal; if we denote the fundamental tensor of the space by g_{ij}, that is, if the linear element is given by the positive definite form

$$(3)\qquad ds^2 = g_{ij}\, dx^i dx^j, \dagger$$

then we may write

$$(4)\qquad g_{rs} \lambda_h|^r \lambda_j|^s = \delta_{hj},$$

---

* Presented to the Society, April 28, 1923.

† An index repeated, once a subscript and once a superscript, is summed from 1 to n.

for $h, j = 1, 2, \cdots, n-k$, where δ_{hj} is Kronecker's delta,

$$\delta_{hj} = \begin{cases} 0, & h \neq j, \\ 1, & h = j. \end{cases} \tag{5}$$

Take any other k congruences, defined by (2) for $h = n-k+1, \cdots, n$, which are mutually orthogonal and orthogonal to the first group. Then (4) will hold for all values of h and j from 1 to n.

The coefficients of rotation, γ_{hij}, of this orthogonal ennuple will be defined by

$$\gamma_{hij} = \lambda_{h|r,s} \lambda_i|^r \lambda_j|^s, \tag{6}$$

where $\lambda_{h|r}$ for $r = 1, 2, \cdots, n$ are the covariant components of the vector $\lambda_{h|}$, that is $\lambda_{h|r} = g_{rs} \lambda_h|^s$, and $\lambda_{h|r,s}$ is the covariant derivative of $\lambda_{h|r}$,

$$\lambda_{h|r,s} = \frac{\partial \lambda_{h|r}}{\partial x^s} - \lambda_{h|t} \Gamma_{rs}|^t. \tag{7}$$

The functions $\Gamma_{rs}|^t$ are the Christoffel symbols of the second kind.

Ricci has shown* that we may solve (6) for $\lambda_{h|r,s}$, and that

$$\lambda_{h|r,s} = \sum_{i,j=1}^{n} \gamma_{hij} \lambda_{i|r} \lambda_{j|s}. \tag{8}$$

If congruences $\lambda_{1|}, \lambda_{2|}, \cdots, \lambda_{n-k|}$ have a family of k-dimensional hypersurfaces as orthogonal trajectories, every direction orthogonal to these $n-k$ congruences must belong to the hypersurface. Hence the equations†

$$X_j(f) \equiv \lambda_j|^r \frac{\partial f}{\partial x^r} = 0, \qquad (j = n-k+1, \cdots, n), \tag{9}$$

must have $n-k$ independent solutions. A necessary and sufficient condition that (9) be a complete system admitting $n-k$ independent solutions is that $(X_iX_j-X_jX_i)f$ be a linear function of the expressions $X_m(f)$. Now

---

* See *Méthodes de calcul différentiel absolu*, Ricci and Levi-Civita, MATHEMATISCHE ANNALEN, vol. 54 (1901), p. 148. As will be seen later, the result of this paper coincides with that of Ricci for the case $k = n-1$.

† Regarding the notation here used. see Goursat, *Les Equations Différentielles du Premier Ordre.*

$$X_iX_j(f) = \lambda_i|^r \frac{\partial}{\partial x^r}\left(\lambda_j|^s \frac{\partial f}{\partial x^s}\right) = f_{,s}\lambda_i|^r \lambda_j|^s + f_s\lambda_j|^s_{,r}\lambda_i|^r$$

where

$$\lambda_j|^s_{,r} = g^{st}\lambda_{j|t,r},$$

where f_r is $\partial f/\partial x^r$, and where f_{rs} is the covariant derivative of f_r. Making use of (8) to eliminate $\lambda_j|^s_{,r}$, we find, by (4) and (9),

$$X_iX_j(f) = f_{rs}\lambda_i|^r\lambda_j|^s + \sum_{l=n-k+1}^{n} \gamma_{jli}X_l + \sum_{h=1}^{n-k}\gamma_{jhi}f_s\gamma_h|^s.$$

Since f_{rs} is symmetric, we have

$$(X_iX_j-X_jX_i)f = \sum_{l=n-k+1}^{n}(\gamma_{jli}-\gamma_{ilj})X_l + \sum_{h=1}^{n-k}(\gamma_{jhi}-\gamma_{ihj})f_s\lambda_h|^s.$$

Hence we may write

$$\sum_{h=1}^{n-k}(\gamma_{jhi}-\gamma_{ihj})f_s\lambda_h|^s = 0.$$

All the coefficients of $f_s\lambda_h|^s$ must vanish, otherwise we would have another relation between the derivatives of f. Consequently, necessary conditions are given by the equations

$$\gamma_{jhi}-\gamma_{ihj} = 0, \tag{10}$$
$$(h = 1, 2, \cdots, n-k;\ i, j = n-k+1, \cdots, n).$$

These conditions are also sufficient, hence we have the following theorem.

THEOREM. *The necessary and sufficient condition that congruences* $\lambda_{1|}, \lambda_{2|}, \cdots, \lambda_{n-k|}$ *have a family of* k*-dimensional hypersurfaces as orthogonal trajectories is that equations* (10) *be satisfied.* When $k = n-1$ this coincides with Ricci's results.*

---

* In a paper presented to the International Mathematical Congress in August, 1924, Ricci, unaware of my work, also obtained this general result using essentially the preceding method.

Suppose every set of $n-k$ congruences of the orthogonal ennuple is normal to a family V_k. Then equations (10) are satisfied when h, i and j take on any distinct values. But Ricci has shown that $\gamma_{hij} = -\gamma_{ihj}$; combining, we find $\gamma_{hij} = 0$ for h, i, j distinct, and this is a sufficient condition that all the congruences be normal. Consequently, *if every set of* $n-k$ *congruences* ($k>1$) *of an orthogonal ennuple has a family of* k *dimensional hypersurfaces as orthogonal trajectories, then all of the congruences are normal.*

PRINCETON UNIVERSITY

LIMITS FOR ACTUAL DOUBLE POINTS OF SPACE CURVES*

BY T. R. HOLLCROFT

1. *Introduction.* Noether† has proved that space curves of maximum genus on non-singular surfaces always exist. In the first part of this paper, such curves will be assumed to exist also when any number of actual double points are added, subject only to the fact that the genus can not be negative for a proper curve and to certain other limitations. The purpose of this paper is to ascertain and define these limitations when all of the actual double points of the space curve are cusps or when all are nodes, and to discuss the existence of space curves with any number of actual double points up to and including the maximum.

From Noether's formula for the maximum genus π_μ of a curve of order n on a non-singular surface of order μ, is obtained the theorem: The minimum number of apparent double points h_μ for a space curve of order n cut out by a surface of order ν on a non-singular surface of order μ, if $\mu\nu \geqq n > \mu(\nu-1)$, $\mu \leqq \nu$, is

---

* Presented to the Society, February 24, 1923.

† M. Noether, *Zur Grundlegung der Theorie der Algebraischen Raumkurven*, ABHANDLUNGEN DER PREUSSISCHEN AKADEMIE DER WISSENSCHAFTEN, 1882, Section 6.

$$h_\mu = \frac{1}{2}(2n-\mu\nu)(\mu-1)(\nu-1).$$

2. *Limits Obtained by Projection.* From an arbitrary point, the apparent double points of a space curve project into plane nodes, but from a point on the developable, from a point on the nodal curve of the developable, or from a triple point of this curve, one, two or three apparent double points respectively are projected into cusps on the plane curve of projection. For $n > 3$ the nodal curve always exists and for $n \geqq 8$ it always has triple points.

Let C_n represent a space curve of order n and maximum genus π_μ with h_μ apparent double points and lying on a non-singular surface of order μ. Let C_n be projected into a plane curve $\bar{C}_n$, for $n \leqq 7$, from any point of the nodal curve of the developable surface. Resulting from the projection of the apparent double points of C_n, the plane curve $\bar{C}_n$ will have two cusps and

$$\delta_1 = h_\mu - 2 = \frac{1}{2}(2n-\mu\nu)(\mu-1)(\nu-1)-2$$

nodes. For $n \geqq 8$ the projection will be made from a triple point of the nodal curve and the plane curve will then have three cusps and $\delta_1 = h_\mu - 3$ nodes.

The maximum number of cusps that may be added to the singularities of this plane curve $\bar{C}_n$ is the number of cusps of $\bar{C}_n$ that can result from the projection of cusps of C_n, that is, it gives an upper limit for the number of cusps of the space curve. The solution of this problem for a plane curve is given in a former paper.* Use the above values of δ_1 in the limits of δ_1 given in the solution of this problem, denote the maximum number of cusps of C_n by β_μ and we have the result:

---

* T. R. Hollcroft, *Singularities of curves of given order*, this BULLETIN, vol. 29 (1923), Problem 1, pp. 409–10.

The maximum number of cusps β_μ for C_n as defined above, provided that $\mu\nu \geqq n > \mu(\nu-1)$, is as follows:

For $n \leqq 7$,

$$\beta_\mu = \frac{1}{2}(n-1)(n-2) - h_\mu; \tag{1}$$

For $n \geqq 8$,

If $h_\mu \geqq \frac{1}{2}n(n-10) + 3 + 2\sqrt{4n+13}$,

$$[\beta_\mu]^* \leqq \frac{3}{8}[n(n-2) - 2(h_\mu + 1)]; \tag{2}$$

If $\frac{1}{2}n(n-10) + 3 + 2\sqrt{4n+13} > h_\mu > \frac{1}{2}n(n-13)$
$+ 14 - \sqrt{16n-23}$,

$$[\beta_\mu] \geqq \frac{1}{6}[2n(n-1) - 4h_\mu - 15 - \sqrt{4n(n-9) - 8h_\mu + 105}\,]; \tag{3}$$

If $h_\mu \leqq \frac{1}{2}n(n-13) + 14 - \sqrt{16n-23}$,

$$[\beta_\mu] \leqq \frac{1}{4}[(n-3)(n+6) - 2(h_\mu + 3)]. \tag{4}$$

An exception to (1) occurs when $\mu = 2$ since a C_7 on a quadric can have but five cusps.

Since $\mu\nu \geqq n > \mu(\nu - 1)$, in (2)

$$h_\mu \geqq \frac{1}{2}n(n-10) + 3 + 2\sqrt{4n+13}$$

only for $\mu + \nu \leqq 8$; and in (3)

$$h_\mu \geqq \frac{1}{2}n(n-13) + 14 - \sqrt{16n-23}$$

only for $\mu + \nu \leqq 14$.

If in the formula for h_μ we let $\mu = 2$ and $\nu = n/2$ or

---

[*] The symbol $[x]$ followed by $\leqq$ ($\geqq$) means the largest (one greater than the largest if not equal) integer contained in the expression on the right.

$(n+1)/2$ we obtain $h_2 = \frac{1}{4}n(n-2)$ or $\frac{1}{4}(n-1)^2$ for n even or odd respectively. The inequality

$$[h_2] \geqq \frac{1}{4}n(n-2)$$

defines the smallest number of apparent double points for a proper space curve of order n.* Then when $\mu = 2$, C_n can have the largest number of actual double points possible for a given order n.

Using the above value of h_2 in the preceding limits, we can determine the values of n for which the several limits of δ_1 hold and thus obtain the following formulas.

For $n \leqq 6$,

$$[\beta_2] \leqq \frac{1}{4}(n-2)^2;$$

For $7 \leqq n \leqq 11$,

$$[\beta_2] \leqq \frac{3}{16}(n^2-2n-4);$$

For $12 \leqq n \leqq 25$,

$$[\beta_2] \geqq \frac{1}{6}\left(n^2-15-\sqrt{2n(n-12)+105}\right);$$

For $n \geqq 26$,

$$[\beta_2] \leqq \frac{1}{8}(n^2+8n-44).$$

With the exception of the limit for $n \leqq 6$, these are only upper bounds for β_2 and can not be attained, as will be shown in the two following sections.

Since every rational plane curve of order $n \geqq 4$ may have three or more cusps, the above considerations do not limit the nodes of a space curve.

3. *Limits given by* $\gamma \geqq 0$. Certain limits for β arise from the fact that none of the characteristic numbers of a

---

* G. Halphen, *Sur quelques propriétés des courbes gauches algébriques*, Bulletin de la Société de France, vol. 2, p. 42.

proper space curve can be negative, and in some cases these limits are more restrictive than those derived above by projection. Of these, the lowest limit for β is given by γ, the number of cusps of the nodal curve. For $\gamma \geqq 0$, there results the inequality

$$[\beta] \leqq \frac{1}{6}[r(n-6)+4(n-\omega-H)-2v],$$

wherein r is the rank, v the number of linear inflections, ω the number of double tangents and H the number of nodes of a space curve. When $\omega = v = 0$, as may happen for curves of any order, the space curve can have a larger number of cusps. There results

$$[\beta] \leqq \frac{1}{6}[r(n-6)+4(n-H)].$$

This serves as an upper bound to β for all space curves.

Space curves that lie on a quadric have $[h] \geqq n(n-2)/4$, so that

$$[r] \leqq \frac{n^2}{2} - 3\beta - 2H.$$

Substitute this limit for r in the foregoing inequality derived from $\gamma \geqq 0$ and we obtain

$$[\beta_2] \leqq \frac{1}{6}[n(n-2)-4H].$$

Since, for a maximum β, $H = 0$ or 1, the limit becomes

$$[\beta_2] \leqq \frac{1}{6} n(n-2).$$

If, when $n(n-2)$ is divided by 6, the remainder is greater than or equal 4, there may be one node. This occurs only when n is of the form $n \equiv 1$, mod 6.

This formula gives as small a limit for β_2 for $7 \leqq n \leqq 26$ as those obtained by projection, or a smaller limit.

For curves on cubic surfaces, we find that the inequality $\gamma \geqq 0$ leads to the limit

$$[\beta_3] \leqq \frac{1}{9}(n-2)(n+3).$$

For $n \leqq 8$, all the actual double points may be cusps. The above limit is more restrictive than those derived by projection for $12 \leqq n \leqq 17$ and gives the same value of β_3 for $n = 9, 10, 11$.

For curves on quartic surfaces, there are no values of n for which the limit derived from $\gamma \geqq 0$ gives a smaller limit for β_4 than those given by the projection formulas.

If in the above inequality for β in terms of n and H, we let $\beta = 0$ instead of H, we obtain

$$[H_2] \leqq \frac{1}{4} n(n-2).$$

This limit, however, is always greater than the maximum genus of C_n on a quadric, so the fact that γ cannot be negative does not restrict the number of nodes when $\beta = 0$. The same is true for curves on surfaces of higher order than two.

4. *Limits Caused by Adjoint Curve.* As was shown by Noether,* the necessary and sufficient condition that a noncomposite plane curve be the projection of a space curve is that the double points of the plane curve which are projections of the apparent double points of the space curve lie on a plane curve of order $n-i-3$, where $i \leqq \mu+\nu-4$ if $\mu\nu = n$ or $i \leqq \mu+\nu-5$ if $\mu\nu > n > \mu(\nu-1)$. This curve of order $n-i-3$ is called an adjoint curve. There are adjoint curves for all values of i less than or equal to the given limits, but since the one for the largest value of i is of least order and thus imposes the greatest number of conditions on C_n, only that one will be considered here.

First consider curves that are complete intersections of the two surfaces of orders μ and ν, that is, let $\mu\nu = n$. For such curves the greatest value of i is $\mu+\nu-4$ and the order of the adjoint curve of least order is $(\mu-1)(\nu-1)$.

---

* M. Noether, loc. cit., § 4, Theorem II.

The number of conditions necessary to determine this adjoint curve is $\frac{1}{2}(\mu-1)(\nu-1)[(\mu-1)(\nu-1)+3]$. Subtracting this from $h_\mu=\frac{1}{2}\mu\nu(\mu-1)(\nu-1)$, we find that there are $(\mu-1)(\nu-1)(\mu+\nu-4)/2$ double points of $\overline{C}_n$ through which the adjoint curve passes gratuitously, and therefore that among the h_μ double points of $\overline{C}_n$ there exist $\frac{1}{2}(\mu-1)(\nu-1)(\mu+\nu-4)$ linear relations.

For a sufficiently large n, these conditions on $\overline{C}_n$ lead to a limit to the number of cusps (and, for a still larger n, to the number of nodes) of C_n. The projections of the h_μ apparent double points of C_n, since they may be projected into $h_\mu-3$ nodes and 3 cusps of $\overline{C}_n$, account at most for $h_\mu+3$ conditions on $\overline{C}_n$. Then the number of degrees of freedom of $\overline{C}_n$ is

$$\frac{1}{2}n(n+3)-h_\mu-3-\frac{1}{2}(\mu-1)(\nu-1)(\mu+\nu-4)$$
$$=\frac{1}{2}[6n-2+(\mu+\nu)(\mu+\nu-5)],$$

all of which may be absorbed by assigning cusps or nodes to $\overline{C}_n$. Lefschetz* has proved that a node accounts for one and a cusp for two invariants, and that for the genus $p\geqq 2$, no more than $\frac{1}{2}n(n+3)-8$ of the $\frac{1}{2}n(n+3)$ conditions necessary to determine a plane curve may be invariants. For the curves $\overline{C}_n$ involved here, more than eight metric conditions are given in each case by the linear relations that exist among the double points of $\overline{C}_n$ which are projections of the apparent double points of C_n, so that all the remaining degrees of freedom of $\overline{C}_n$ may be absorbed by assigning nodes or cusps to $\overline{C}_n$.

An upper limit to the number of cusps of C_n is, therefore,

$$[\beta\mu]\leqq\frac{1}{4}[2(3n-1)+(\mu+\nu)(\mu+\nu-5)],$$

and an upper limit to the number of nodes is

$$H_\mu = 3n-1+\frac{1}{2}(\mu+\nu)(\mu+\nu-5).$$

---

* S. Lefschetz, *On the existence of loci with given singularities*, Transactions of this Society, vol. 14 (1913), pp. 23–41.

When C_n is not a complete intersection, for curves with the minimum number of apparent double points, $\mu\nu > n > \mu(\nu-1)$, $i = \mu+\nu-5$ and the number of linear conditions among the h_μ double points of $\overline{C}_n$ is, for $\mu \leqq \nu$,

$$\frac{1}{6}[i(i+1)(i+2)-(i-\mu+1)(i-\mu+2)(i-\mu+3)]$$

for $\mu \leqq 4$, and

$$\frac{1}{6}[i(i+1)(i+2)-(i-\mu+1)(i-\mu+2)(i-\mu+3) \\ -(i-\nu+1)(i-\nu+2)(i-\nu+3)],$$

for $\mu \geqq 5$.

An upper limit to the cusps of C_n is found as before by subtracting the sum of each of the foregoing expressions and $h_\mu+3$ from $\frac{1}{2}n(n+3)$, which gives, respectively,

$$[\beta_\mu] \leqq \frac{1}{4}[(n-1)(n+2)+(\mu+\nu-\mu\nu)(2n+\mu+\nu-\mu\nu-7) \\ -\frac{1}{3}(\mu-2)(\mu-3)(\mu-4)]$$

for $\mu \leqq 4$, and for $\mu \geqq 5$,

$$[\beta_\mu] \leqq \frac{1}{4}[(n-1)(n+2)+(\mu+\nu-\mu\nu)(2n+\mu+\nu-\mu\nu-7)].$$

Since the additional terms in the first limit by which it differs from the second vanish identically for values of μ within the limit $\mu \leqq 4$ for which the formula holds, the second limit for β_μ holds for all values of μ, $\mu \leqq \nu$ and $\mu\nu > n > \mu(\nu-1)$.

Also as before, the upper limit to the number of nodes of C_n for $\mu \leqq \nu$ and $\mu\nu > n > \mu(\nu-1)$ becomes

$$H_\mu = \frac{1}{2}[(n-1)(n+2)+(\mu+\nu-\mu\nu)(2n+\mu+\nu-\mu\nu-7)].$$

These limits for β_μ are more restrictive than projection limit (2) for $\mu+\nu \geqq 9$, but since limit (2) holds only for $\mu+\nu \leqq 8$, it is not superseded by this new limit.

The new limits are more restrictive than projection limit (3) for $\mu+\nu\geqq 9$. Since projection limit (3) holds for $\mu+\nu\leqq 14$, it is superseded by the new limits for all values of μ and ν such that $\mu+\nu\geqq 9$. Projection limit (4) is entirely superseded by the new limits for all values of μ and ν.

5. *Summary of Limits.* Finally, for any proper space curve of order n which is formed by the intersection of a surface of order ν with a non-singular surface of order μ such that $\mu\nu\geqq n>\mu(\nu-1)$, there exist the following upper limits to the number of cusps it may possess·

When $n\leqq 7$ ($n\leqq 6$ for $\mu=2$),

$$\beta_\mu = \frac{1}{2}(n-1)(n-2)-h_\mu;$$

When $n\geqq 8$ and $\mu+\nu\leqq 8$,

For $h_\mu\geqq\frac{1}{2}n(n-10)+3+2\sqrt{4n+13}$

$$[\beta_\mu]\leqq\frac{3}{8}[n(n-2)-2(h_\mu+1)];$$

For $\frac{1}{2}n(n-10)+3+2\sqrt{4n+13}>h_\mu>\frac{1}{2}n(n-13)+14-\sqrt{16n-23}$

$$[\beta_\mu]\geqq\frac{1}{6}\left[2n(n-1)-4h_\mu-15-\sqrt{4n(n-9)-8h_\mu+105}\right];$$

When $\mu+\nu\geqq 9$,

If $\mu\nu=n$,

$$[\beta_\mu]\leqq\frac{1}{4}[(\mu+\nu)(\mu+\nu-5)+2(3n-1)];$$

If $\mu\nu>n>\mu(\nu-1)$,

$$[\beta_\mu]\leqq\frac{1}{4}[(n-1)(n+2)+(\mu+\nu-\mu\nu)(2n+\mu+\nu-\mu\nu-7)].$$

In the limits derived from the properties of the adjoint curve, if the equality sign holds, the curve C_n with the maximum number of cusps can have no nodes. If the inequality sign holds, it may have one, but only one.

Under the same general conditions as above, when $\beta_\mu = 0$ the following limits exist for the number of nodes H_μ:

When $\mu\nu = n$, $\mu \leqq \nu$,

For $\mu = 2$, $\nu \geqq 15$; $\mu = 3$, $\nu \geqq 11$; all values of μ such that $4 \leqq \mu \leqq 7$ and $\mu + \nu \geqq 13$; all values of $\nu \geqq \mu > 7$;

$$H_\mu = 3n - 1 + \frac{1}{2}(\mu + \nu)(\mu + \nu - 5);$$

When $\mu\nu > n > \mu(\nu - 1)$, $\mu \leqq \nu$,

For $\mu = 2$, $\nu \geqq 16$; $\mu = 3$, $\nu \geqq 13$; $\mu = 4$, $\nu \geqq 11$ if ν is of form $(n+1)/4$ or $(n+2)/4$, or $\nu \geqq 12$ if ν is of form $(n+3)/4$; $\mu = 5, 6, 7$, $\nu = (n+c)/\mu$, if $c = 1$, $\mu + \nu \geqq 14$ and if $c > 1$, $\mu + \nu \geqq 15$; and for all values of $\nu \geqq \mu > 7$;

$$H_\mu = \frac{1}{2}[(n-1)(n+2) \\ + (\mu + \nu - \mu\nu)(2n + \mu + \nu - \mu\nu - 7)].$$

For $\mu = 2$ and n even, the adjoint curve is of order $\frac{1}{2}(n-2)$ and $\frac{1}{8}(n-2)(n-4)$ conditions are imposed on the $\frac{1}{4}n(n-2)$ double points of $\overline{C}_n$. For $7 \leqq n \leqq 26$ the limit

$$[\beta_2] \leqq \frac{1}{6} n(n-2)$$

is lower than that given by the projection limits. The number of conditions imposed by these cusps and nodes on $\overline{C}_n$ together with the linear relations existing among them amount to $(82n - 5n^2 - 96)/24$. This is negative for $n \geqq 16$, so that for $n \geqq 16$

$$[\beta_2] \leqq \frac{1}{6} n(n-2) - \frac{1}{48}(5n^2 - 82n + 96) = \frac{1}{16}(n^2 + 22n - 32).$$

The limit for n odd is found in a similar way. These limits can also be obtained by substituting $\mu = 2$ and $\nu = n/2$ or $(n+1)/2$ respectively in the general limits for β_μ.

For $\mu = 3$ and $12 \leqq n \leqq 17$ the limit

$$[\beta_3] \leqq \frac{1}{9}(n-2)(n+3)$$

is lower than that given by the projection limits. By taking $\mu = 3$ and $\nu = n/3$, $(n+1)/3$, $(n+2)/3$, respectively, and following each through as for $\mu = 2$, we find that within these limits for n the above limit is lower also than the limits derived from the conditions imposed on $\overline{C}_n$ by the adjoint curve.

For $\mu = 3$, the following limits exist.

For $n \leqq 7$,

$$[\beta_3] \geqq \frac{1}{6}(n-1)(n-2);$$

For $8 \leqq n \leqq 11$,

$$[\beta_3] \leqq \frac{1}{8}(n^2-9);$$

For $12 \leqq n \leqq 17$,

$$[\beta_3] \leqq \frac{1}{9}(n-2)(n+3);$$

For $n \geqq 18$,

If $\nu = \frac{n}{3}$, $\quad [\beta_3] \leqq \frac{1}{36}(n^2+57n-72)$;

If $\nu = \frac{n+1}{3}$, $[\beta_3] \leqq \frac{1}{36}(n^2+65n-116)$;

If $\nu = \frac{n+2}{3}$, $[\beta_3] \leqq \frac{1}{36}(n^2+61n-98)$.

For $\nu = \frac{n}{3}$ and $n \geqq 33$,

$$H_3 = \frac{1}{18}(n^2+57n-72);$$

For $\nu=\frac{n+1}{3}$ and $n \geqq 38$,

$$H_3=\frac{1}{18}(n^2+65n-116);$$

For $\nu=\frac{n+2}{3}$ and $n \geqq 37$

$$H_3=\frac{1}{18}(n^2+61n-98).$$

When $\mu=2$, in view of the fact that space curves with the least number of apparent double points for a given n lie on a quadric, this theorem results:

Every proper space curve with the maximum number of cusps for a given order n lies on a quadric and the maximum for each order is given by the following limits.

(I) For $n \leqq 6$,

$$[\beta_2] \leqq \frac{1}{4}(n-2)^2;$$

(II) For $7 \leqq n \leqq 15$,

$$[\beta_2] \leqq \frac{1}{6}n(n-2);$$

(III) For n even and $n \geqq 16$,

$$[\beta_2] \leqq \frac{1}{16}(n^2+22n-32);$$

(IV) For n odd and $n \geqq 17$,

$$[\beta_2] \leqq \frac{1}{16}(n^2+24n-41).$$

For $n=7$ the curve may be rational, that is, it may have one node in addition to the five cusps. For $n>7$, space curves with the maximum number of cusps for a given n are irrational. For some values of n they may have one node, for others, none.

With regard to nodes, a similar statement may be made, viz., every proper space curve with the maximum number of nodes for a given order n lies on a quadric, and the maximum for each order is given by the following limits (no limit except genus for $n < 30$).

For n even and $n \geqq 30$,

$$H_2 = \frac{1}{8}(n^2 + 22n - 32);$$

For n odd and $n \geqq 31$,

$$H_2 = \frac{1}{8}(n^2 + 24n - 41).$$

6. *Existence of Curves.* Since for $n \geqq 4$, plane curves with three cusps and any number of nodes up to the maximum allowed by the genus exist, and since any two surfaces may have as many ordinary contacts as the apparent genus of their curve of intersection, there are no further questions as to the existence of space curves with any number of nodes up to and including the maximum, except those relating to the adjoint curve and they will be considered in the discussion of the existence of space curves with cusps.

The fact proved by Noether that space curves with only apparent double points, of maximum genus and lying on non-singular surfaces always exist has been mentioned above. The addition of cusps requires that the two surfaces containing the C_n have that number of stationary contacts. So far as the surfaces are concerned, they may have as many stationary contacts as the apparent genus of C_n. In regard to C_n itself, the addition of cusps reduces all the other characteristic numbers, some very rapidly, notably γ, which serves as an effective limit on β. Finally, when the maximum β has been so chosen that all these characteristic numbers are non-negative (with the single exception of the genus of the nodal curve, which may be negative, since the nodal curve may be composite when the cuspidal curve is not) there appears to be nothing more appertaining only to the space curve itself to interfere with its existence.

If, however, we attempt to project C_n upon a plane when β is limited only as above, we find that for n greater than certain limits, no proper plane curve exists into which C_n may be projected and therefore that C_n itself does not exist as a proper space curve. In order that the plane curve of projection exist, two conditions must be satisfied—first, the number of cusps and nodes assigned to it must not exceed the maximum number a plane curve of that order can possess, and second, the nodes and cusps of the plane curve which are projections of the apparent double points of the space curve, must lie on the adjoint curve.

The existence of plane curves with any number of cusps up to and including the maximum has been proved by both Lefschetz* and Coolidge† for $p \leqq p_0$, where p_0 is the genus associated with a given n on the assumption that the bitangents and inflections are both as near zero as possible. Although lacking a formal proof as yet, the existence of curves for $p > p_0$ with any number of cusps up to and including the maximum is practically assured.

In all the preceding, we have used the condition that the double points of the plane curve of projection, which are projections of the apparent double points of the space curve, must lie on an adjoint curve only as the necessary condition that the plane curve be the projection of the space curve. But Noether‡ and Valentiner§ have both proved that this condition is also sufficient—that such a plane curve is always the projection of a space curve.

Therefore the existence of space curves with cusps up to and including the maximum as given by the preceding limits is established, if plane curves for $p > p_0$ with any number of cusps up to and including the maximum exist.

WELLS COLLEGE

---

* S. Lefschetz, loc. cit., pp. 33–39.

† J. L. Coolidge, *On the existence of curves with assigned singularities* this BULLETIN, vol. 28 (1922), pp. 451–55.

‡ M. Noether, loc. cit., § 3, Theorem I.

§ H. Valentiner, *Bidrag til Rumcurvenes Theori*, Inaugural Dissertation, Kopenhagen, 1881.

SOME MEAN-VALUE THEOREMS CONNECTED WITH COTES'S METHOD OF MECHANICAL QUADRATURE

BY D V. WIDDER

1. *Introduction.* It is the purpose of this paper to prove several mean-value theorems whose importance in the problem of mechanical quadrature will appear later.

2. *Extension of a Theorem of Birkhoff.* The following theorem has been proved by G. D. Birkhoff* for the cases $n = 1, 2, 3, 4, 5$. The proof was made by computations based on tables computed by Cotes. The method is not applicable to the general case.

THEOREM I. *A function $u(x)$ is continuous with its first $2n+2$ derivatives in an interval (a, b), and its $(2n+2)$th derivative itself possesses a derivative for every value of x between a and b. If the function vanishes at the points a and b, and if its first derivative vanishes at $2n+1$-points of the interval, equally spaced and including the end points a and b, then the $(2n+3)$th derivative of the function vanishes between a and b.*

By a successive application of Rolle's theorem it may be inferred directly that $u^{(2n+1)}(x)$ must vanish in the interval (a, b), and that without use of the hypothesis that the points in question are equally spaced. It is important to note, however, that if the points are not equally spaced the conclusion of the theorem is not valid. This fact is made clear by a simple example·

$$u(x) = (x^2-4)^2(4x-1).$$

Here

$$u(2) = u(-2) = u'(2) = u'(-2) = u'(1) = 0.$$

Yet the fifth derivative of $u(x)$ is a constant not zero.

---

* G. D. Birkhoff, *General mean-value and remainder theorems*, TRANSACTIONS OF THIS SOCIETY, vol. 7 (1906), p. 131.

Suppose that the $2n+1$ equally spaced points have coordinates $x_{-n}, x_{-n+1}, \cdots, x_0, x_1, \cdots, x_n$, and suppose the points arranged from left to right on the line in order of increasing indices.

Then

$$x_{-n} = a, \qquad x_n = b.$$

We first make a linear transformation of the independent variable transforming the points in question into the points $-n, -n+1, \cdots, -1, 0, 1, \cdots, n-1, n$. Clearly, if the theorem is valid for this set of points, it is valid for the original set. We may now base our proof on a lemma proved by J. F. Steffensen.*

LEMMA. *If m and n are any positive integers, then the function*

$$Q(x) = \int_{-m}^{x} x(x^2-1)(x^2-2^2)\cdots(x^2-n^2)\,dx \tag{1}$$

does not vanish in the interval $-m < x < m$.

The proof may readily be supplied by geometrical considerations. We may state our hypotheses on $u(x)$ as follows:

$$u(-n) = u(n) = u'(k) = 0,$$
$$k = -n,\ -n+1,\ \cdots,\ -1, 0, 1,\ \cdots,\ n-1, n.$$

Since $u(-n) = 0$,

$$u(x) = \int_{-n}^{x} u'(x)\,dx.$$

Now $u'(x)$ vanishes at the same points as the function

$$P(x) = x(x^2-1)(x^2-2^2)\cdots(x^2-n^2).$$

Determine a function $R(x)$ by the equation

$$u'(x) = P(x)R(x),$$

---

* J. F. Steffensen, CONFÉRENCES DU CINQUIÈME CONGRÈS DES MATHÉMATICIENS SCANDINAVES, p. 126.

$R(x)$ being defined at an integer r by the equation

$$R(r) = \lim_{x\to r} \frac{u'(x)}{P(x)}.$$

Integrating this equation from $-n$ to n, and recalling that $u(-n) = u(n) = 0$, we have

$$u(n)-u(-n) = \int_{-n}^{n} P(x) R(x)\, dx = 0.$$

On integrating the right hand side of this equation by parts, and setting

$$Q(x) = \int_{-n}^{x} P(x)\, dx,$$

it follows that

$$0 = Q(x)R(x)\Big|_{-n}^{n} - \int_{-n}^{n} R'(x)Q(x)\, dx = \int_{-n}^{n} R'(x)Q(x)\, dx.$$

Now by the lemma, $Q(x)$ is a function of one sign in the interval $-n \leqq x \leqq n$, so that we may apply the first mean-value theorem for integrals. We have thus

$$R'(\zeta)\int_{-n}^{n} Q(x)\, dx = 0, \qquad -n<\zeta<n.$$

Now

$$\int_{-n}^{n} Q(x)\, dx \neq 0,$$

since $Q(x)$ does not change sign in the interval $(-n, n)$ and is not identically zero. Hence it follows that

$$(2) \qquad R'(\zeta) = 0.$$

We may now make use of Rolle's theorem and of equation (2) to obtain the result desired through the following device. Form the function

$$\Phi(x) = P(x)[R(x)-R(\zeta)].$$

On account of the relation (2), it is seen that the quantity in brackets vanishes at least twice in the point $x=\zeta$. $P(x)$ vanishes $2n+1$ times in the interval $-n \leqq x \leqq n$. Hence, allowing for multiple zeros, we see that $\Phi(x)$ vanishes at least $2n+3$ times in that interval, whether or not ζ is an integer (that is, a point at which $P(x)$ vanishes). Now by successive applications of Rolle's theorem* we see that $\Phi^{(2n+2)}(x)$ vanishes at least once in the interval $(-n, n)$.

Since the $(2n+2)$th derivative of $P(x)$ is identically zero, and since $P(x)\cdot R(x)$ is $u'(x)$, it follows that

$$\Phi^{(2n+2)}(\xi) = u^{(2n+3)}(\xi) = 0, \qquad -n<\xi<n.$$

The theorem is thus established.

3. *A Companion Theorem.* The second theorem is similar in character to the first, but deals with an even number of points where the first theorem dealt with an odd number. The conclusion, however, is essentially different in that the order of the highest derivative whose vanishing can be inferred is one less than the number of conditions imposed on $u(x)$, while in Theorem I these numbers were equal.†

THEOREM II. *A function $u(x)$ is continuous with its first $2n$ derivatives in an interval (a, b), and its $2n$th derivative itself possesses a derivative for every value of x between a and b. If the function vanishes at a and b, and if its first derivative vanishes at $2n$ points of the interval, equally spaced and including the end points a and b, then the $(2n+1)$th derivative of the function vanishes between a and b.*

No loss of generality will be incurred by taking the points as $-2n+1, -2n+3, \cdots, -3, -1, 1, 3, \cdots, 2n-3, 2n-1$. Then $a=-2n+1$, $b=2n-1$. Since $u(a)=u(b)=0$, we have

$$\int_a^b u'(x)\,dx = u(b)-u(a) = 0.$$

---

* See, for example, de la Vallée Poussin, *Cours d'Analyse*, vol. 1, p 66 (5th edition).

† From the point of view of the general theory of mean-values, as set forth by Birkhoff, this is an essential distinction, Theorem II coming under the general case and Theorem I being an exceptional case.

Set

$$P(x) = (x^2-1)(x^2-3^2)\cdots(x^2-\overline{2n-1}^2),$$

and determine $R(x)$ by the equation

$$u'(x) = P(x)R(x).$$

Then

$$\int_a^b P(x)R(x)\,dx = 0. \tag{3}$$

Now set

$$q(x) = \frac{P(x)}{x-2n+1}, \qquad Q(x) = \int_a^x q(x)\,dx.$$

We can show that $Q(x)$ is a function of one sign in the interval (a, b). By the lemma, $Q(x)$ is a function of one sign in the interval $(a, 2n-3)$. (The lemma is geometrical, and is applicable after a change of unit or a shift of origin.) $Q(x)$ vanishes at a and at $2n-3$* but is positive at intermediate points, $q(x)$ is positive in the interval $(2n-3, b)$, and consequently $Q(x)$ is never negative in the interval (a, b).

From equation (3) we have

$$\int_a^b P(x)\,R(x)\,dx = \int_a^b (x-2n+1)\,q(x)\,R(x)\,dx = 0.$$

Integrating by parts, we obtain

$$(x-2n+1)R(x)Q(x)\Big|_b^a - \int_a^b \frac{d}{dx}[(x-2n+1)R(x)]\,Q(x)\,dx = 0.$$

The first term on the left hand side drops, so that we have on applying the first mean-value theorem for integrals

$$\frac{d}{dx}\Big[(x-2n+1)R(x)\Big]_{x=\zeta}\cdot\int_a^b Q(x)\,dx = 0, \qquad a<\zeta<b.$$

Since $\int_a^b Q(x)\,dx \neq 0$, it follows that

$$\frac{d}{dx}\Big[(x-2n+1)R(x)\Big]_{x=\zeta} = 0. \tag{4}$$

---

* This follows because $q(x-1) = q(-x-1)$.

We now employ a device similar to that used in the proof of Theorem I. Set

$$\begin{aligned}\Phi(x) &= P(x)R(x) - R(\zeta)(\zeta - 2n + 1)q(x)\\ &= q(x)[R(x)(x - 2n + 1) - R(\zeta)(\zeta - 2n + 1)].\end{aligned}$$

Now on account of the relation (4) it is seen that the quantity in brackets has at least two zeros in the point $x = \zeta$; and $q(x)$ has $(2n - 1)$ zeros. It follows then that $\Phi(x)$ has at least $(2n + 1)$ zeros in the closed interval (a, b). By Rolle's theorem $\Phi^{(2n)}(x)$ must vanish at least once in that interval. Since $P(x)R(x)$ is $u'(x)$, and since the $2n$th derivative of $q(x)$ is identically zero, it follows that

$$\Phi^{(2n)}(\xi) = u^{(2n+1)}(\xi) = 0, \qquad a < \xi < b.$$

The theorem is thus established.

4. *Applications to Mechanical Quadrature.* Let us now apply these theorems to the problem of mechanical quadrature. Suppose a function $f(x)$ is known at equally spaced points $x_1, x_2, x_3, \cdots x_{2n+1}$, and suppose it be required to find an approximate expression for

$$\int_{x_1}^{x_{2n+1}} f(x)\,dx.$$

Let $F(x)$ be the polynomial of degree $2n$ at most, taking on the known values of $f(x)$ at the given points. Then form the function

$$u(x) = \int_{x_1}^{x} [f(x) - F(x)]\,dx - A\int_{x_1}^{x} P(x)(x - x_{n+1})\,dx,$$

where

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_{2n+1}),$$

and where A is to be so determined that $u(x_{2n+1}) = 0$. This is possible since

$$\int_{x_1}^{x_{2n+1}} P(x)(x - x_{n+1})\,dx \neq 0.^*$$

---

* This may be seen by writing

$$\int_{x_1}^{x_{2n+1}} P(x)(x - x_{n+1})\,dx = -\int_{x_1}^{x_{2n+1}} dx \int_{x_1}^{x} P(t)\,dt$$

and then applying the Lemma.

Now $u(x)$ satisfies all the conditions of Theorem I (making proper assumptions regarding the continuity of $f(x)$, and consequently

$$u^{(2n+3)}(\xi) = 0; \qquad x_1 < \xi < x_{2n+1}.$$

That is,

$$f^{(2n+2)}(\xi) - A(2n+2)! = 0, \quad A = f^{(2n+2)}(\xi)/(2n+2)!$$

Since $u(x_{2n+1}) = 0$, we have

$$\begin{aligned} \int_{x_1}^{x_{2n+1}} f(x)\,dx = \int_{x_1}^{x_{2n+1}} F(x)\,dx \\ + \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \int_{x_1}^{x_{2n+1}} P(x)(x-x_{n+1})\,dx, \end{aligned} \tag{5}$$

a remainder formula for mechanical quadrature.

In case the number of points is even, we use theorem II and obtain

$$\begin{aligned} \int_{x_1}^{x_{2n}} f(x)\,dx = \int_{x_1}^{x_{2n}} F(x)\,dx \\ + \frac{f^{(2n)}(\xi)}{(2n)!} \int_{x_1}^{x_{2n}} (x-x_1)\cdots(x-x_{2n})\,dx. \end{aligned} \tag{6}$$

These formulas were given by Birkhoff in the paper already cited, but were established only for small values of n. Steffensen established formula (5) by other methods in the article above cited.

5. *Extensions.* The methods here employed might be used to prove certain other mean-value theorems involving equally spaced points, and to which Rolle's theorem is not directly applicable. For example, I have proved the following theorem.

THEOREM III. *A function $u(x)$ having the same amount of continuity as that of Theorem* I *vanishes at the points a and b. If (a, b) is divided into $2n+2$ equal parts, and if $u'(x)$ vanishes at the interior points of division, then $u^{(2n+3)}(x)$ vanishes at an interior point of (a, b).*

HARVARD UNIVERSITY

THE GEOMETRY OF FREQUENCY FUNCTIONS*

BY DUNHAM JACKSON

1. *Introduction.* The Pearson coefficient of correlation, calculated for a finite number of observations, has a geometric interpretation which is simple and almost immediate.† The same thing may be said of the corresponding expression formed for a pair of functions of a continuous variable.‡ When the distribution of the observed quantities is thought of as given by a frequency function, the geometric interpretation of the correlation coefficient is not so obvious. It is the purpose of this paper to show one form that such an interpretation may take.§ The geometric configurations are exactly the same as in the other cases mentioned; the difference is in the manner of setting up the association. This is accomplished by defining an appropriate correspondence between an arbitrary point of a plane, or of space, and an arbitrary linear combination of the variables subjected to measurement.

There will be no assumption that the distributions involved are "normal", in the sense of the Gaussian law. There will be incidental reference to frequency functions having properties that correspond to those of normal orthogonal sets of functions, as the terms are used in the theory of the development of arbitrary functions in series; but the word "normal" will not be used in this connection, because of the possibility of misunderstanding. As the paper is concerned primarily with the establishment of certain formal relations, questions of convergence will be avoided for the most part by assuming that the functions considered are different from zero only over a finite range; the concluding paragraph will deal briefly with distributions that "tail off" to zero at infinity.

---

* Presented to the Society, October 25, 1924.

† Cf., e. g., D. Jackson, *The trigonometry of correlation,* AMERICAN MATHEMATICAL MONTHLY, vol. 31 (1924), pp. 275–280; also the paper cited in the next footnote.

‡ Cf., e. g., D. Jackson, *The elementary geometry of function space*; recently submitted to the AMERICAN MATHEMATICAL MONTHLY.

§ For another form, cf. James McMahon, *Hyperspherical goniometry; and its application to correlation theory for n variables,* BIOMETRIKA, vol. 15 (1923), pp. 173–208. The fundamental idea of attaching a geometric meaning to the correlation coefficient appears to be due to Pearson himself.

The order of presentation in the main part of the paper is closely parallel to that followed in the author's article on *The elementary geometry of function space*, to which reference has been made in a previous footnote. There is enough difference in detail, however, to call for an independent treatment at some length.

2. *Frequency Functions in Two Variables.* To begin with the case of two variables, let $\varphi(x, y)$ be defined and continuous as a function of its arguments over a finite region R of the plane; let it be everywhere positive or zero, and not identically zero. There is no essential loss of generality, and there is some gain of simplicity, in assuming that

$$\int\int_R \varphi(x, y)\, dx dy = 1.$$

For preliminary consideration, let it be supposed further that φ satisfies the conditions*

$$(1) \qquad \int xy\varphi = 0, \qquad \int x^2\varphi = \int y^2\varphi = 1,$$

where $\int xy\varphi$ is an abbreviated notation for the double integral

$$\int\int_R xy\,\varphi(x, y)\, dx\, dy,$$

and where the other integrals are to be interpreted similarly.

Let a and b be any two real numbers. Then we have $\int(ax+by)^2\varphi = a^2+b^2$, because of (1); if (a, b) are taken

---

[*] A method of constructing an infinite variety of functions φ satisfying the conditions imposed will appear presently.

as the coordinates of a point, the value of the integral is the square of the distance of the point from the origin. If P_1 and P_2 are two points with coordinates (a_1, b_1) and (a_2, b_2), and O the origin, the distance P_1P_2 is the square root of the quantity

$$\int [(a_2-a_1)x+(b_2-b_1)y]^2\varphi = (a_2-a_1)^2+(b_2-b_1)^2,$$

and the cosine of the angle P_1OP_2 is

$$\frac{a_1a_2+b_1b_2}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}} = \frac{\int (a_1x+b_1y)(a_2x+b_2y)\varphi}{\sqrt{\int (a_1x+b_1y)^2\varphi}\sqrt{\int (a_2x+b_2y)^2\varphi}}.$$

Thus the fundamental geometric measures can be expressed, somewhat indirectly and artificially to be sure, in terms of integrals involving the function φ.

In transition, let φ be a function subject to the same hypotheses as before, except that the last two of the conditions (1) are not imposed; and let*

$$\sqrt{\int x^2\varphi} = \sigma, \qquad \sqrt{\int y^2\varphi} = \tau.$$

If new variables ξ, η are introduced by means of the relations $\xi = x/\sigma$, $\eta = y/\tau$, so that, incidentally, $d\xi = dx/\sigma$, $d\eta = dy/\tau$, and if the function $\sigma\tau\varphi(x, y)$ is designated, with regard to its dependence on the new variables, by $\Phi(\xi, \eta)$, it is found that

$$(2)\left\{\begin{aligned} &\int\int \Phi(\xi,\eta)d\xi\, d\eta = 1, \qquad \int\int \xi\eta\,\Phi(\xi,\eta)d\xi\, d\eta = 0,\\ &\int\int \xi^2\Phi(\xi,\eta)d\xi\, d\eta = \int\int \eta^2\Phi(\xi,\eta)d\xi\, d\eta = 1,\end{aligned}\right.$$

---

* Since $\int\varphi = 1$, the quantities σ and τ are the standard deviations of x and y, when φ is interpreted as a frequency function, if $\int x\varphi = \int y\varphi = 0$; these last conditions have no bearing on the formal work in hand, apart from its statistical interpretation, but they may be thought of as included among the conditions imposed on φ, if it appears that the notation is likely to cause confusion otherwise.

the integrals being extended over the region of definition of Φ. A simple substitution has replaced φ by a function satisfying all the original hypotheses.

Now let φ, while retaining the other properties that have been assigned to it, be relieved of the restrictions (1) altogether. Let

$$\sqrt{\int x^2\varphi}=\sigma,\quad \int xy\varphi=p,\quad y'=y-(p/\sigma^2)x,\quad \sqrt{\int y'^2\varphi}=\tau'.$$

It is found by substitution that

$$\int xy'\varphi=\int xy\varphi-\frac{p}{\sigma^2}\int x^2\varphi=0.$$

As a matter of notation, let $x'=x$, $\sigma'=\sigma$, $\varphi(x,y)=\bar{\varphi}(x',y')$. Since the functional determinant of x' and y' with respect to x and y is 1, three of the above relations may be written in the form

$$\int\int x'^2\bar{\varphi}(x',y')\,dx'dy'=\sigma'^2,\qquad \int\int y'^2\bar{\varphi}(x',y')\,dx'dy'=\tau'^2,$$
$$\int\int x'y'\bar{\varphi}(x',y')\,dx'dy'=0,$$

the range of integration now being that over which $\bar{\varphi}(x',y')$ is defined. The present $\bar{\varphi}$ then corresponds to the function φ in the preceding paragraph. To obtain a Φ satisfying (2), it is sufficient to let

$$\xi=x'/\sigma',\qquad \eta=y'/\tau',\qquad \sigma'\tau'\bar{\varphi}(x',y')=\Phi(\xi,\eta).$$

The original variables x,y are expressed in terms of ξ,η by the equations

$$x=\sigma'\xi,\qquad y=\frac{p}{\sigma'}\xi+\tau'\eta, \tag{3}$$

which are of the general form

$$x=a_1\xi+b_1\eta,\qquad y=a_2\xi+b_2\eta. \tag{4}$$

The particular determination of the coefficients specified in (3) is only one of an infinite variety of determinations which will serve essentially the same purpose. For if ξ

and η are subjected to a further transformation defined by setting

$$\xi' = \alpha_1\xi + \beta_1\eta, \qquad \eta' = \alpha_2\xi + \beta_2\eta,$$

where

$$\alpha_1^2 + \beta_1^2 = \alpha_2^2 + \beta_2^2 = 1, \qquad \alpha_1\alpha_2 + \beta_1\beta_2 = 0,$$

and if $\Phi(\xi, \eta) = \overline{\Phi}(\xi' \eta')$, equations of the form (2) are satisfied in terms of the new variables ξ' and η', because of the equations (2) themselves, in terms of ξ and η, and the fact that the functional determinant* of the transformation is ± 1. There are accordingly infinitely many transformations of the form (4) which yield a Φ satisfying (2), if $\Phi(\xi, \eta)$ is equal to $\varphi(x, y)$ multiplied by the absolute value of the determinant of the coefficients. Any transformation (4) fulfilling this requirement may be taken as a basis for the work that is to follow, and it will be understood that the symbols a_1, b_1, a_2, b_2 refer to the coefficients in such a transformation.

With this understanding, it is seen at once that

$$\int x^2\varphi = \int\int (a_1\xi + b_1\eta)^2 \Phi(\xi, \eta)\, d\xi d\eta = a_1^2 + b_1^2,$$

$$\int y^2\varphi = a_2^2 + b_2^2,$$

$$\int xy\varphi = \int\int (a_1\xi + b_1\eta)(a_2\xi + b_2\eta)\, \Phi(\xi, \eta)\, d\xi d\eta$$
$$= a_1 a_2 + b_1 b_2;$$

when the variables of integration are not expressly indicated, it is intended always that the integration shall be performed with regard to x and y. *The equations* (4) *serve, when the function* φ *is given, to associate* x *and* y *with two points* P_1 *and* P_2 *in the* (ξ, η)*-plane, having the coordinates* (a_1, b_1) *and* (a_2, b_2) *respectively. Because of the degree of arbitrariness retained by the coefficients in* (4), *the association may*

---

[*] To repeat the familiar proof,

$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}^2 = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = \begin{vmatrix} \alpha_1^2 + \beta_1^2 & \alpha_1\alpha_2 + \beta_1\beta_2 \\ \alpha_1\alpha_2 + \beta_1\beta_2 & \alpha_2^2 + \beta_2^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

be carried out in an infinite variety of ways, but in all cases the expressions

$$(5)\qquad \sqrt{\int x^2\varphi},\quad \sqrt{\int y^2\varphi},\quad \frac{\int xy\varphi}{\sqrt{\left(\int x^2\varphi\right)\left(\int y^2\varphi\right)}},$$

which are themselves independent of the coefficients, represent respectively the distance OP_1, the distance OP_2, and the cosine of the angle P_1OP_2. The last expression, in case $\int x\varphi = \int y\varphi = 0$, is the coefficient of correlation between two variables x and y distributed according to the frequency function φ, and a geometric meaning has thus been assigned to the correlation coefficient.

The correspondence that has been established renders it possible to prove analytical theorems involving a frequency function by the mere interpretation of geometric facts. Consider, for example, the problem of determining a coefficient λ so as to make the value of the integral $\int (y-\lambda x)^2\varphi$ a minimum. Since this integral is equal to

$$\int\int [(a_2-\lambda a_1)\xi + (b_2-\lambda b_1)\eta]^2\,\Phi(\xi,\eta)\,d\xi d\eta,$$

it is the square of the distance between the points $(\lambda a_1, \lambda b_1)$ and (a_2, b_2). The points (a_1, b_1) and (a_2, b_2) have already been denoted by P_1 and P_2; let Q stand for the point $(\lambda a_1, \lambda b_1)$, and θ for the angle P_1OP_2. Then Q is on the line OP_1, the distances OQ and OP_1 being in the ratio of λ to 1. In order that the distance QP_2 may be a minimum, Q must be the foot of the perpendicular from P_2 on OP_1. This means that $\overline{OQ} = \overline{OP_2}\cos\theta$, and

$$\lambda = \frac{\overline{OQ}}{\overline{OP_1}} = \frac{\overline{OP_2}}{\overline{OP_1}}\cos\theta$$
$$= \frac{\sqrt{\int y^2\varphi}}{\sqrt{\int x^2\varphi}}\cdot\frac{\int xy\varphi}{\sqrt{\left(\int x^2\varphi\right)\left(\int y^2\varphi\right)}} = \frac{\int xy\varphi}{\int x^2\varphi}.$$

If $\int x\varphi = \int y\varphi = 0$, the value of $\int \varphi$ being 1 throughout, the quantities

$$\sigma = \sqrt{\int x^2\varphi}, \qquad \tau = \sqrt{\int y^2\varphi}$$

are the *standard deviations* of x and y respectively, while, as already mentioned, $\cos\theta = r$ is the *coefficient of correlation* of x and y. With this notation, it is seen that $\lambda = (\tau/\sigma)r$. The minimum distance QP_2 is equal to

$$\overline{OP_2}\sin\theta = \tau\sqrt{1-r^2}.$$

The familiar determination of the coefficient of regression of y on x and of the root-mean-square deviation from the line of regression will be recognized at once.

3. *Frequency Functions in Three Variables.* The geometric treatment can be extended to three dimensions without difficulty. To begin, as before, with a preliminary inspection of a special case, let $\varphi(x, y, z)$ be defined throughout a finite region of space, continuous, everywhere positive or zero, not identically zero, and so constituted that

$$(6) \quad \int xy\varphi = \int xz\varphi = \int yz\varphi = 0, \\ \int \varphi = \int x^2\varphi = \int y^2\varphi = \int z^2\varphi = 1,$$

the integrals being triple integrals with regard to x, y, and z, extended over the region of definition of φ. If P_1 and P_2 are two points in space, with coordinates (a_1, b_1, c_1) and (a_2, b_2, c_2), O being the origin, it is found directly that the squares of the distances OP_1, OP_2, P_1P_2 are represented by the integrals

$$\int (a_1x + b_1y + c_1z)^2\varphi, \qquad \int (a_2x + b_2y + c_2z)^2\varphi,$$
$$\int [(a_2 - a_1)x + (b_2 - b_1)y + (c_2 - c_1)z]^2\varphi.$$

A corresponding evaluation of the quotient

$$\frac{\int (a_1x+b_1y+c_1z)\,(a_2x+b_2y+c_2z)\,\varphi}{\sqrt{\int (a_1x+b_1y+c_1z)^2\varphi}\ \sqrt{\int (a_2x+b_2y+c_2z)^2\varphi}}$$

gives the familiar formula of solid analytic geometry for the cosine of the angle P_1OP_2.

If the last three equations in (6) are replaced by

$$\sqrt{\int x^2\varphi} = \sigma, \qquad \sqrt{\int y^2\varphi} = \tau, \qquad \sqrt{\int z^2\varphi} = \omega,$$

the unit values of the integrals may be restored by setting

$$\xi = x/\sigma, \quad \eta = y/\tau, \quad \zeta = z/\omega, \quad \sigma\tau\omega\varphi(x, y, z) = \Phi(\xi, \eta, \zeta).$$

If φ is not subject to (6) at all, except for the condition $\int\varphi = 1$, a reduction may be performed as follows. Let

$$x' = x, \qquad \sqrt{\int x'^2\varphi} = \sigma', \qquad \int x'y\varphi = p'_{12},$$

$$y' = y - (p'_{12}/\sigma'^2)x', \ \sqrt{\int y'^2\varphi} = \tau', \ \int x'z\varphi = p'_{13}, \int y'z\varphi = p'_{23},$$

$$z' = z - (p'_{13}/\sigma'^2)x' - (p'_{23}/\tau'^2)y', \quad \sqrt{\int z'^2\varphi} = \omega',$$

$$\varphi(x, y, z) = \bar{\varphi}(x', y', z').$$

It is readily verified that

$$\int\int\int x'y'\bar{\varphi}(x', y', z')dx'dy'dz'$$
$$= \int\int\int x'z'\bar{\varphi}(x', y', z')dx'dy'dz'$$
$$= \int\int\int y'z'\bar{\varphi}(x', y', z')dx'dy'dz' = 0.$$

It remains to set

$$\xi = x'/\sigma', \qquad \eta = y'/\tau', \qquad \zeta = z'/\omega',$$
$$\sigma'\tau'\omega'\bar{\varphi}(x', y', z') = \Phi(\xi, \eta, \zeta),$$

and the desired reduction is accomplished; that is, Φ has the properties that

$$
\begin{aligned}
&\int\int\int \Phi d\xi d\eta d\zeta = 1,\\
&\int\int\int \xi\eta\Phi d\xi d\eta d\zeta = \int\int\int \xi\zeta\Phi d\xi d\eta d\zeta\\
(7)\qquad &\qquad = \int\int\int \eta\zeta\Phi d\xi d\eta d\zeta = 0,\\
&\int\int\int \xi^2\Phi d\xi d\eta d\zeta = \int\int\int \eta^2\Phi d\xi d\eta d\zeta\\
&\qquad = \int\int\int \zeta^2\Phi d\xi d\eta d\zeta = 1.
\end{aligned}
$$

Let it be understood henceforth that x, y, z are related to a new set of variables ξ, η, ζ, by equations of the form

$$
\begin{aligned}
x &= a_1\xi + b_1\eta + c_1\zeta,\\
y &= a_2\xi + b_2\eta + c_2\zeta,\\
z &= a_3\xi + b_3\eta + c_3\zeta,
\end{aligned}
$$

so that the conditions (7) are satisfied, when $\Phi(\xi, \eta, \zeta)$ is equal to $\varphi(x, y, z)$ multiplied by the absolute value of the determinant of the coefficients. The coefficients may be determined by the calculation of the preceding paragraph, or in any one of an infinite variety of other ways, the passage from one determination to another amounting to a rotation of coordinate axes, with or without a reflection in one of the coordinate planes. Let the point with coordinates (a_i, b_i, c_i) be denoted by P_i, for $i = 1, 2, 3$, and the origin, as usual, by O. The expressions (5) once more represent the distances OP_1 and OP_2 and the cosine of the angle P_1OP_2, while the corresponding expressions obtained by permuting the variables are to be similarly interpreted.

A foundation is thus laid for geometric reasoning on a more extensive scale. If U and V are any two linear combinations of x, y, and z,

$$U = l_1x + m_1y + n_1z, \qquad V = l_2x + m_2y + n_2z,$$

they can be expressed in the form

$$U = A_1\xi + B_1\eta + C_1\zeta, \qquad V = A_2\xi + B_2\eta + C_2\zeta,$$

and then $\int U^2\varphi$ is the square of the distance of the point (A_1, B_1, C_1) from the origin, $(\int UV\varphi)/\sqrt{(\int U^2\varphi)(\int V^2\varphi)}$ is the cosine of the angle subtended at the origin by the points (A_1, B_1, C_1) and $A_2, B_2, C_2)$, and so on. Any linear combination of x, y, and z corresponds to a definite point in three-dimensional space.

Suppose λ and μ are determined so as to minimize the integrals $\int(y-\lambda x)^2\varphi$ and $\int(z-\mu x)^2\varphi$. The points Q_2 and Q_3 corresponding to λx and μx are on the line OP_1, where this line is met by the perpendiculars from P_2 and P_3 respectively. Let $u = y-\lambda x$, $v = z-\mu x$; the coordinates of the points corresponding to u and v are the components of the vectors Q_2P_2 and Q_3P_3. Consequently the expression $(\int uv\varphi)/\sqrt{(\int u^2\varphi)(\int v^2\varphi)}$ is equal to the cosine of the dihedral angle between the planes P_1OP_2 and P_1OP_3. If $\int x\varphi = \int y\varphi = \int z\varphi = 0$, it is at the same time the coefficient of partial correlation between y and z. The geometric figure is exactly the same as in the case of a finite number of observations, and there is no need of repeating the steps by which the coefficients of partial correlation and of double correlation are calculated in terms of ordinary correlation coefficients.* It may be verified that the determinants

$$\begin{vmatrix} \int x^2\varphi & \int xy\varphi \\ \int yx\varphi & \int y^2\varphi \end{vmatrix}, \qquad \begin{vmatrix} \int x^2\varphi & \int xy\varphi & \int xz\varphi \\ \int yx\varphi & \int y^2\varphi & \int yz\varphi \\ \int zx\varphi & \int zy\varphi & \int z^2\varphi \end{vmatrix}$$

represent respectively the square of the area of the parallelogram determined by OP_1 and OP_2, and the square of the volume of the parallelepiped determined by OP_1, OP_2, and OP_3. The values of the determinants, which have the character of Gramian determinants, are therefore always positive.*

---

* Cf. the papers on *The trigonometry of correlation* and *The elementary geometry of function space,* previously cited.

4. *Functions Defined over an Infinite Range.* As was said in the introduction, it is not the purpose of this paper to enter into an elaborate discussion of the questions of convergence that arise if the frequency functions considered are supposed defined and different from zero over an infinite range, in a highly arbitrary manner. There is no difficulty, however, in arriving at a formulation general enough to cover the most important statistical applications. Suppose, for example (to concentrate attention on the case of three variables), that

$$\varphi(x,y,z) < Ke^{-|x|-|y|-|z|}$$

for all values of x, y, and z, K being a constant. Then all the integrals that appear in the course of the discussion are absolutely convergent, so that the integral over a finite region approaches a definite limit as the boundary of the region recedes to infinity in any way whatever. To justify the transformations of variable in the infinite integrals, it is sufficient to carry out the transformations over corresponding finite regions, a sphere, for example, in one set of variables, and an ellipsoid in the other set, and then to allow these regions to expand to infinity. Furthermore, though this is not essential to the argument, all the triple integrals can be evaluated as iterated integrals.

THE UNIVERSITY OF MINNESOTA

---

* The fact that the two-rowed determinant is positive is equivalent to the fact that the value of a coefficient of correlation is always between -1 and $+1$. It is to be noticed that the extreme values $+1$ and -1 can not occur, under the hypotheses on which this paper has been based; a frequency function corresponding to perfect positive or negative correlation is not continuous as a function of the two or more variables involved.

MATHEMATICS FOR ENGINEERS

Lezioni di Analisi Matematica. By Guido Fubini. Torino, Società Tipografico-Editrice Nazionale, 1923. viii+470 pp.

Grundriss der Differential Rechnung. By Ludwig Kiepert. Hannover, Helwingsche Verlagsbuchhandlung, 1923. viii+348 pp. II. Band: *Einige grundlegende Untersuchungen aus der Algebra und Funktionen von mehreren unabhängigen Veränderlichen.*

Both of these books were written primarily for engineering and technical students. The former follows the course as given in the Polytechnic school at Turin, and the latter is written by a trained engineer for engineering students. It has been remarked so often that our scientific students do not know enough mathematics to enable them to go very far, particularly is this true of our physicists and engineers. A Ph. D. in chemistry complains that he cannot read beyond the first hundred pages of Sommerfeld's *Structure of the Atom and Spectral Lines*, because there is too much mathematics. How far this is true may be judged by comparing the mathematical equipment of the above courses with one obtained in a calculus course in an American university.

The present edition of the first book is the fourth; previous editions appeared in 1913, 1915, 1920, which indicates its wide use in Italy. The difference in content between these texts and our American texts is not so marked. There is perhaps a greater emphasis on applications of calculus to algebraic problems in these foreign books, but the chief difference is in the point of view.

It is rather strange that we should be laboring under severe handicaps in the teaching of the pivotal course of our mathematical curriculum, the calculus. Our difficulties are two-fold. First, there is in vogue, what I may call the "double standard" amongst us. Mathematicians of repute will not hesitate to pursue a policy in their teaching which amounts to keeping their right hands in ignorance of what their left hands are doing. Two examples will illustrate this difficulty. An author of a calculus text writes in a footnote that the equation in the paragraph above "to be strictly accurate" requires some modification. Again, how often do we find theorems stated as true, which in the proof make use of a uniformity condition, which bobs up serenely from nowhere, and which disappears again in applications to nowhere.

Our second difficulty is that too many of us bury our heads ostrichlike in the sand and keep the eyes of our students closed to the fundamental assumptions on which the basic principles of the course are founded. This tends to superficiality and the "hand-book" method of teaching the calculus. As a student, the reviewer recalls emerging from the first calculus course with no other knowledge of what it was all about than being able to differentiate x^3 and integrate x^2dx. Our first course does not go deeply enough into the facts.

If the "small zeros" of the engineer are ever to be banished from the face of the earth, it will be due to the fact that our teaching will from the very beginning have taught him the significance of relative size. No opportunity will be given him to form bad mathematical habits and to acquire erroneous ideas. No longer will we write equations whose only interpretation is "that one variable is approaching another *variable* as a limit".

It is quite a contrast to regard the first book, especially, that by Fubini, and to see what its bearing is on our situation. The author aims at the outset to set forth the fundamental notions of number, angle, area and volume. There are no extensive proofs given, but the facts are stated correctly and interestingly. There follows about 100 pages of algebra, principally the theory of equations, which an American student would normally acquire in his freshman or sophomore year. The remainder of the book deals with the usual topics of the calculus. The meaning of the principles is profusely illustrated by means of examples; taken chiefly from the field of mechanics on account of the future interest of the student. The formal work of differentiation is not given very much space, but all the essential formulas are developed. The applications of the derivative are the usual ones to curve sketching, indeterminate forms, multiple roots of an algebraic equation, maxima and minima and points of inflexion. The first and second laws of the mean for derivatives are proved.

The definite integrals, simple and multiple, are introduced by means of additive functions of a one-dimensional, two-dimensional, or three-dimensional region, as the case may be. The proof for integrability is based on an argument which is very similar to the one given by the reviewer,* except that Fubini's theorem† is not as general. His final result stated on page 329 requires uniformity and is equivalent to Osgood's‡ form of Duhamel's theorem.

The latter half of the course includes what would be included in a course of advanced calculus in this country. The existence theorem due to Cauchy for differential equations is stated, but its proof is omitted. Space geometry, envelopes, line and surface integrals, transformation of a double integral, Fourier series, and functions of a complex variable are touched upon and treated to a slight extent in this part of the book.

---

* H. J. Ettlinger, *A simple form of Duhamel's theorem and some new applications*, AMERICAN MATHEMATICAL MONTHLY, vol. 29 (1922), p. 246 ff. See also a paper by the reviewer, *On the integrability of a continuous function*, AMERICAN MATHEMATICAL MONTHLY, vol. 31 (1924), p. 319.

† Cf. pp. 316–17 and footnote of the book under review.

‡ W. F. Osgood, *The integral as a limit of a sum and a theorem of Duhamel's*, ANNALS OF MATHEMATICS, (2), vol. 4 (1903), pp. 164–65.

There is one list of problems thirty-nine in number on pp. 55–61 for the chapter on theory of equations. The exercises in the rest of the book are few and far between. It is probably true that long lists of calculus problems are available for class use from other independent sources.

The printing of the book is unusually good; only two misprints have been noticed. On page 123, near the bottom of the page, the third term of the geometric progression should be $1/2^2$ instead of $1/3$. On page 170, the term $1/x$ is omitted from the formula for the derivative of $\log_e x$.

The book by Kiepert deals with the applications of the calculus to algebra, particularly the solution of equations numerically to any required degree of approximation. In the latter half of the book, functions of several variables are treated with applications to space analytic geometry, including what is ordinarily included in differential geometry in this country.

Throughout the book the complete solution of a great many examples is given to illustrate the principles used. The last chapter is devoted to maxima and minima of functions of several variables. There is a summary of the important formulas proved in this volume at the end of the book.

The plan of both of these books commends itself very highly to those who are desirous of improving the teaching of the calculus not alone for those who use mathematics as a tool, but also for those whom we may look to as the future searchers after mathematical truth.

H. J. ETTLINGER

STUDY ON VECTORS AND INVARIANTS

Einleitung in die Theorie der Invarianten linearer Transformationen auf Grund der Vektorenrechnung. Erster Teil. By E. Study. Braunschweig, Vieweg und Sohn, 1923. 268 pp.

Like other writings of Professor Study the present work is interesting from several different angles. It is a development of the theory of invariants of ternary forms, based upon what he calls vectors; it is an exposition of his thesis that mathematics is the study* of "natural (positive, integral) numbers and everything which can be based upon them, but nothing else"; it is a running commentary and criticism of many things and many mathematicians. One needs occasionally a background of other papers of his dealing with questions of a semi-philosophical character. He particularly takes to task the investigators who have developed systems of vector analysis because they have not contented themselves with utilising the theory of invariants, as developed by Clebsch, Aronhold, etc. He says (p. 3) "Whether one develops vector analysis for the sake of its applications or as a self-contained discipline, there must always come first, as stated, a consideration of certain topics of algebra, namely invariants of certain groups of linear transformations: all expressions considered are invariants of orthogonal transformations, or invariants of groups closely related to the group of orthogonal transformations. But for more than fifty years we have had a highly developed theory of the group of all linear transformations, and for more than twenty-five years at least the fundamentals of a theory of invariants of the other groups just referred to (GES. WISS. LEIPZIG, MATH.-PHYS. 1897, p. 443 ff.). But not a glimmer of light seems to have fallen from these investigations upon the highly beloved "Vector Analysis" of today. Indeed old problems have been handled as if never treated before, and we are thus lagging far behind what has long been a guaranteed possession of the science. So far as I know, the question is never raised in such writings, as to what are all possible algebraic, and in particular, rational, invariants of these groups; and yet there can be no doubt that this is the problem, which from the very nature of things, lies at the heart of the matter." And again "As for the majority of authors it is not evident that they lived in a generation when the theory of groups was in full bloom. We even see the algorithm of H. Grassmann, which was a mark of progress in his own time, and which stands as a monument to his originality, yet which has long ago been absorbed into the more profound theory of invariants, hailed as the acme of the attainable, or even as a panacea, which certainly it is not; while on the other hand algebraists like Aronhold and Clebsch, who, differing from their predecessors, have set out from precisely stated problems, and have gone much farther, do not seem to have existed for these same authors. In short we are behind the times, and nothing less."

---

* *Mathematik and Physik*, p. 6.

The Introduction, of twelve full pages, is labeled "Problems and methods", and is an insistence upon the importance of problems and the relative unimportance of methods. "Those who construct vector analysis and related algorithms whose essence is method (extensive algebra, calculus of matrices, quaternions, etc.) usually consider their subject-matter as a world in itself. Doubtless there is much justification for this. Such a claim and the related desire for purity of method contain a wholesome pressure towards thoroughness". There are esthetic considerations but there is also the usefulness and ease of manipulation of an algorithm to be taken into account. A happy notation is a great advantage in the progress of a new branch of mathematics, but for that very reason one must hesitate to introduce novelties merely for their own sake, or from a mere desire not to "violate the spirit of the order". The reader will not usually learn a new notation thoroughly, and will in the end give up trying to translate it into what he is familiar with. This also has to be said with regard to the invention of new terms, as, for instance, calling ternary bilinear forms, or the system of coefficients that belong to such a form, by the new terms: tensor of the second kind, affinor, dyadic, tensortriple, complete dyad, asymmetric tensor, diatensor, vector homography, and special terms related, as deviator, antitensor, axiator, idemfactor, versor, perversor, etc. And "what has long been sought in vector calculus—a system of symbols most convenient for the subject,—has been in existence a long time, in a form much more convenient than anything devised since by the various partisans. I shall make use here with minor alterations (whose motive should not be missed) of the notation based upon the older theory of invariants of linear transformations, which is uniquely determined in all essentials by the subject itself. A special notation for vectors, which only too easily may become an obstacle in the free road, seems to me superfluous if not harmful. The 'world for itself' will nevertheless come into its full rights." Professor Study thus comes to the conclusion that the problems of vector calculus are merely invariant problems, and the methods of invariants are the most satisfactory for all purposes of vectors. Thus the raison d'être of vector algorithms has disappeared!

Professor Study also has no uncertain opinion with regard to many ideas of many mathematicians. On page 3, Grassmann is said to have been "absorbed". On page 4 an unnamed writer* is criticised for apologizing for the deluge of formulas he introduces. On page 5 another unnamed writer is criticised for his remark relative to Hilbert's theorem on the finiteness of systems of forms: "How fine that we need not bother any more with invariants." One must feel that perhaps the point was missed. On page 8, Boltzmann is mentioned as saying: "One should leave elegance to cobblers and tailors". On page 11, Beck is criticised for his *Coordinate Geometry*, though he certainly is in sympathy with Study's ideas. On page 19 and page 21, Lie is mentioned as illogical for his term "continuous group", since Study desires to use the term in another sense. On page 52, Coolidge gets favorable mention, as do Veblen and Young on page 59. On page 101, Grassmann's Lückenprodukte are condemned. On the same page the mathematical ability of Einstein is questioned, and Weyl's notations are condemned. On page 118, the right of authors to invent terms is questioned. On page 127, Frobenius comes in for a share of criticism, along with unnamed mathematicians who use the English language, and spend time on hypernumbers. The criticism of Frobenius' standpoint extends over to page 129. On page 238, authors of analytic geometries are criticised. On page 257, Rabinovitch and the Italian school get their deserts.

---

* See Weyl, MATHEMATISCHE ZEITSCHRIFT, vol. 20 (1924), p. 131.

Professor Study's definitions remove geometry from the study of space, as usually conceived, completely. For him a point is merely a triplex of numbers, or in general a multiplex of N numbers, and a vector is precisely the same thing. The term multiplex, it may be explained, means a set of numbers in a specified order. It is exactly the idea of Hamilton over again, who first dealt with couples, then with "sets" of numbers. Of course Study would and does repudiate the Hamilton relation of number to "time", but as a matter of fact Hamilton's "time" is nothing more than the linear continuum. He does not point out, however, that it was Hamilton who first made a study of the couple, multiplex set, etc., as an entity. He did not call it a vector, defining (for the first time) this to mean a geometric segment with a definite direction. One who insists on the correct use of established terminology should have retained the very useful terms of Hamilton. The reviewer agrees perfectly with Professor Study that good existing terminologies should be preserved, and new ones avoided; therefore he desires Professor Study to follow the same caution. The same remark applies to changing the terminology of Lie. And it is to be noted further that one does not escape axioms and postulates by merely referring everything to the system of natural numbers. If that were possible he could go still further and start with Whitehead and Russell's *Principia*. We might grant that it is possible to set up a useful isomorphism between points or vectors in space and triplexes, but it is easily evident that the isomorphism will work both ways, and that arithmetic can be based upon space just as much as space can be based upon arithmetic. A good deal of so-called simplification and "going to the foundations" consists in substituting one abstraction for another.

But we also have a conviction that the theory of vectors is not the same by any means as the theory of invariants. Professor Study seems to have forgotten the origin of quaternions, and from it that of the various systems of vectors. If one reads Hamilton carefully he will find that Hamilton was concerned with the extensions of number along the lines of algebra, starting with negative numbers and the so-called imaginary. In other words he was studying the theory of hypernumbers and made considerable progress along this line. These are not invariants, any more than other numbers are invariants, and their development antedates that of invariants by several years. The idea of linear transformations was superimposed upon them and is not the outcome of their study. It is only in a very superficial sense that one could consider that the idea of negative, or of $\sqrt{-1}$, was an outcome of natural number. These notions may be called with some justification an outcome of operations upon number, but operations upon number and number itself are very different entities. Further to represent $\sqrt{-1}$ as a linear substitution, as Peano does, is again only an isomorphism, and should not be confused with an identity. Keeping this in mind we will see that a system of vectors is a system of algebraic numbers in an extended sense, and all formulas are algebraic formulas, when algebra is thus understood. They do not primarily deal with multiplezes at all.* A quaternion is an algebraic number of the type of the ordinary complex number, and so long as we deal only with its powers and their combinations with positive and negative numbers, a quaternion differs in no sense from a complex number. Further it is an individual entity and not a set of four numbers. This Hamilton very well understood though he used sets of four positive or negative numbers in the various representations of quaternions. When this fact is clearly understood all the nonsense about geometric numbers and the like vanishes. The expressions in the formulas of such hypernumbers are not there because of certain invariancies, but because of the properties of these algebraic numbers. That they turn out to be invariants for orthogonal or other transformations of the so-called coefficients is due to such facts as that in any quaternion formula as $S\alpha\beta$, or Sqr, we have such relations as

$$S\cdot t\alpha t^{-1}t\beta t^{-1} = S\cdot\alpha\beta, \qquad S\cdot tqt^{-1}trt^{-1} = Sqr.$$

Therefore when Professor Study laments the fact that investigators have not tried to list the sets of "fundamental invariants" he might as well be concerned over the fact that not much study has been made of the arithmetics that are developed in such hypernumber systems, though the latter line of development is now started by Dickson. Hence the charge that the theory of invariants has not existed for such investigators is not well-founded. And further in the application of such hypernumbers to geometric problems no concern need be felt over invariants, for where the expressions do not depend upon coordinate planes, they are ipso facto all invariant expressions. The reviewer has pointed this out elsewhere* with reference to the Einstein relativity theory, showing that when such theory is stated in terms of an N-dimensional vector calculus, there are no coordinates involved, hence a change of coordinate references has nothing to do with the problem. The thesis that all physical phenomena should be so represented that the same expressions will hold whatever the system of coordinates is actually realised by removing all systems of coordinates. Indeed the difficulties dragged in by coordinates have no business there at all. And to assert, as Professor Study does, that there is no geometry of a sound logical character without coordinates, is to ignore the whole of Euclid, or any other system of synthetic geometry. And to assert further that this is mixing up the phenomenal world, and questions of epistemology, with geometry, is to show that one has not yet clearly seen what is going on in the study of synthetic geometry. Space forbids entering into a full explanation, but we may point out that the mind studies many types of ideal or non-material "constructions", and among these we find the ones that enter geometry alongside of the "arithmetic", and one has the same kind of reality or validity as the other.

---

* The „multiplex" character means merely that from *one* equation between vectors arise an *infinity* of equations between ordinary numbers, all dependent on any N of the set which are linearly independent.

The reviewer is quite well aware of all the notions that have clung to that of vector calculus, and has always maintained that those systems that were mere shorthand or symbolisms for coordinate expressions were not worthy the name of vector calculus. Nothing can be accomplished by them which cannot also be done equally well without them, and in so far Professor Study is right in insisting that if one utilises the algebra of invariants he can get along just as well. There is of course some small gain in the use of the vectors but it is a matter of taste whether one prefers to write either of the two sets of expressions (the first is Study's; the second is quaternions)

$$(AP) \text{ or } m_1(\varphi),\ (AP')(A'P) \text{ or } m_1(\varphi^2),\ (AP')(A'P'')(A''P) \text{ or } m_1(\varphi^3),$$
$$\tfrac{1}{6}(AA'A'')(PP'P'') \text{ or } m_3(\varphi),\ (XU) \text{ or } S\rho\tau,\ (XA)(PU) \text{ or } S\rho\varphi\tau,$$
$$(XA)(PA')(P'U) \text{ or } S\rho\varphi^2\tau,\ \tfrac{1}{2}(XPP')(AA'U) \text{ or } S\rho\psi(\varphi)\tau,\ (SX)^3$$
$$\text{or } S\rho\varphi\rho\varphi^2\rho,\ (\Sigma U)^3 \text{ or } S\tau\breve{\varphi}\tau\breve{\varphi}^2\tau \text{ etc.}$$

One form is no more intelligible than the other, and expresses no more than the other so long as we are planting our feet on the artificial ground of coordinates. But when we remember that the invariant symbols have no meaning for metrical work till the coordinate planes are given and the coordinates known, while the quaternion symbols are directly interpretable without any such artificial reference, the two sets come to have a quite different significance. There is at least as much advantage in the quaternion case as in the use of trigonometric solutions over these of rectangular coordinate geometry.

---

* *General vector algebra*, TRANSACTIONS OF THIS SOCIETY, vol. 24 (1922), pp. 195–244.

The essence of the matter is that when one is looking at every problem in geometry from the projective point of view, then projective invariants will be useful. When, however, he is looking at geometry from the metric point of view, then the direct metric analysis is far more advantageous. To use the artificial relations Study has on pages 35 et seq. is to be forgetful of the fact that mathematics is not so poverty-struck as to possess but one garment.

In order to be absolutely clear and definite, a word will not be amiss as to the significance of expressions to a quaternionist (whatever they may mean to others who merely dabble in vector analysis). The expression

$S\alpha\beta$ means the lengths of α and β multiplied by the cosine of their exterior angle;

$V\alpha\beta$ means the vector perpendicular positively to the plane α, β, whose length is the product of their lengths by the sine of the exterior angle;

$S\alpha\beta\gamma$ is the volume of the parallelepiped whose edges are α, β, γ;

$\varphi\rho$ is a linear vector function of ρ, that is, is such that we have $\varphi(x\alpha + y\beta) = x\varphi\alpha + y\varphi\beta$.

In other words $S\alpha\beta$ *does not mean* $-ax-by-cz$, $S\alpha\beta\gamma$ *does not mean* a three-rowed determinant, etc. This is what vector analysts like Burali-Forti and Marcolongo and their predecessors mean by "*autonomous expressions*", in the sense of not being related in any way to coordinate systems. Apparently some men have not yet seized this simple idea.

As far as the work before us has a constructive purpose, — that of developing a systematic text which shall be an introduction or even a manual of the theory of invariants under the linear homogeneous group or some of its subgroups, using the notion of triplex as a foundation,— we find a successful accomplishment of the purpose. The author has shown, what other writers on invariants have not, that invariants can be used, together with the symbolic notation peculiar to invariants, for the study of any geometric problems expressible by ternary forms. In a succeeding volume he expects to continue into n-ary forms. Professor Study has memoirs of long standing on these subjects, and has produced a very satisfactory book for these uses. A detailed list of contents is not necessary; the treatment is quite complete. If any book can resurrect the theory of invariants this one will.

J. B. Shaw

SHORTER NOTICES

Cours d'Analyse Infinitésimale. By Ch. J. de la Vallée Poussin. Vol. I, 5th ed. Paris, Gauthier-Villars, 1923.

An earlier edition of this Cours was reviewed in this BULLETIN of 1915. This latest edition has been very largely recast and the first volume only deals with the classic topics treated in the standard French *Cours.* The matter printed in large type in the older editions constitutes the bulk of this volume and only Riemann integrals are considered. The material referring to point-set theory, measurable functions and Lebesgue integrals which was printed in small type in the old edition will find place in a subsequent volume.

This arrangement seems much better than the old one for it will permit of a homogeneous development more in harmony with the historical evolution of the ideas involved.

Even in following the conventional order of the French treatises, de la Vallée-Poussin displays his usual elegance and simplicity of presentation so that the most hackneyed matters acquire a new interest. Thus on page 39 will be found a neat and simple proof that the familiar functional equations

$$f(x+y) = f(x)+f(y), \qquad \Phi(x+y) = \Phi(x)\Phi(y)$$

have unique solutions of the form $f(x) = ax$ and $\Phi(x) = A^x$ provided f and Φ are *borné* in an arbitrarily small interval (O, e).

The treatment of indeterminate forms is the best the reviewer knows of and extremes of functions of one or more variables are carefully handled.

Chapter III on derivatives and differentials of functions of several variables is extremely well done and the conditions of Young and Schwarz for the inversion of the order of derivation are neatly and briefly established.

Even the usually arid subject of formal integration gains with his deft treatment an elegance and beauty rarely found in text books on the calculus. Multiple integrals, change of variable, the theorems of Green and Stokes are carefully treated and conventions of sign, so often loosely passed over, are fully gone into. Attention is called to the clever evaluation of the error in Simpson's rule.

The book ends with a chapter on series where the usual tests including Kummer's are set forth. In conclusion it may be said that this is one of the most valuable handbooks on modern analysis in any language and an English translation of it would be a welcome addition to our literature of the subject.

M. B. PORTER

Lezioni sulla Teoria dei Numeri Algebrici. By Luigi Bianchi, Bologna, Nicola Zanichelli, 1923, 640 pp.

After an examination of the contents of this book the reviewer feels that it serves a double purpose. On account of the very comprehensive treatment of the subject the book is an excellent work for reference. Moreover, on account of the manner in which the author coordinates the various notions used in the development, it will undoubtedly prove an excellent work for any one who wishes to gain a clear insight into the theory of algebraic numbers.

A brief résumé of the contents of the various chapters will serve to show the ground covered and also to give an idea of how the various topics are correlated in the development.

Chapter one deals with the elements of the theory of matrices and their applications to systems of linear equations and systems of linear forms. It also contains a brief study of forms relative to a modular system $(f_1, f_2, \ldots, f_n)$ where the f_i are linear forms. This leads to the proof of Minkowski's well known theorem regarding linear forms. Some space is devoted to the geometric interpretation.

In the second chapter the author develops the elements of the theory of numbers for the Gaussian field $k(i)$, $(i^2 = -1)$. This development contains the proofs of the fundamental theorems whose generalizations are contained in the succeeding chapters. The chapter concludes with the consideration of further examples of quadratic number fields, showing how the law of unique factorization, which was seen to hold for $k(i)$, breaks down.

The third chapter contains the development of the algebraic side of the general theory of algebraic number fields.

Chapter four is devoted to the theory of units in an algebraic number field. The author shows the existence of units, and of fundamental systems of units.

The fifth chapter contains the development of the theory of ideals and chapter six is devoted to congruences with respect to an ideal modulus with special consideration of prime ideal moduli. In this connection the author treats the subject of quadratic residues which he has already treated, in Chapter 2, for the field $k(i)$. Quadratic residues in a quadratic field are given special consideration.

Chapter seven contains the theory of equivalence of ideals and the separation of ideals into classes. The group of classes and its invariants are studied and a proof of the finiteness of the number of classes is given. This chapter also contains a consideration of the correspondence between ideals and decomposable forms and the relations between classes of ideals and classes of forms, and the multiplication of ideals and composition of forms.

Chapter eight contains the proofs of the general theorems regarding the factorization of the rational primes in an algebraic number field and its relation to the decomposition of $f(x)$ with respect to the modulus p, where $f(x) = 0$ is an equation one of whose roots generates the field. The last part of the chapter considers the prime factors of the field discriminant and also the factorization of the rational primes in cyclotomic fields.

The first part of chapter nine contains the theory of orders (Dedekind, Ordnung; Hilbert, Ring) in a field, and the ideals of any order. The second part of the chapter is devoted to Galois domains and their sub-domains.

Chapter ten deals with the analytic theory of algebraic numbers and develops the transcendental expression for the number of classes of ideals in a field.

Chapter eleven contains the development of the expression for the number of classes of the quadratic and cyclotomic fields and the proof of Dirichlet's theorem regarding arithmetic progressions.

G. E. WAHLIN

Report on Radiation and the Quantum-Theory By J. H Jeans. London, Fleetway Press, Ltd., 1924. 86 pp.

The first edition of this report was written in 1914 and was published by the Physical Society of London. In preparing the new edition the writer has omitted some of the parts in which he says there is an *apologia* for the defects and inconsistencies of the theory and has made numerous additions which add to the charm and lucidity of the presentation. The author's remark "that the quantum theory need no longer be considered on the defensive" aptly describes the present situation, in fact some writers have a feeling that the apologia must now be made for the classical electrodynamics.

In the first chapter Jeans points out that the smallness of the total density of radiant energy in temperature-equilibrium with matter compared with that of the heat-energy in the matter cannot be explained on the basis of Newtonian mechanics or by a supposed analogy with a vibrating elastic system immersed in a fluid such as air or water, for it is known by experience that the energy of the elastic system is finally transformed into heat energy of the fluid. The relative smallness of the density of the radiation at 0° C is quite startling in the case cited, being 4×10^{-5} ergs per cubic centimeter as compared with 8×10^{9} ergs per cubic centimeter in the matter.

In the next chapter it is shown how the classical electrodynamics in combination with the kinetic theory of gases leads to the radiation formula which is associated with the names of Rayleigh and Jeans. This formula is correct in the region of long wave-lengths or at very high temperatures when the density of radiation is very large and one of the guiding principles of quantum theory in its later developments is that a formula of quantum theory should reduce to that given by the classical theory under the conditions just mentioned.

Jeans then gives an account of the development of the quantum-theory, describing, in particular, the derivation of Planck's formula which was given by Einstein and the investigations of Poincaré and Fowler regarding the nature of the laws of motion that can lead to Planck's formula.

Chapters IV and V are devoted to Bohr's theory of spectra and to Einstein's theory of the photoelectric effect while Chapter VI contains an account of Debye's theory of the specific heat of solids. Much additional matter is contained in Chapter VII which deals with the dynamics of the quantum theory. The articles on conditionally periodic systems and Bohr's principle of correspondence will be found useful as an introduction to the more complete works mentioned in the text.

The report ends with a short discussion of the physical basis for the quantum theory in which the views expressed are practically the same as in the first edition. The idea of light-quanta is regarded with disfavor but it is perhaps unfortunate that the new report should have been completed just when a big development of the theory of light-quanta was taking place. The beautiful discoveries of Compton and Duane have shown that the phenomena of the scattering and diffraction of X-rays are not incompatible with the idea of light-quanta and the development of the idea of light-molecules by Wolfke, Bothe and de Broglie has shown that Planck's law of radiation is indeed obtained by slight modifications of arguments which previously led to Wien's law. Thus Pauli's remarkable application of the Compton-Debye theory of scattering to the equilibrium between free electrons and cavity radiation has been completed in this way by Bothe and Schrödinger and it now appears that the distribution of energy among the different wave-lengths may be unaltered by collisions of various types, including even the collisions between waves of light and sound which were considered first by Brillouin.

H. BATEMAN

Bibliographie des Déterminants à plus de Deux Dimensions. By Maurice Lecat. Louvain, 1924. 16 pp.

Those interested in determinants will welcome this short bibliography by an author who has made the subject peculiarly his own. It consists of three parts; a bibliography with 121 entries arranged alphabetically, an interesting chronological table, and a list of minor contributions to the subject. It is worthy of note that 50 out of the titles listed are by M. Lecat himself.

J. H. M. WEDDERBURN

Handy Methods of Geodetic Astronomy for Land Surveyors. By T. G. Gribble. London, Potter, 1921. 27 pp.

The particular feature of this pamphlet is to show how, "by means of some rapidly made diagrams, the stars available at any time and place can be at once identified, and their observation facilitated". Two sets of diagrams are used: I. Diagrams for a fixed latitude, II. Diagrams for all latitudes.

Under I, the variable laid off horizontally is the declination of the celestial body in question, its altitude, azimuth or hour angle being laid off vertically. Curves are shown giving for objects of various declinations (*a*) the altitude when on the meridian, (*b*) the altitude and hour angle when in azimuth 90° E or W, (*c*) the altitude and azimuth when in hour angle 90°, (*d*) the altitude, azimuth and hour angle of a circumpolar star at elongation.

Under II. the variable laid off horizontally is again the declination, and two sets of curves, for various latitudes at intervals of 5°, are drawn giving for objects of different declinations (*a*) the altitude and hour angle of objects due east or due west, (*b*) the altitude and azimuth of objects whose hour angle is six hours, (*c*) the altitude, azimuth and hour angle of circumpolar stars at elongation.

The curves are sketched in the usual way through points whose positions have been computed by the usual formulas of spherical trigonometry. It is suggested that a 20-inch slide rule be used for this computation and that the diagrams be drawn on a large scale. The method has the advantages and disadvantages always associated with graphical methods. The general relations involved are clearly shown, but it is doubtful whether sufficiently accurate values could be obtained in this way, and the labor involved in the construction of the diagrams would be considerable.

C. H. CURRIER

Differential Equations in Applied Chemistry. By F. L. Hitchcock and C. S. Robinson. New York, Wiley and Sons, 1923. vi+110 pp.

The purpose of this little text is to introduce the student in chemistry to the processes of building up differential equations where he has to use them, and to carry out the integrations necessary for the complete solution. It is full of examples from the subject and will be interesting for this reason to the student of mathematics as well. The actual equations solved belong to a very limited number of types but are of the kind a chemist meets. The book fills a gap in this part of applied mathematics. Its importance to the physical chemist depends upon what the latter has as his goal.

J. B. SHAW

The Alpha, Beta, Gamma Navigation Tables. By H. B. Goodwin. London, Potter, 1921. 4+54 pp.

The Alpha and Beta Tables, arranged in parallel columns, give respectively the natural cosines and logarithmic cosines of angles at intervals of 1′ from 0° to 90°. They are intended for use in cases where formulas of the type $\cos z = \cos a \pm \cos b$ where a and b are given and $\log \cos z$ is required for further computation. To facilitate the addition or subtraction of the natural cosines of a and b, each natural cosine in the tables has been multiplied by 1,000, and the resulting product it regarded as so many seconds of arc. Thus $\cos 60° = \cdot 500$, which is regarded as 500 seconds of arc, and is given in the table as 8′ 20″. It is claimed that it is easier to add or subtract quantities thus expressed in minutes and seconds of arc than to add or subtract four-place decimals, and this claim seems reasonable. After the natural cosines have been combined in this way, the parallel column arrangement makes it possible to get at once the logarithmic cosine of z without the trouble of taking out the angle itself and looking up its logarithmic cosine in a separate table.

The Gamma Tables give the value of the logarithm of $1/\text{vers}\, h$ (where h is the hour angle) to four decimal places for values of h between 0 hours and 24 hours at intervals of 5 seconds. It is claimed that these tables simplify the solution of the problem of fixing a ship's position either by the Marcq system or by the older Sumner process. More generally they facilitate the solution of problems which involve the two cases of spherical trigonometry:

(1) Given three sides of a triangle to find an angle.

(2) Given two sides and the included angle to find the third side.

In both these cases the necessary formulas can be reduced to forms involving the sums and differences of cosines, and $1/\text{vers}\, h$.

C. H. CURRIER

Zentral-Perspektive, by Hans Freyberger. Revised by J. Vonderlinn. Sammlung Göschen, No. 57, 2d edition, Berlin and Leipzig, Walter de Gruyter & Co. 1923. 148 pp.

This little book originally written by an architect, emphasises the practical and artistic aspect of the subject and may very well serve as a first introduction to this fascinating chapter of applied geometry. The figures are well drawn and the exercises well selected, so that the reader who wishes to acquire a working knowledge of perspective can readily accomplish this by reconstructing the figures on a larger scale in connection with a comprehensive study of the text.

ARNOLD EMCH

NOTES

The second number of volume 26 of the TRANSACTIONS OF THIS SOCIETY (April, 1924) contains the following papers: *On the expansion of analytic functions in series of polynomials,* by J. L. Walsh; *Operations with respect to which the elements of a boolean algebra form a group,* by B. A. Bernstein; *Isometric W-surfaces,* by W. C. Graustein; *Space time continua of perfect fluids in general relativity,* by L. P. Eisenhart; *Equivalent rational substitutions,* by J. F. Ritt; *Extension of Bernstein's theorem to Sturm-Liouville sums,* by E. Carlson; *An existence theorem,* by E. Hille, *On the complete independence of the postulates for betweenness,* by W. E. Van de Walle; *A new set of postulates for betweenness, with proof of complete independence.* by E. V. Huntington.

The third number of volume 46 of the AMERICAN JOURNAL OF MATHEMATICS contains: *Geometric aspects of the abelian modular functions of genus four,* by A. B. Coble, *The curve of ambience.* by F. Morley; *On a class of invariant subgroups of the conformal and projective groups in function space,* by I. A. Barnett.

The second number of volume 25, series 2, of the ANNALS OF MATHEMATICS contains. *A contribution to the theory of closed chains,* by A. Arwin; *On the characterization of the set of points of λ-continuity,* by H. Blumberg; *On the Ampère-Cauchy derived functions,* by H. L. Smith; *Note on certain seminvariants of n-lines,* by L. P. Copeland; *The Fredholm theory of Stieltjes integral equations,* by C. A. Fischer; *Rectilinear congruences referred to special surfaces,* by M. C. Foster; *Parallel propagation of a vector around an infinitesimal circuit in an affine manifold,* by J. L. Synge.

In 1918 the Mathematical Society of Charkow found it impossible to continue the publication of its COMMUNICATIONS. After an interval of more than five years a new journal has now been founded, under the auspices of this Society and edited by Professor S. Bernstein; it is called SECTION MATHÉMATIQUE DES ANNALES SCIENTIFIQUES DES INSTITUTIONS SAVANTES DE L'UKRAINE

Another new mathematical journal, ARCHIMEDE, RASSEGNA BIMESTRALE DI MATEMATICA, FISICA E INGEGNERIA, has recently been founded at Palermo, as the organ of the Società Universitaria di Scienze Matematiche. It is edited by G. Priulla.

At the meeting of the British Association for the Advancement of Science at Toronto in August, 1924, Professor Horace Lamb was elected president, for the meeting to be held in Southampton in 1925. Sir W. H. Bragg, as president of the section of mathematical and physical sciences, delivered an address on *Crystal structure.* On the occasion of this meeting, the University of Toronto conferred the honorary degree of doctor of science on Sir Ernest Rutherford, retiring president of the Association.

Among the officers of the National Research Council for 1924–25 are the following: Dr. G. E. Hale, honorary chairman; Professor A. A. Michelson, first vice-chairman; Professor R. A. Millikan, third vice-chairman and chairman of the division of foreign relations; Professor J. S. Ames, chairman of the division of physical sciences.

Professor J. C. McLennan, of the University of Toronto, has been elected president of the Royal Society of Canada.

On the occasion of his seventy-fifth birthday, Professor Felix Klein was elected honorary member of the Deutsche Mathematiker-Vereinigung and of the Gesellschaft für angewandte Mathematik und Mechanik, and received the honorary degree of doctor rerum politicarum from the University of Berlin.

Professor R. A. Millikan, of the California Institute of Technology, has received the honorary degree of doctor of science from Trinity College, Dublin.

Professor A. S. Eddington, of Cambridge University, has been elected an honorary member of the American Astronomical Society.

Professor M. I. Pupin, of Columbia University, has received the honorary degree of doctor of science from Princeton University.

The private scientific library of the late Professor A. G. Webster, of Clark University, has been purchased by the Riverbank Laboratories, Geneva, Illinois, and is now housed there as a separate collection.

The two Benjamin Peirce Instructorships in Mathematics at Harvard University (see this BULLETIN, vol. 21, p. 315) are again open for general competition. Applications for the year 1925–26, accompanied by the necessary papers, should reach Professor Birkhoff not later than February 15, 1925, and all inquiries relating to these appointments should be addressed to him.

Professors M. Amaldi, of the University of Padua, and L. Silla, of the University of Genoa, have been transferred to the Higher School of Architecture, recently established at Rome.

Associate Professor R. Wavre has been promoted to a full professorship at the University of Geneva.

The following have been admitted as privat-docents: D. Gigli, at the University of Pavia; A. Hammerstein, at the University of Berlin.

Dr. B. B. Baker, of the University of Edinburgh, has been appointed professor of mathematics at Royal Holloway College, University of London.

Dr. S. Brodetsky, of the University of Leeds, has been promoted to the chair of applied mathematics.

Professor Sydney Chapman, of the University of Manchester, has been appointed successor to Professor A. N. Whitehead as chief professor of mathematics at the Imperial College of Science and Technology, South Kensington.

Dr. T. M. Cherry has been appointed associate professor of applied mathematics at the University of Manchester.

Mr. E. A. Milne, of the Solar Physics Laboratory at Cambridge, has been appointed to the Beyer chair of applied mathematics at the University of Manchester.

Miss Katherine S. Arnold, for the past three years professor of mathematics at Constantinople Woman's College, has been appointed a member of the administrative staff of the American Association of University Women, effective September, 1924.

Mr. E. F. Baxter has been appointed assistant lecturer in mathematics at the University of Sheffield.

At Bryn Mawr College, Professor Charlotte A. Scott has retired; she is succeeded as head of the department of mathematics by Professor Anna J. Pell. Dr. D. V. Widder has been appointed associate in mathematics.

Mrs. Ethelwyn R. Beckwith has been appointed professor of mathematics at the College for Women, Western Reserve University, for 1924-25, to serve during the sabbatical leave of Professor Anna H. Palmié

Dr. H. E. Bray, of Rice Institute, has been promoted to an assistant professorship of mathematics.

Miss Evelyn T. Carroll, of Wells College, has been promoted to an assistant professorship of mathematics.

Professor E. C. Coker, of Winthrop College, has been appointed professor of mathematics and astronomy at the University of South Carolina.

Assistant Professor A. R. Crathorne, of the University of Illinois, has been promoted to an associate professorship of mathematics.

Professor Ira M. De Long, who retires this year, on a Carnegie pension, as Professor and Head of the Department of Mathematics at the University of Colorado, has been elected President of the Mercantile Bank and Trust Company of Boulder, Colorado.

Miss Leslie Gaylord, of Agnes Scott College, has been promoted to an assistant professorship of mathematics.

Assistant Professor Lachlan Gilchrist, of the University of Toronto, has been promoted to an associate professorship of physics.

Assistant Professor W. H. Hill, of the State Teachers' College of Pittsburg, Kansas, has been promoted to an associate professorship of mathematics.

Professor J. M. Howie, of Alma College, has been appointed head of the department of mathematics at Nebraska Wesleyan University.

Dr. J. H. Jeans, of Cambridge University, has been appointed research associate at Mount Wilson Observatory.

Associate Professor A. S. Merrill, of the University of Montana, has been promoted to a full professorship of mathematics.

Miss Georgia E. Robinson, of the University of Missouri, has been appointed acting head of the department of mathematics at Shorter College, Rome, Georgia.

Mr. J. C. Tinner has been appointed professor of mathematics at Wilberforce University.

Mr. W. M. Whyburn, of the South Park Junior College, Beaumont, Texas, has been appointed assistant professor of mathematics at Texas Agricultural and Mechanical College.

The following appointments to instructorships are announced:

Amherst College, Mr. C. S. Porter;
Brooklyn Polytechnic Institute, Mr. D. H. MacPherson;
Bryn Mawr College, Miss Anna M. Lehr;
Coker College, Miss Nellie C. Stokes;
Colorado School of Mines, Mr. E. P. Martinson;
Cornell University, Mr. H. A. Hoover;
University of Iowa (astronomy) Dr. D. H. Menzel;
Hood College, Miss Evelyn P. Wiggin;
Lehigh University, Mr. C. A. Balof;
University of Oklahoma, Mr. A. E. Anderson;
Purdue University, Mr. J. C. Bennett;
University of Wisconsin, Mr. H. S. Pollard.

Professor August Gutzmer, of the University of Halle, died May 10, 1924, at the age of sixty-four. He had been on the editorial staff of the Jahresbericht der Deutschen Mathematiker-Vereinigung since 1894, and was sole editor from 1901 to 1921.

Lucien de la Rive, the Swiss mathematical physicist, died at Geneva May 4, 1924, at the age of ninety years.

Professor T. Kotof, of the University of Charkow, died in 1923, at the age of twenty-eight years.

Professor R. H. Jude, of Rutherford College, Newcastle-on-Tyne, died June 1, 1924, at the age of seventy.

Mr. Charles Leudesdorf, registrar of Oxford University and author of mathematical papers, died August 10, 1924, at the age of seventy-one.

Professor J. G. Longbottom, of the department of mechanics at the Royal Technical College, Glasgow, died June 6, 1924, at the age of fifty-four.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ABSALONNE (A.). Cours de trigonométrie rectiligne et sphérique. Namur, Berhin, 1923. 183 pp.

BRICARD (R.). Petit traité de perspective. Paris, Vuibert, 1924. 87 pp

CRELLE (A. L.). Rechentafeln. Neue Ausgabe besorgt von O. Seeliger. Berlin, de Gruyter, 1923. 7 + 501 pp.

CRONE (C.). Matematisk Forening gennem 50 Aar. København, Gjellerup, 1923. 91 pp.

DESCHARMES (R.). Bibliographie des travaux scientifiques (sciences mathématiques, physiques et naturelles) publiée par les sociétés savantes de la France. Dressée sous les auspices du Ministère de l'Instruction publique, 1922. 4to. 7 + 235 pp.

ENGEL (F.). See LIE (S.).

FUETER (R.). Vorlesungen uber die singulären Moduln und die komplexe Multiplikation der elliptischen Funktionen. Teil 1. Leipzig, Teubner, 1924. 8 + 142 pp.

HALPHEN (G. H.). Oeuvres. Tome IV. Paris, Gauthier-Villars, 1924. 5 + 660 pp.

KRAITCHIK (M.). Recherches sur la théorie des nombres. Paris, Gauthier-Villars, 1924. 16 + 272 pp.

LAMOUCHE (A.). La méthode générale des sciences pures et appliquées Paris, Gauthier-Villars, 1924. 298 pp.

LIE (S.). Gesammelte Abhandlungen. Band 5: Abhandlungen uber die Theorie der Transformationsgruppen. Abteilung 1, herausgegeben von F. Engel. Leipzig, Teubner, und Kristiania, Aschehoug, 1924. 12 + 776 pp.

LOVITT (W. V.). Linear integral equations. New York, McGraw-Hill, 1924. 14 + 253 pp.

MOLONY (B. C.). An elementary course of analytical geometry. London, Bell, 1924. 8 + 167 + 16 pp.

RENAULD (J. F.). See RUSSELL (B.).

ROTHE (H.). Einfuhrung in die Tensorrechnung. Wien, Seidel, 1924. 4 + 179 pp.

RUSSELL (B.). Les problèmes de la philosophie. Traduit de J. F. Renauld. Paris, Alcan, 1923. 6 + 176 pp.

SEELIGER (O.). See CRELLE (A. L.).

PART II. APPLIED MATHEMATICS.

ACHALME (D.). Les édifices physico-chimiques. Tome III: La molécule minérale. Paris, Payot, 1924. 350 pp.

VAN ABKEL (P. C.). Bijdrage tot de toepassing van de waarschijnlijkheidsrekening in de natuurkunde. (Dissertation, Utrecht.) Utrecht, van Druten, 1922. 80 pp.

BUTTGENBACH (H.). Notions élémentaires de cristallographie géométrique et optique de minéralogie et de pétrographie. Liège, Vaillant-Carmanne, 1922. 100 pp.

CAMPBELL (N. R.). Théorie quantique des spectres. La relativité. Traduit de l'anglais par A. Corvisy. Paris, Hermann, 1924. 237 pp.

CHOISNARD (P.). Les probabilités en science d'observation. Paris. Alcan, 1923. 4+168 pp.

CORVISY (A.). See CAMPBELL (N. R.).

COURANT (R.) und HILBERT (D.). Methoden der mathematischen Physik. Band I. (Die Grundlehren der mathematischen Wissenschaften, Band XII.) Berlin, Springer, 1924. 14+450 pp.

DOUBLET (E.). Histoire de l'astronomie. Paris, Doin, 1923. 550 pp.

EGERER (H.). Ingenieur-Mechanik. Band I: Graphische Statik starrer Korper. Manualdruck. Berlin, Springer, 1924. 8+380 pp.

FOPPL (A.) und FOPPL (L.). Drang und Zwang. 2te Auflage. Band I Munchen und Berlin, Oldenbourg, 1924. 11+359 pp.

FOPPL (L.). See FOPPL (A.).

FORSYTH (C.). An introduction to the mathematical analysis of statistics. New York, Wiley, 1924. 8+241 pp. $2.25

GRAETZ (L.). Der Aether und die Relativitatstheorie. Stuttgart, Engelhorn, 1923. 6+80 pp.

—— Le nuove teorie atomiche e la costituzione della materia. Prima traduzione italiana da C. Rossi. Milano, Hoepli, 1924. 264 pp

HAHN (K.). Mathematische Physik. Ausgewählte Abschnitte und Aufgaben aus der theoretischen Physik. Leipzig, Teubner, 1924.

HILBERT (D.). See COURANT (R.).

JEFFREYS (H.). The earth. Its origin, history and physical constitution. Cambridge, University Press, 1924. 12+278 pp.

JOUGUET (E.). Lectures de mécanique. La mécanique enseignée par les auteurs originaux. 2 volumes. Paris, Gauthier-Villars, 1924 8+206+8+284 pp.

VON KÁRMÁN (T.) und LEVI-CIVITA (T.). Hydro- und Aerodynamik. Berlin, Springer, 1924. 4 + 251 pp.

KEARTON (W. J.) and WOOD (G.). Alignment charts for engineers and students. A text-book explaining the theory and construction of alignment charts. London, Griffin, 1924. 8 + 220 pp.

KENNELLY (A. E.). Electrical vibration instruments. An elementary textbook on the behavior and tests of telephone receivers, oscillographers, and vibration galvanometers. New York, Macmillan, 1923. 10 + 450 pp.

VON LAUE (M.). La théorie de la relativité. Tome 1: Le principe de relativité et la transformation de Lorentz. Traduit d'après la 4e édition allemande par G. Letang. Paris, Gauthier-Villars, 1924. 16 + 332 pp.

LEA (F. C.). Hydraulics for engineers and engineering students. 4th edition. London, E. Arnold, 1923. 8vo. 606 pp.

LETANG (G.). See VON LAUE (M.).

LEVI-CIVITA (T.). See VON KÁRMÁN (T.).

LODGE (O.). Atoms and rays. An introduction to modern views on atomic structure and radiation. London, Benn, 1924. 208 pp.

LOEWY (A.). Versicherungsmathematik. 4te neubearbeitete und durch Hinzunahme der Invalidenversicherung erweiterte Auflage. Berlin, Springer, 1924. 6 + 224 pp.

LORIA (G.). Complementi di geometria descrittiva. Visibilità, ombre, chiaroscuro, prospettiva lineare. Milano, Hoepli, 1924. 12 + 192 pp.

MARCOLONGO (R.). Meccanica razionale. 3a edizione. 2 volumi. Milano, Hoepli, 1923. 15 + 325 + 12 + 414 pp.

MARCUS (H.). Die Theorie elastischer Gewebe und ihre Anwendung auf die Berechnung biegsamer Platten. Unter besonderer Berücksichtigung trägerloser Pilzdecken. Berlin, Springer, 1924. 8 + 368 pp.

MIERZEJEWSKI (H.). Metrologja techniczna. Lwow, Ksaiznica-Atlas, 1924. 7 + 206 pp.

MOYER (J. A.) and others. Elements of engineering thermodynamics. 2d edition. London, Chapman and Hall, 1924. 224 pp.

MURAKAMI (H.). On the hydrodynamic group-waves and the flux of the wave-energy. Tokyo, Irie, 1923. 34 pp.

PETITET (A.). See ROUSSELET (L.).

PETROVITCH (M.). Durées physiques indépendantes des dimensions spatiales. Paris, Blanchard, 1924. 28 pp.

PIA (J.). Untersuchungen über die Tektonik der Lessinischen Alpen und uber die Verwendung statistischen Methoden in der Tektonik. Teil 1. Leipzig und Wien, Deuticke, 1923. 229 pp.

PIERNET (E.). Theorie générale sur les courants alternatifs. Paris, Gauthier-Villars, 1924. 100 pp.

DE PISTOYE (H.). Etude mécanique et usinage des machines électriques. Paris, Baillière, 1924. 802 pp.

PROBST (E.). Vorlesungen über Eisenbeton. Band 1. 2te umgearbeitete Auflage. Berlin, Springer, 1923. 11 + 620 pp.

ROUSSELET (L.) et PETITET (A.). Déformations des constructions usuelles. Paris, Dunod, 1924. 8 + 290 pp.

RIETZ (H. L.) and others. Handbook of mathematical statistics. Prepared under the auspices of the National Research Council. Boston, Houghton-Mifflin, 1924. 8 + 221 pp. $4.00.

ROMOLI-VENTURI (D.). Elettrotermia. Trattato teorico-pratico. Milano, Hoepli, 1924. 11 + 300 pp.

ROSSI (C.). See GRAETZ (L.).

SCHMIEDEL (K.). Die Prüfung der Elektrizitätszähler, Messeinrichtungen, Messmethoden und Schaltungen. 2te, vermehrte und verbesserte Auflage. Berlin, Springer, 1924. 160 pp.

SCHOKLITSCH (A.). Graphische Hydraulik. Leipzig, Teubner, 1923.

SCHWEIKERT (G.). Innere Ballistik. Leipzig, Barth, 1923. 107 pp.

SOULEYRE (A.). Les oscillations océaniques et les oscillations climatiques dans le passé et dans le présent. Le rythme de contraction du soleil. Bône, Imprimerie Emile Thomas, 1922. 19 + 389 pp.

STORMER (C.). Fra verdensrummets dybder til atomernes indre. Kristiania, Gyldendalske Bokhandel, 1923. 152 pp.

SWINDIN (N.). Modern theory and practice of pumping. A treatise on the application of the Reynolds-Stanton law of viscous flow to modern pumping problems and the flow of liquids through pipes. London, Benn, 1924. 364 pp.

TALLQVIST (H.). Energi och entropi. Stockholm, Natur und Kultur, 1923. 160 pp.

WESTIN (O. E.). Mechanical questions. Stockholm, Norstedt & Soner, 1922. 5 + 59 pp.

WITZ (A.). Traité théorique et pratique des moteurs à gaz, à essence et à pétrole. 5e édition. 2 volumes. Paris, A. Michel, 1923. 1300 pp.

WOOD (G.). See KEARTON (W. J.).

ZENNECK (J.). Elektronen- und Ionen-Ströme. Berlin, Springer, 1923. 4 + 48 pp.

CALCUTTA UNIVERSITY LIBRARY
Acc. No. ......
Book No. 97 ......

THE OCTOBER MEETING IN NEW YORK

The two hundred thirty-seventh regular meeting of the Society was held at Columbia University on Saturday, October 25, 1924, extending through the usual morning and afternoon sessions. The attendance included the following fifty members of the Society:

Alexander, Archibald, Babb, C. R. Ballantine, J. P. Ballantine, Barnum, W. M. Bond, R. W. Burgess, B. H. Camp, Cole, L. D. Cummings, Dantzig, Doak, Eisenhart, Fite, Fort, Philip Franklin, Gafafer, Garretson, Gehman, Gill, Glenn, Gronwall, Hille, Himwich, Dunham Jackson, Joffe, Kline, Langman, Lefschetz, Longley, MacColl, Meder, C. L. E. Moore, Northcott, Pell, R. G. Putnam, Rainich, Reddick, R. G. D. Richardson, Ritt, Ruger, Seely, C. E. Smith, J. H. Taylor, J. M. Thomas, Veblen, M. E. Wells, Wiener, R. G. Wood.

At the May Meeting of the Society a new type of membership was created and designated Sustaining Membership. This action was taken on the conviction that there are individuals and institutions who realize the fundamental part that mathematical research plays in our modern civilization and who are willing to render financial assistance to its publication. At the present meeting the Secretary took great pleasure in announcing the following list of Sustaining Members who had been solicited by the Committee on Endowment:

Allyn and Bacon, Boston;
Babcock and Wilcox Company, New York,
Dartmouth College, Hanover;
General Electric Company, Schenectady; Patron;
Ginn and Company, Boston,
Insull Interests, Chicago (comprising the Commonwealth Edison Company, the People's Gas Light and Coke Company, the Public Service Company, the Middle West Utilities Company, and the Chicago, North Shore and Milwaukee Railroad); Patron;
Mr. E. W. Rice, Jr., General Electric Company, Schenectady;
Union Central Life Insurance Company, Cincinnati;
University of Washington, Seattle;
Western Electric Company, New York; Patron;
Westinghouse Electric and Manufacturing Company, New York; Patron.

The Secretary announced also that the following persons had been elected to ordinary membership in the Society by the Council since April, 1924, either by mail vote[*] or at a meeting:

Mr. Anantanarayana Aiyar, Judson College, Rangoon;
Mr. William Leake Ayres, University of Texas;
Miss Wealthy Babcock, University of Kansas;
Mr. Frank Swan Beale, University of Maine;
Professor Ella Edna Bernstorf, Friends University;
Professor Harold Blair, Western State Normal College, Kalamazoo;
Mr. Leonard Marcot Blumenthal, University of Michigan;
Mr. Robert Bogue, Omaha;
Professor Zechariah Chafee, Harvard University Law School;
Professor Tsaihsin H. Chen, Yen-Ching University, Peking,
Mr. Starling W. Childs, Jr., Yale University;
Professor Jung Lu Chin, Nankai University, Tientsin;
Superintendent William Horner Cocke, Virginia Military Institute;
Mr. Redmond S. Colnon, St. Louis;
Miss Mae Elizabeth Conn, University of Southern California;
Sister Mary Cordia, Notre Dame College;
Professor Moses Eugene Cox, Agricultural and Mechanical College of Texas;
Professor Marian Elizabeth Daniells, Iowa State College,
Professor Robert Dodds Daugherty, Iowa State Teachers College;
Professor Raymond Mark Deming, Upper Iowa University;
Mr. Howard Theodore Engstrom, University of Maine;
Professor Claire Beatrice Fisher, Potomac State College,
Mr. Donald Alexander Flanders, University of Pennsylvania;
Mr. Roger Lee Flanders, Norwich University;
Mr. Alfred Douglas Flinn, Engineering Foundation, New York,
Miss Neda Belle Freeman, Irving College,
Professor Benjamin Curtis Glover, Otterbein College;
Professor George Wollam Gorrell, University of Denver;
Professor Vernon Guy Grove, Michigan Agricultural College;
Mr. Ira James Gwinn, Morningside College;
Professor John Alexander Hardin, Centenary College of Louisiana;
Mr. Floyd S. Harper, University of Nebraska;
Professor Nathan W. Harter, Thiel College;
Professor Frederick Charles Hartwick, University of Dayton,
Mr. Hubert Banks Huntley, University of New Hampshire;
Miss Mabel Hutchins, Blue Mountain College;
Mr. Saburo Isayama, Niigata High School for Girls;
Professor Ralph Lent Jeffery, Acadia University,
Professor Elijah Newton Johnson, Butler College;
Professor Jesse Brelande Johnson, Baylor University;
Miss Dora Evelyn Kearney, University of Minnesota;
Mr. Ernest George Keller, University of Wisconsin;
Mr. Arthur J. Koeppe, Union Central Life Insurance Company, Cincinnati;
Mr. Abraham Boris Kouperman, University of Chicago;
Mr. William H. Kurzin, Crane Junior College;
Mr. Benjamin George Lehenbauer, Union Central Life Insurance Company, Cincinnati;
Mr. John J. Lichter, St. Louis;
Mr. Hugh Gray Lieber, Columbia University;

---

[*] See this BULLETIN, vol. 30, p. 481.

CALCUTTA UNIVE
Acc. No.........
Book No..........
LIBRARY.

Professor William Thomas MacCreadie, Norwich University;
Professor Anna E. Many, Newcomb College;
Mr. Thomas Midgley, General Motors Chemical Company, Dayton;
Professor Kevin Nowlan, S. J., Loyola University;
Professor Yoshitomo Okada, Tôhoku Imperial University;
Professor Earle Kerr Paxton, Washington and Lee University;
Mr. Walter Otis Pennell, Southwestern Bell Telephone Company, St. Louis;
Professor Sue Scott Perkins, Missouri State Teachers College, Springfield;
Mr. Howard Manchester Phillips, DeKalb Junction. N. Y.;
Dr. Robert Garfield Putnam, New York University;
Miss Margaret Ramsey, Linfield College;
Professor Caroline M. Reaves, Coker College;
Dr. Bernard Paul Reinsch, University of Illinois;
Dr. Norman Hurd Ricker, Humble Oil and Refining Company, Houston;
Professor Joseph Alphonsus Rooney, Mount St. Charles College;
Mr. Norman Eby Rutt, University of Pennsylvania;
Professor Samuel Thomas Sanders, Louisiana State University;
Dr. Paul Schnurmann, Pittsburgh;
President John Stephen Sewell, Alabama Marble Company, Birmingham;
Professor Charles E. Shull, Bridgewater College;
Professor Elmer McClellan Stahl, Midland College;
Professor Will B. Stokes, Southern Louisiana Institute;
Miss Lucretia Mae Switser, University of Kansas;
Dr. Brandreth Symonds, Mutual Life Insurance Company;
Mr. Keizo Takahashi, Shonan Middle School, Banagawa-Ken;
Professor William Duane Tallman, Montana State College;
Mr. Ward Hastings Taylor, University of Arkansas;
Mr. Benjamin Franklin Tillson, New Jersey Zinc Company;
Mr. Paul S. Wagner, Johns Hopkins University;
Miss Esther May Weaver, Northwestern University;
Dr. David Vernon Widder, Bryn Mawr College;
Mr. Frank Harry Mead Williams, Drexel Institute.

Ordinary Members nominated by Sustaining Members:

Mr. W. L. Abbott, Commonwealth Edison Company, Chicago;
Mr. R. J. Andrus, Twin State Gas and Electric Company, Boston;
Mr. C. C. Argabrite, Interstate Public Service Company, Indianapolis;
Mr. H. E. Bates, Peoples Gas Light and Coke Company, Chicago;
Mr. L. V. Bewley, General Electric Company, Schenectady;
Mr. J. E. Black, Babcock and Wilcox Company, New York;
Mr. E. J. Blair, Chicago Rapid Transit Company;
Mr. L. F. Blume, General Electric Company, Pittsfield;
Mr. G. T. Bogard, Kentucky Utilities Company, Louisville;
Mr. Aram Boyajian, General Electric Company, Pittsfield;
Mr. B. O. Buckland, General Electric Company, Schenectady;
Mr. B. I. Budd, Chicago, North Shore and Milwaukee Railroad Company;
Mr. F. H. Buller, General Electric Company, Pittsfield;
Mr. C. M. Burrill, General Electric Company, Schenectady;
Mr. J. R. L. Carrington, Union Central Life Insurance Company, Cincinnati;
Mr. P. R. Cassidy, Babcock and Wilcox Company, New York;
Miss E. Clarke, General Electric Company, Schenectady;
Mr. S. B. Cushing, Public Service Company of North Illinois, Chicago;
Mr. G. F. Davis, General Electric Company, Schenectady;

Dr. Saul Dushman, General Electric Company, Schenectady;
Mr. J. W Evers, Commonwealth Edison Company, Chicago;
Mr. C. L. Fortescue, Westinghouse Company, East Pittsburgh;
Mr. E. J. Fowler, Commonwealth Edison Company, Chicago;
Mr. R. F. Franklin, General Electric Company, Schenectady;
Mr. G. W. Hamilton, Middle West Utilities Company, Chicago;
Mr. E. E. Hardcastle, Union Central Life Insurance Company, Cincinnati;
Mr. R. B. Harper, Peoples Gas Light and Coke Company, Chicago;
Mr. Isaac Harter, Babcock and Wilcox Company, New York;
Mr. J. L. Hecht, Public Service Company of North Illinois, Chicago;
Mr. G. J. Heimberger, Babcock and Wilcox Company, Bayonne;
Mr. Walter Helmer, Chicago Rapid Transit Company;
Mr. R. E. Hellmund, Westinghouse Company, East Pittsburgh;
Mr. Jesse Hyatt, Chicago, North Shore and Milwaukee Railroad Company;
Mr. E. E. Johnson, General Electric Company, Schenectady;
Mr. H. A. Johnson, Chicago Rapid Transit Company;
Mr. Charles Jones, Chicago Rapid Transit Company;
Mr. W. A. Jones, Babcock and Wilcox Company, Bayonne;
Mr. E. W. Kellogg, General Electric Company, Schenectady;
Mr. H. J. Kerr, Babcock and Wilcox Company, New York;
Mr. A. L. Kimball, General Electric Company, Schenectady;
Mr. C. H. Koch, General Electric Company, Schenectady;
Mr. A. J. Koeppe, Union Central Life Insurance Company, Cincinnati,
Mr. L. L. Langvand, Babcock and Wilcox Company, Barberton;
Mr. E. S. Lee, General Electric Company, Schenectady;
Mr. N. E. Lewis, Babcock and Wilcox Company, New York;
Mr. D. W. McLenegan, General Electric Company, Schenectady;
Mr. Campbell MacMillan, General Electric Company, Schenectady;
Mr. H. B. Marvin, General Electric Company, Schenectady;
Mr. S. T. Maunder, General Electric Company, Schenectady;
Mr. M. S. Mead, General Electric Company, Schenectady;
Mr. F. W. Merrill, General Electric Company, Fort Wayne;
Mr. C. W. Middletown, Babcock and Wilcox Company, New York;
Mr. Harold Mott-Smith, General Electric Company, Schenectady;
Mr. Harold Otis, Chicago Rapid Transit Company;
Mr. W. E. Paul, General Electric Company, Schenectady;
Mr. C. W. PenDell, Public Service Company of North Illinois, Chicago;
Mr. J. F. Peters, Westinghouse Company, East Pittsburgh;
Mr. A. D. Pratt, Babcock and Wilcox Company, New York,
President A. G. Pratt, Babcock and Wilcox Company;
Mr. C. W. Rice, Schenectady;
Mr. J. D. Roberts, Central Illinois Public Service Company, Springfield,
Mr. E. L. Robinson, General Electric Company, Schenectady;
Mr. R. F. Schuchardt, Commonwealth Edison Company, Chicago;
Mr. E. O. Schweitzer, Commonwealth Edison Company, Chicago;
Mr. E. B. Shand, Westinghouse Company, East Pittsburgh;
Mr. I. H. Summers, General Electic Company, West Lynn;
Mr. H. D. Taylor, General Electric Company, Schenectady;
Mr. C. E. Thompson, Chicago, North Shore and Milwaukee Railroad Company;
Mr. Stephan Timoshenko, Westinghouse Company, East Pittsburgh;
Mr. L. A. Umansky, General Electric Company, Schenectady;
Mr. G. B. Warren, General Electric Company, Schenectady;
Mr. Hosea Webster, Babcock and Wilcox Company, New York;
Mr. R. W. Wieseman, General Electric Company, Schenectady.

During the summer, the Council voted thanks for the following gifts to the Endowment Fund: from Stone and Webster, $1000; from the Cutler-Hammer Company, $250; from Mr. C. H. Coffin, ex-president and founder of the General Electric Company, $2000. It was voted that sustaining members may be elected by the Council at the time of the first presentation of their names, and that not more than five persons may be nominated to ordinary membership by a sustaining member for each $100 of dues paid.

The President made the following appointments to represent the Society: at the Pan-American Scientific Congress at Lima, Professors E. V. Huntington and D. N. Lehmer; at the inauguration of President Vinson of Western Reserve University, Professor W. G. Simon; at the celebration of the one hundredth anniversary of the Rensselaer Polytechnic Institute, Dr. G. A. Campbell; at the International Mathematical Congress at Toronto, Professor G. A. Bliss and the Secretary; at the inauguration of President Irving Maurer, of Beloit College, Professor H. H. Conwell.

The President appointed as Committee to Administer the Eliakim Hastings Moore Fund, Professors Arnold Dresden (chairman), J. W. Alexander, and H. E. Slaught. Mr. Robert Henderson, of the Equitable Life Assurance Society, was designated to deliver the second Josiah Willard Gibbs Lecture at the coming Annual Meeting in Washington; his subject will be *Life insurance as a social service and as a mathematical problem.*

A committee consisting of Mr. S. A. Joffe and Professor Tomlinson Fort was appointed to audit the accounts of the Treasurer for the current year. A list of nominations of officers, members of the Council and members of the Board of Trustees was adopted and ordered printed on the official ballot for the annual election. An invitation was received from the University of Pennsylvania to hold the summer meeting at that institution in 1926.

The report of the Committee on the Cole Prize (see this BULLETIN, vol. 29, p. 14) recommending that the prize be offered for the most notable advance in the theory of Linear Algebras was adopted. Further details will be published shortly by the Committee of which Professor H. S. White is chairman.

It has been decided to continue the printing of the BULLETIN and TRANSACTIONS in Hamburg for the year 1925.

Titles and abstracts of the papers read at this meeting follow below. President Veblen presided at both sessions. Mr. Huber was introduced by Professor G. E. Wahlin. The papers of Alexander, Cole, Franklin (second paper), Garabedian, Hille (first paper), Huber, Hutchinson, Michal, Pennell, Walsh, and Williams were read by title.

1. Professor B. H. Camp: *Note on a transformation of the hypergeometric series.*

The sum of the first t terms of the hypergeometric series which occurs in the theory of probability is* $(1/n^{(r)})\sum_{t=0}^{t=t}(r^{(t)}/t!)\,y^{(s)}(n-y)^{(t)}$, where $s=r-t$, and $r^{(t)}=r(r-1)\cdots(r-t+1)$. This may be transformed to the form: $(r^{(t+1)}/n^{(r)}t!)\sum_{y=s-1}^{y=y-1}y^{(s-1)}(n-y-1)^{(t)}$, the summation now being with respect to y. The process by which this is obtained may be of value in the transformation of other sums. The details will appear in a paper on *Probability integrals for a hypergeometric series,* in BIOMETRIKA.

2. Professor Louise D. Cummings: *A new type of double sextette closed under a binary* (3,3) *correspondence.*

This paper shows that cyclic double sextettes closed under a binary (3,3) correspondence are reducible to two non-congruent types. Type I was determined in a previous paper, and type II is here exhibited.

3. Dr. J. H. Taylor:† *Reduction of Euler's equations to a canonical form.*

This paper will appear in full in an early issue of this BULLETIN.

---

* Cf. Elderton, *Frequency Curves and Correlation,* p. 37.

† National Research Fellow.

4. Mr. H. M. Gehman: *Necessary and sufficient conditions that every closed and connected subset of a continuous curve be a continuous curve. Second paper.*

The conditions previously announced to the Society (Chicago Meeting, April 19, 1924) are contained in this more general theorem: *A necessary and sufficient condition that every closed and connected subset of a continuous curve M be a continuous curve is that M contain no closed and connected point set N, containing a continu of condensation W, such that $N-W$ is uniformly connected im kleinen "relative to W."* Another necessary and sufficient condition is that, given any positive number ε, M contains at most a finite number of mutually exclusive closed and connected sets of diameter greater than ε. A condition which is sufficient but not necessary is that, given any positive number ε, M can be expressed as the sum of a finite number of closed and connected sets each of diameter less than ε, each pair having at most a finite number of points in common.

5. Dr. Philip Franklin: *The rotating disc.*

A homogeneous incompressible disc of radius r, when caused to rotate with constant angular velocity ω, suffers a contraction in radius, according to the theory of relativity. Eddington, assuming that the geometry on the disc remains euclidean, finds the new radius to be, approximately, $r' = r(1-\omega^2 r^2/8)$. The author discusses the results when this assumption is dropped, certain others being retained, and finds for the new radius approximately $r' = r(1-\omega^2 r^2/14)$. Some properties of the geometry on the disc are also given.

6. Dr. Philip Franklin: *The electric currents in a network.*

If constant E. M. F.'s are inserted in a network of conductors, the resulting currents are uniquely determined. Their explicit expression in terms of the resistances of the conductors and the topological properties of the network was given by Kirchhoff in 1847. In this paper the result is obtained by a briefer method, based on the matrices for a linear graph.

7. Professor F. N. Cole: *On simple groups of low order.*

This paper appeared in full in the November-December, 1924, number of this BULLETIN.

8. Professor J. I. Hutchinson: *On the roots of the Riemann zeta function.*

The object of this paper is to simplify the methods and formulas in use in connection with numerical investigations on the zeta function. The limits of an empirical law noticed by Gram, regarding the separation of the roots, are determined and the results used in locating 59 additional roots, thus completing the list of those smaller than 300. The values of 14 roots are calculated, thus including the remaining unknown roots smaller than 100. The number of positive roots less than 500 is calculated and found to be 269.

9. Professor Einar Hille: *A note on regular singular points.*

This paper contains a discussion of the differential equation $w'' + G(z)w = 0$ in the neighborhood of the origin, which is supposed to be a regular singular point. The discussion is based upon the transformation $Z(z; z_0) = \int_{z_0}^{z} G(z)dz$; $W(Z) = [G(z)]^{-1/4}w(z)$, and involves among other things a characterization of the nature of the singular point by means of the geometrical singularity at the origin of the curve net $X = \text{const.}$, $Y = \text{const.}$ The orientation of the zeros and the extrema of the solutions upon a Riemann manifold in the neighborhood of the origin is studied, and regions are determined in which these points are regularly distributed. The results are incomplete in the case in which one of the roots of the indicial equation is greater than $+1$. The investigation is completed with a discussion based on the method of zero-free regions.

10. Professor C. A. Garabedian: *Four methods for solving the problem of the rectangular beam.*

Upon applying to beams of rectangular section the series already employed by the author in connection with plates and rods[*], one is led to four distinct methods. Method A deals with the broad shallow beam, while method B is concerned with the beam that is narrow and deep. Methods AA and BB (announced in the closing paragraph of the author's note on *Thick rectangular plates*[†]) are analogous to methods A and B respectively. All four methods treat the beam rigorously and in three dimensions, despite the two-dimensional aspect of the problems. Methods A and B excel in rigor, yet AA and BB yield interesting solutions expressible in polynomial form. It is possible to vary the depth in A and the width in B, and in methods AA and BB both width and depth may be varied. A note concerning these methods appeared in the COMPTES RENDUS of August 18, 1924; the author hopes later to publish the further developments in extenso.

---

[*] See abstract in this BULLETIN, vol. 30, p. 295, with accompanying references.

[†] COMPTES RENDUS, Feb. 11, 1924, and this BULLETIN, loc. cit.

11. Professor J. L. Walsh: *Some two-dimensional loci.*

This paper solves the following problem: If two points of the plane, A and B, have as their respective loci the interiors of two circles, determine the locus of the point P such that (I) the angle APB is a prescribed constant, (II) the ratio $PA : PB$ is a prescribed constant. In each case, the locus of P is a region bounded in whole or in part by a cartesian oval or an arc thereof.

12. Professor K. P. Williams: *A uniqueness theorem for the Legendre and the Hermite polynomials.*

Let

$$\varphi(y) = a_0 + a_1 y + (a_2/2!)y^2 + (a_3/3!)y^3 + \cdots,$$

and put

$$\varphi(2xz + z^2) = P_0 + P_1(x)z + P_2(x)z^2 + \cdots.$$

In this paper it is shown that a simple recursion relation between three successive polynomials of the set $P_0, P_1, P_2, \cdots$ will exist only in case we have to do essentially with the generalized polynomials of Legendre or the polynomials of Hermite.

13. Mr. C. M. Huber: *On the prime divisors of the cyclotomic functions.*

In this paper the author deals with a generalization of the test of Sylvester as to the character of the primes as divisors or non-divisors of the cyclotomic functions. As preliminary, a theorem is given for determining the prime ideal factorization of the rational primes in cyclotomic sub-fields. A necessary and sufficient condition is then developed for a rational prime to be a divisor or non-divisor of the cyclotomic functions. Particular examples are then discussed in application of the test found.

14. Mr. A. D. Michal: *Functional invariants, with a continuity of order p, of one-parameter Fredholm and Volterra transformation groups.*

This paper considers functionals of a function $y(x)$ and its first p derivatives $y'(x), \cdots, y^{(p)}(x)$ admitting a given arbitrary one-parameter *Fredholm* group of linear functional transformations. Sufficient conditions for invariance are given in the form of completely integrable functional equations with partial functional derivatives when the kernel of the transformation is of a certain type. In the case of the linear functional necessary and sufficient conditions for invariance are given. The paper concludes with a theorem showing the unique rôle played by a linear functional of $y(x)$ and its derivative $y'(x)$ in the invariant theory of linear functionals, of continuity order p, which admit a given arbitrary one-parameter *Volterra* group of linear functional transformations.

15. Professor J. W. Alexander: *On certain new topological invariants.*

The author derives certain topological invariants analogous to but in general different from the Betti numbers and coefficients of torsion.

16. Mr. W. O. Pennell: *The interpretation of non-integral exponents with notes on the theory of subponents.*

In this paper the author makes use of the following vector notation: a_α represents a vector in the XY plane of length a making an angle α with the X axis. He calls α the subponent of the vector X, and points out that in multiplication subponents follow laws similar to those of exponents.

17. Professor Norbert Wiener: *The representation of bounded functions by trigonometric integrals.*

The author investigates the condition under which a bounded function may have a trigonometric expansion analogous to a Fourier integral. He studies the convergence in the mean of this expansion both over a finite range and (in the sense of Bohr) over an infinite range.

18. Professor Einar Hille: *A general type of singular points.*

The present paper gives a unified treatment of a fairly general type of singular points of linear differential equations of the second order, based upon the transformation of Liouville. It is assumed that there exists a region abutting upon the singular point which is carried over into a right half-plane by the transformation and, further, that in this half-plane the transformed equation is asymptotic (in an easily characterized sense) to a sine-equation. Such an equation can be integrated by the method of successive approximations and the form of the integral gives directly the asymptotic properties of solutions of the transformed equation in the half-plane. In particular, the distribution of the zeros can be read off easily. Any irregular singular point can be handled by this method, also a wide class of regular singular points, though not all. Further, any singular point which is transformable into one of the previously mentioned types by a change of independent variable can be attacked directly by this method, the detour over the intermediate transformation being unnecessary.

19. Professor Dunham Jackson: *The geometry of frequency functions.*

This paper has appeared in the January-February issue of this BULLETIN.

20. Professor J. F. Ritt: *Elementary functions and their inverses.*

The chief item of this paper is the determination of all elementary functions whose inverses are elementary. The elementary functions are understood to be those obtained by performing algebraic operations and taking exponentials and logarithms. For instance, the function $\tan\left[e^{z^2}-\log_z(1+\sqrt{z})\right]+[z^z+\log\arcsin z]^{1/2}$ is elementary. It is proved that *if $F(z)$ and its inverse are both elementary, there exist n functions $\varphi_1(z), \varphi_2(z), \cdots, \varphi_n(z)$, where each $\varphi(z)$ with an odd index is algebraic, and each $\varphi(z)$ with an even index is either e^z or* $\log z$, *such that $F(z)=\varphi_n\varphi_{n-1}\cdots\varphi_2\varphi_1(z)$, each $\varphi_i(z)$ $(i<n)$ being substituted for z in $\varphi_{i+1}(z)$.*

This paper is a continuation of Liouville's work of almost a century ago on the classification of the elementary functions, on the possibility of effecting integrations in finite terms, and on the impossibility of solving certain differential equations, and certain transcendental equations, in finite terms.

21. Professor J. F. Ritt: *On the impossibility of solving certain differential equations in finite terms.*

Liouville, in 1841, proved that the solutions of Bessel's equation $z^2w'' + zw' + (z^2 - \nu^2)w = 0$ cannot be expressed in terms of z by performing elementary operations* and quadratures. But it is customary to regard a differential equation as solved if one finds a finite equation in w and z; one does not generally try to express w itself in finite terms. In the present paper it is shown that the solution of Bessel's equation does not satisfy any equation in w and z found by performing elementary operations upon those variables and integrating exact differentials.

22. Mr. G. Y. Rainich: *On singularities in analytical physics.*

The introduction, in place of the electromagnetic tensor and its dual, of their sum w, which is a complex tensor, and of complex vectors simplifies calculations. The energy tensor is the norm of w, i. e., w multiplied by its conjugate. Residues of an electromagnetic field are introduced as values of surface integrals surrounding singular lines (electron paths); they are complex numbers only the moduli of which are determined by space-time. Residues can be used to complete the determination of the electromagnetic field, which, in a region free from singularities, is given, by the curvature of space-time, only to a constant of integration (see the author's paper read before the Society in May, 1924). To determine this constant we can agree to make all the residues real, which can be done because the arguments of the residues of every electromagnetic field differ only by multiples of π, proved by the fact that magnetic charges do not exist. The analogy of the electromagnetic field with the analytic function on a minimal surface (see abstract of the author's second paper, below) is discussed.

---

* See preceding abstract.

23. Mr. G. Y. Rainich: *The analytic function on a minimal surface.*

The Codazzi equations, to which the tensor usually represented by the second differential form is subjected, have in the case of a minimal surface the form of the Cauchy-Riemann equations which are satisfied by an analytic function. To make the analogy complete, we have to consider an analytic function as defining in each point of the plane a linear vector transformation. The Cauchy integral theorem is immediately applicable to the function on a minimal surface, and this leads to the definition of a residue. The Gaussian curvature at each point is equal to minus the square of the modulus of the function at that point. If a surface is given by its first form and the coefficients satisfy a certain differential equation, the analytic function is defined to a constant of integration, the values of which correspond to associated surfaces. Certain of the results are extended to surfaces of constant mean curvature.

24. Professor Dunham Jackson: *On the gradient of a functional.* Preliminary communication.

This paper is concerned with a definition of the directional derivative of a functional, the determination of the direction in function space for which the value of this directional derivative is a maximum, and the evaluation of the derivative in any other direction as the product of the above-mentioned maximum value by the cosine of the angle between the two directions in question.

25. Dr. T. H. Gronwall: *Extension of a theorem due to Lerch.*

Lerch has shown that when $f(x)$ is continuous for $0 < x < 1$ and $\int_0^1 x^n f(x)dx = 0$ for $n = 0, 1, 2, \cdots$, then $f(x)$ vanishes identically. In the present note it is shown that when $f(x)$ is absolutely integrable and the above equation holds, $f(x)$ vanishes identically on any sub-interval where it is continuous.

26. Dr. T. H. Gronwall: *On the Cesàro summability of Fourier's and Laplace's series.*

Fejér has shown that any Cesàro sum of order one of the Fourier series of a continuous function takes only values between the extreme values of the function, and also that the same property holds for the sums of order two of the Laplace series. Considering also fractional values of the order k of the sums, the present paper proves that $k=1$ and $k=2$, respectively, are the smallest values of k with the above property.

27. Professor J. R. Kline: *A note concerning closed non-dense linear sets which are enumerable.*

In both the first and the second editions of Hobson's *Theory of the Functions of a Real Variable*, the author attempts to prove that a non-dense closed set is enumerable if its complementary intervals are such that every one abuts on another at each of its ends. Professor R. L. Moore gave an example of a set satisfying the conditions of the hypothesis of Hobson's theorem, which is not enumerable. In the present paper it is shown that in case it is stipulated that the non-dense closed set and all its derived sets have the property of having their complementary intervals each abut on another at each of its ends, then the set is enumerable.

R. G. D. RICHARDSON,
Secretary.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION

The forty-fourth regular meeting of the San Francisco Section was held at the University of California on October 25, 1924. In the absence of the Chairman of the Section, Professor Carpenter, the meeting was called to order by the Secretary of the Section. Professor Hedrick was elected temporary chairman. The total attendance was thirty, including the following twenty-five members of the Society:

Alderton, Allardice, Andrews, Bell, Bernstein, Blichfeldt, Growe, Haskell, E. R. Hedrick, Hotelling, Irwin, Lehmer, Sophia Levy, McCarty, McFarland, F. R. Morris, Moreno, Noble, Pehrson, T. M. Putnam, Robertson, Schmiedel, Pauline Sperry, A. R. Williams, Wong.

The following officers were elected for the coming year: Chairman, Professor E. R. Hedrick; Secretary, Professor B. A. Bernstein; Program Committee, Professors E. T. Bell, H. F. Blichfeldt, B. A. Bernstein.

It was decided to hold regular meetings of the Section during 1925 as follows: at Stanford University on April 4, at the University of Oregon on June 19, and at the University of California in the fall, the exact dates and places to be determined by the officers of the Section.

Titles and abstracts of papers read at this meeting follow. In the absence of the author, Professor Bateman's paper was read by Mr. H. P. Robertson. Mr. Reimer was introduced by Professor Bernstein.

1. Professor Harry Bateman: *Numerical solution of an integral equation.*

Hafen has shown that the distribution of electricity on the circular plates of a parallel plate condenser can be found by solving a linear integral equation of the second kind. In the present paper the author solves this equation numerically by a method which is in some respects analogous to the method of least squares, and discusses the method in a general way, remarking on the question of convergence.

2. Professor E. T. Bell: *An algebra with singular zero.*

This algebra, which is that of the symbolic calculus for dealing with relations between the individual terms of sequences, is abstractly identical with common algebra, when properly interpreted, except that zero, defined by $a-a=0$, where a is any element of the algebra, has singular properties, such as $b+a-a\neq b$ (b an element of the algebra), $b(a-a)\neq 0$. An outline is given of the applications of this powerful tool for investigating the properties of numbers defined by sequences, also of the Bessel coefficients.

3. Professor E. T. Bell: *On certain functions of two variables and their integrals related to the Bessel coefficients.*

The functions $L_n(\lambda, \mu)$ in question are the coefficients of $t^n/n!$ ($n = 0, \pm 1, \pm 2, \cdots$) in the Laurent expansion, absolutely convergent for $|t| \neq 0$, of $\exp(\lambda t + \mu t^{-1})$. By taking the coefficients of $t^n/n!$, and not of t^n, it is feasible to apply the umbral calculus of Blissard to the study of these functions and their integrals. For $\lambda = -\mu = \frac{1}{2}z$ the functions become the Bessel coefficients $J_n(z)$. Thus the relations between the $J_n(z)$ and the integrals dependent on them are obtained here in special cases. The L_n are readily expressed in terms of the J_n for irrational arguments. Many of the integrals thus evaluated in terms of Bessel coefficients appear to be new. The symbolic method applies equally well to the functions defined by $\exp(F(t))$, where $F(t)$ is any finite or infinite polynomial in t, t^{-1}. These also are expressible in terms of Bessel coefficients. The use of the symbolic method cuts the computations to a minimum.

4. Professor E. T. Bell: *Modular equations and quadratic forms.*

The elliptic modular equation associated with the transformation of order n implies and is implied by each of several elementary arithmetical theorems concerning the representations of integers in certain quadratic forms with integer coefficients, of which at least one is divisible by n. For $n = 7$, in many respects the simplest case, there are 256 associated theorems, of which 4 refer to binary forms, 56 to ternary, and 196 to quaternary. These are given in full, in a condensed form.

5. Professor E. T. Bell: *On generalizations of the Bernoullian functions and numbers.*

All generalizations are segregated into two classes, (1) those obtained by modifications of the difference equations of definition, (2) those derived from modifications of the symbolic equations of definition. An example of (1) is Nörlund's memoir in ACTA MATHEMATICA, vol. 43, pp. 121–196. Here (2) only is discussed and illustrated by an extensive generalization of the "ultra-Bernoullian functions" of Krause (LEIPZIGER BERICHTE, vol. 55, pp. 39–63) and Appell (ARCHIV FÜR MATHEMATIK UND PHYSIK, vol. 4 (1902), pp. 292–5). The number of attainable generalizations is infinite. Suggestions are given for classifying this infinity.

6. Professor E. T. Bell: *Transformations of relations between numerical functions.*

From consideration of functions whose arguments are the elements of an abelian semi group, Cesàro (ANNALI DI MATEMATICA, (2), vol. 13, p. 155, (12), (13)) obtained a remarkable inversion of series, "le plus général que l'on connaisse," and states that the well known Mobius inversion is a special case. It is here shown that either one of the Cesàro or Möbius inversions implies the other, and that both are contained in an extremely general theorem of reciprocity, of which the Cesàro-Möbius inversion is the simplest example possible.

7. Professor E. T. Bell: *A new type of class number relations.*

The novelty is that the sums of class number functions are expressible as functions of the real divisors alone of a single integer when and only when the fixed integer in the arguments of the class number functions is a square. This paper appears in the October, 1924, number of the TRANSACTIONS.

8. Professor B. A. Bernstein: *A general theory of representation of finite operations and relations.*

At the International Mathematical Congress at Toronto, the author presented a method of representing arithmetically any abstract binary operation and any dyadic relation in a finite class. The method is now extended to the case of operations other than binary and relations other than dyadic. The representations bring out clearly the nature of an operation or relation.

9. Professor Florian Cajori: *Fanciful hypotheses on the origin of the forms of the Arabic numerals.*

The author describes numerous attempts, which have been made during the past nine centuries, to explain the origin of the forms of our nine numerals and the zero from the composition of short lines, or of angles, or of dots, and points out that all the hypotheses advanced proved fruitless because they failed to coordinate the known facts relating to the forms of the numerals at different times and in different countries, and failed to suggest useful new lines of further research.

10. Professor M. W. Haskell: *Note on the self-dual quintics.*

At the Mathematical Congress at Toronto, the author developed a method for deriving an unlimited number of curves which are autopolar with respect to a finite number of conics. In the present paper he exhibits two such curves of the fifth order, viz. (*a*) The curve whose equation in polar coordinates is $8\varrho^5 \cos 5\varphi - 15\varrho^4 + 10\varrho^2 - 3 = 0$ is a quintic with five cusps, invariant under a dihedral group of 10 linear substitutions, and hence autopolar with respect to six conics; (*b*) The curve whose equation is $(9\varrho^5 - 5\varrho^3) \cos 3\varphi - 15\varrho^4 + 15\varrho^2 - 4 = 0$ is a quintic with three cusps and three double points, invariant under a dihedral group of 6 linear substitutions, and hence autopolar with respect to four conics.

11. Professors E. R. Hedrick and Louis Ingold: *On certain attempted extensions of the theory of functions to three dimensions.*

The authors discuss attempts made by themselves and by others to extend the ordinary theory of functions of a complex variable to three dimensions by means of definitions of the product or of the quotient of two three-dimensional vectors. The nature of the difficulties encountered and the reasons for them are pointed out. Another road toward extension of the theory was finally adopted, and has been presented to the Society (see this BULLETIN, vol. 29, p. 437), but the reasons for abandoning the more obvious path have not been stated, and it seems not without interest to state them.

12. Dr. Harold Hotelling: *Three-dimensional manifolds of states of motion.*

To every two-dimensional dynamical problem with constant total energy there corresponds a three-dimensional manifold, two coordinates determining the position of the moving particle on the surface given by the problem, and the third the direction of its motion. The topology of this manifold, which is important in the study of periodic orbits, depends only upon that of the portion of the given surface on which the particle may move. Characterizations of all of these manifolds and formulas for their principal invariants are given.

13. Dr. F. R. Morris: *Development of annuity formulas without series.*

The formula $s = (1+r)^n$ may be taken as the definition of the amount on one unit with interest at rate r per period for n periods. Then $(1+r)^n-1$ is the interest on one unit. $(1+r)^n-1$ is also the value of an annuity with periodic rent r. The author makes use of these facts to derive the well known annuity formulas without the help of series.

14. Mr. E. H. Reimer: *Application of Weyl's geometry to differential geometry in two dimensions.* Preliminary report.

If neighboring vectors of zero length are equivalent, the geometries of Weyl and Eddington coincide for the surface $S(u_1, u_2)$. For Eddington's $\lambda g_{ij} = R_{ij}$, the Gauss equations give $\lambda = R$. Either of two vectors $V_i(u_1, u_2)$ $(i = 1, 2)$ in the asymptotic directions of length $\sqrt{R}$ gives a Weyl vector. Accents denoting covariant derivatives, the Codazzi equations become $V_1' = \pm V_2'$, according as the parameters are (u_1, u_2) or (u_2, u_1).

15. Mr. H. P. Robertson: *On certain solutions of the cosmological equations.*

This paper concerns itself with certain solutions of Einstein's equations in a space-time which may be described by orthogonal coordinates. A solution is found which gives the field due to a line or plane distribution of matter and electricity. It is proved that certain types of line element lead only to the central-symmetric field of Schwarzschild. Other solutions are found which give rise to theorems of interest in differential geometry and the new physics; in these cases simple data of observation are the criteria of the form of space-time.

B. A. BERNSTEIN,
Secretary of the Section.

THE SEVENTEENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION

The seventeenth regular meeting of the Southwestern Section of this Society was held at Iowa State College on Saturday, November 29, 1924. The following thirty members of the Society were present at the meeting:

E. S. Allen, Ashton, Birkhoff, Brenke, Chittenden, Coffin, J. T. Colpitts, Daniells, C. W. Emmons, Gaba, Gouwens, Herr, Louis Ingold, J. V. McKelvey, Martha McKelvey, Pattengill, Pierce, J. F. Reilly, H. L. Rietz, M. M. Roberts, Roever, Rusk, E. R. Smith, Helen Smith, Snedecor, Stouffer, Tappan, J. S. Turner, W. H. Wilson, Roscoe Woods.

Professor E. R. Smith occupied the chair, being relieved during the afternoon session by Professor Birkhoff. The morning session and part of the afternoon session were devoted to the reading of the papers listed below.

During the afternoon session Professor William H. Roever gave a special lecture by invitation of the program committee on *Some phases of descriptive geometry*. This paper will appear in full in an early issue of this BULLETIN.

It was voted to hold the next meeting of the Southwestern Section at Kansas City, Missouri, in conjunction with the meeting of the American Association for the Advancement of Science. The following program committee was elected: Professors R. L. Moore (chairman), Louis Ingold, and E. B. Stouffer (secretary).

The titles and abstracts of the papers read are given below. In the absence of the authors the papers by Dr. Green and Professor Moore were read by title.

1. Professor G. D. Birkhoff: *On the structure of matter.*

The author developes a theory of matter, and electricity attached to matter, on the assumption that the matter is a perfect adiabatic fluid in which every disturbance travels with the velocity of light.

2. Professor E. W. Chittenden: *Note on a compact set which is not perfectly compact.*

R. L. Moore has called attention to the fact that the class of transfinite ordinals is compact but not perfectly compact. The present note contains an example of an enumerable set which is compact but not perfectly compact.

3. Professor W. H. Wilson: *On two general functional equations.*

Several special cases of the functional equations

$$f(x+y)+g(x-y) = h(x)\,k(y),$$

and

$$F(x+y)\,G(x-y) = H(x)+K(y),$$

where x and y are independent, have been considered. The author assumes no relationships between the functions and places no restrictions on them. He expresses these eight functions directly in terms of the functions which satisfy the Cauchy equations $\varphi(x+y) = \varphi(x)+\varphi(y)$ and $\varphi(x+y) = \varphi(x)\,\varphi(y)$. By virtue of the results obtained in this paper, any information concerning these basic Cauchy equations may be applied immediately to the general equations proposed above or to special cases of these general equations.

4. Professor W. H. Wilson: *On a certain functional equation.*

The author discusses the functional equation

$$f(x+y)-f(x-y) = yg_k(x)+y^3g_{k-1}(x)+\cdots+y^{2k-1}g_1(x),$$

in which x and y are independent (not necessarily real), and $f(x)$, $g_1(x)$, $g_2(x)$, $\cdots$, $g_k(x)$ are functions to be determined. It is shown that $g_i(x)$, for each value $i=1, 2, \cdots, k$, satisfies the equation

$$\text{(A)} \qquad \sum_{j=0}^{2i}(-1)^j\,{}_{2i}C_j\,g_i[x+(2i-j)y] = 0,$$

where ${}_{2i}C_j$ is the coefficient of a^j in the expansion of $(1+a)^{2i}$. In addition to this it is obvious that

$$f(2x) = f(0)+xg_k(x)+x^3g_{k-1}(x)+\cdots+x^{2k-1}g_1(x).$$

This separation of the unknown functions is effected without restricting the functions (as to continuity, etc.). Equations of form (A) have been discussed by the writer (AMERICAN JOURNAL, vol. 40 (1918), pp. 263 et seq.), in the light of various restrictions on the functions $g_i(x)$.

5. Professor J. S. Turner: *The spherical analogue of central forces.*

If a particle is constrained to move on a sphere, the projection of its acceleration at P upon the tangent plane at P is here called its tangential acceleration. The line of action of the tangential acceleration touches a certain great circle; if O is a point on this great circle, the tangential acceleration at P is said to be directed to O. A tangential acceleration which is always directed to a fixed point O is called a central tangential acceleration. It is proved that if a particle moving with a central tangential acceleration directed to O describes a curve on the sphere, then $v \sin p$ is constant, where v is the velocity at P, and p is the arcual perpendicular from O on the arcual tangent to the curve at P. It is also proved that if the particle describes a spherical conic under a central tangential acceleration directed to a focus S, this acceleration is inversely proportional to $\sin^2 SP$; and a simple expression is found for the velocity at P.

6. Professor J. S. Turner: *An extension of the property of the circle known as "Simson's line".*

Let ABC be a triangle inscribed in a circle, and let P, Q be two inverse points with respect to the circle. Take L, M, N, the images (symmetric points) of P in BC, CA, AB, and join QL, QM, QN, cutting BC, CA, AB in X, Y, Z respectively. Then the points X, Y, Z are collinear.

7. Professor Julia T. Colpitts: *Entire functions defined by series of the form* $f(z) = \sum f(n)(z^n/n!)$.

In the first part of the paper, $f(n)$ is taken as a polynomial in n. It is proved that the necessary and sufficient condition that a function of genus one have a finite number of zeros is that $f(n)$ be the product of α^n(α constant) and a polynomial in n. In the second part, $f(n) = 1/(n+a)^s$, $s > 0$ and rational, and a any real number except a negative integer or zero. The nature of the zeros and some approximations to their values are determined.

8. Professor T. A. Pierce: *An explicit solution of a congruence.*

A. Hurwitz employed an interpolatory function (mod p) in finding the number of real roots of a congruence. Professor Dickson used the same function in constructing a theory of modular invariants. The author shows how the interpolatory function may be used to furnish an explicit solution of a congruence whose modulus is a prime.

9. Dr. C. F. Green: *On the summability and regions of summability of a general class of series of the form* $\sum c_n g(x+n)$.

The first part of the paper is devoted to the determination of the character of the regions of summability of the series for Cesàro type of summability of order r. The regions are half planes bounded by straight lines whose abscissas are dependent upon the real part of the exponents in the asymptotic representation of the series. A certain class of auxiliary series (a generalized Dirichlet series) which possesses the same regions of summability as the above series is used as an aid in the determination of the abscissas of summability. Relations between the regions of summability are discussed for the general series and also for particular series of the above type.

10. Professor Louis Ingold: *Extensions of the Gauss and Codazzi relations.*

Relations connecting certain differential parameters of the first and second orders are obtained which are analogous to the well known relations of Gauss and Codazzi connecting the first and second fundamental quantities of surface theory. The extension to n dimensions is also given.

11. Professor E. S. Allen: *The game of "bell and hammer".*

The game of "bell and hammer" is begun by the "sale" of five cards at auction. M. Jequier of Neuchatel investigated the probable value of these cards, neglecting the last phase of the game; the present paper completes the investigation. Among other questions, that of the probable number of throws needed to obtain a certain sum exactly (either by one throw or by a succession) is studied.

12. Professor R. L. Moore: *Concerning upper semi-continuous collections of continua which do not separate a given continuum.*

A collection of continua G is defined to be an *upper semi-continuous* collection provided it is true that, if p is a continuum of the collection G and $p_1, p_2, p_3, \cdots$ is a sequence of continua of the collection G such that the lower distance from p_n to p approaches zero as a limit as n approaches infinity, then so does the upper distance from p_n to p. It is shown that if, in a plane S, M is a bounded continuum which does not separate S and G is a countable upper semi-continuous collection of sub-continua of M, no one of which separates M, then the sum of all the continua of the set G does not separate M.

13. Professor R. L. Moore: *Concerning sets of segments which cover a point set in the Vitali sense.*

In volume 5 of FUNDAMENTA MATHEMATICAE, J. Splawa Neyman showed that if, in space of one dimension, a closed point set K of measure zero is covered in the Vitali sense by a set of segments G then, for every position number e, G contains a subset G_e such that G_e covers K and such that the sum of the lengths of the segments of G_e is less than e. Sierpinski has raised the question whether this theorem remains true if the requirement that K be closed is omitted. This question here is answered in the negative. But if the requirement in question is replaced by the requirement that there do not exist infinitely many segments of the set G, all of length greater than the same positive number, then the resulting proposition is true.

14. Professor F. Reilly: *On Lidstone's demonstration of the osculatory interpolation formula.*

The author shows that the osculatory interpolation formula of Lidstone can be generalized so that it will involve finite differences of order $2h+1$, and its graph will have contact of order k with the partial interpolation curves, in a manner analogous to that previously employed in generalizing the formula as demonstrated by Sprague. The two generalized formulas are, in general, not identical.

E. B. STOUFFER,
Secretary of the Section.

REMARKS ON THE FOUNDATIONS OF GEOMETRY*

BY OSWALD VEBLEN

1. *Relation between Matter and Space.* The foundations of geometry must be studied both as a branch of physics and as a branch of mathematics. From the point of view of physics we ask what information is given by experience and observation as to the nature of space and time. From the point of view of mathematics, we ask how this information can be formulated and what logical conclusions can be drawn from it.

It is from the side of physics that has come the most important contribution in the last two decades. The experimental physicists and the astronomers have uncovered facts which thus far can only be reconciled with one another by the method due to Einstein. This method consists essentially in regarding the events of space-time as a four-dimensional manifold in which there is a Riemann metric. The method is radically different in principle from that of the classical geometry and mechanics. Yet the quantitative results which flow from it are so closely in agreement with those of the older theory that they can be distinguished experimentally only in a few cases and with great difficulty.

To understand the exact way in which the euclidean geometry and the Newtonian mechanics can be regarded as a first order approximation to the Einstein theory of gravitation seems to me to be of great importance in understanding the physical significance of geometry altogether. For it requires us to be clear as to what we mean physically by straight lines, axes of inertia, absolute rotation, and so on. Therefore I propose, with your kind indulgence, to make my first remarks on this subject. It seems to me that it is a suitable question to discuss in this sort of an address, in which one is not expected to pile new bricks on the edifice of science but rather to make a few observations as to how the brick-layers are getting on with their work.

---

* Presidential address delivered before the Society, December 31, 1924.

In the Newtonian theory of gravitation the motion of a particle is determined by the interplay of two forces, the attraction (according to the inverse square law) of all other particles, and the particle's own inertia, i. e., its tendency to move uniformly in a straight line. The inertia of a particle is an unexplained and unchanging property of the particle itself, although it clearly requires a euclidean geometry and a theory of time to state what is meant by the phrase "uniformly in a straight line".

The lack of a physical basis for inertia was seen by Mach, if not by others before him, as a weakness of the Newtonian mechanics, and Mach put forward the idea that the inertia of a particle as well as the gravitational attraction on it is determined by all the rest of the matter in space. This conception was very influential in developing Einstein's gravitation theory and is taken account of in this theory by merging the ideas of inertia and gravitational attraction into a new idea. The motion of a particle is completely described by saying that its world-line is a geodesic of the differential form

$$ds^2 = g_{ij}dx^i dx^j. \tag{1}$$

The effect of the rest of the matter in the universe is to determine the values of the functions g_{ij}, and the functions g_{ij} in turn determine the motion of the particle.

This motion can be described in terms of inertia, gravitation, etc., if we employ a properly chosen euclidean metric as an approximation to the Riemann metric of the differential form (1). The approximation may be described as follows.

Consider an event in our galaxy well separated in space and time from any large mass of matter. A set of coordinates with this event as origin may be chosen by a well known method (Riemann normal coordinates) so that (1) becomes

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{2}$$

at the origin, and the equations of the geodesics through the origin become

$$x = as, \quad y = bs, \quad z = cs. \tag{3}$$

Since there is no mass of matter near the origin it follows that the formula (2) which holds rigorously only at the origin is also valid to a high degree of approximation for a large region R. In this region the geodesics which do not pass through the origin, as well as those which do, will be represented to a high degree of accuracy by first degree equations.

In other words special relativity holds with high accuracy throughout R. Any two coordinate systems with regard to which the conditions (2) and (3) hold good are related by a Lorentz transformation and are said to be in uniform motion, the one with regard to the other. The three-dimensional locus of a moving particle is the same as the three-dimensional locus of a ray of light. This is the physical content of the statement that a particle moves along a straight line.

For regions in the immediate neighborhood of masses of matter such as the sun or the earth, it is not true that the equations of geodesics are linear. But this does not prevent us from using the coordinate system (x, y, z, t) all around, and all through, the earth and sun. In this region as well as in the region R a straight line is a curve whose equation is of the first degree with regard to the coordinates x, y, z.

The coordinate system was undetermined up to a Lorentz transformation. We now determine it uniquely by the condition that it shall be at rest with respect to the sun.

This particular coordinate system we call an inertial frame of reference F. For small velocities, the Lorentz transformation is very closely approximated by the transformation

$$(4)\qquad \begin{cases} \bar{x} = a_{11}x + a_{12}y + a_{13}z + c_1 t + d_1, \\ \bar{y} = a_{21}x + a_{22}y + a_{23}z + c_2 t + d_2, \\ \bar{z} = a_{31}x + a_{32}y + a_{33}z + c_3 t + d_3, \end{cases}$$

in which the coefficients are all constants and the numbers a_{ij} form an orthogonal matrix. Any frame of reference which arises from F by a transformation of this sort is also called an inertial frame.

With respect to an inertial frame the path of light is still very approximately linear even close to the earth or the sun, but the path of a material particle is more nearly a second degree curve. This divergence of the particle paths from the light paths is accounted for by introducing the concept of force and the Newtonian mechanics. This mechanics says that in the absence of forces, particles would move uniformly in straight lines—i. e., in light paths. But the matter composing the earth brings an attracting force to bear on any particle and consequently the actual path diverges from that of light.

Thus the Newtonian mechanics is to be regarded as a device for separating out the local influence of matter (using an astronomal scale for the term "local") from the influence of matter as a whole. All the matter in the universe determines what lines are straight, what pairs of lines are perpendicular or parallel, what triangles are similar to each other, and so on. It also determines whether any given type of motion has or does not have acceleration; for example, it determines the plane in which a Foucault pendulum swings and thus gives an absolute magnitude for the rotation of the earth. On the other hand the local influence of the earth, the sun, and the planets is taken care of by means of forces which show how the actual paths of particles are diverted from what they would be without the presence of these bodies.

Consider the state of affairs inside a spherical shell of matter. The Newtonian theory assures us that the shell exerts no force of attraction on a particle inside it, a statement which must have seemed paradoxical to all of us so long as we took it to mean that the shell exerts no influence on the particle. We now see that the shell has a share in determining the very geometry according to which the particle moves, and that if the shell is set into rotary motion, say with regard to another shell which contains it, there must be a tendency to rotate the axes of inertia and thus to modify the whole geometry of the interior.

A related question might conceivably arise in astronomy. Suppose we have a very large but very sparsely distributed mass of matter which is far away from other masses. It is conceivable that such a mass might have a certain degree of autonomy in the determination of its axes of inertia. To see this, let us suppose an observer to start somewhere in the interstellar space of our galactic system and to travel without ever coming close to a large mass of matter until he arrives somewhere in the interstellar or intermolecular space of the distant mass. Anywhere he stops and sets up a coordinate system of the type we have been considering, he will find that he has, to good approximation, an inertial system extending over a quite extensive region. But as he travels, the functions g_{ij}, expressed in the original coordinate system, are gradually changing and therefore the relation between the original coordinate system and the inertial system is changing. The relation between his first inertial system and his last one will not in general be capable of representation by a Lorentz transformation. Hence the one system will appear to be accelerated with respect to the other. Such an acceleration would be expected to show itself most obviously in one or all of three ways: in a linear acceleration of the mass from or toward us, in an acceleration toward or from its center, in a rotation of its axes of inertia relative to ours.

There is good reason to believe that the domain of our axes of inertia reaches well out toward the limits of the visible universe. For the velocities of the stars are small and there is nothing known about their motion which conflicts with a Newtonian mechanics based on our inertial axes. It is not impossible, however, that the spiral nebulae might be masses of the sort to which we are trying to attribute partially autonomous inertia. They are well isolated, enormously large, and enormously distant, as recent work by Hubble establishes more clearly than ever before. Moreover their radial velocities relative to the sun are very large, they have an expansive motion relative to their own centers and they have large rotary motion. Indeed, measurements on the internal motions of certain spirals have been made by Van Maanen which Jeans found he could explain by Newtonian mechanics relative to the usual axes only if he assumed a law of attraction different from the inverse square law. Of this law Jeans says "The general effect of the modified gravitational force appears to be one of seizing the particle and compelling it to revolve about the nucleus of the nebula with an angular velocity which approximates to that of the nucleus itself rather than to that demanded by the conservation of angular momentum."

Referring to the work of Jeans, E. W. Brown asked me whether there might not be a possible relativity explanation of the apparent rotation of axes of reference of the nebula. My reply was that the relativity theory provides a physical reason for expecting this rotation, provided the nebula is sufficiently large to have a degree of autonomy with respect to inertia. In this case we should expect to describe the internal motions of the nebula in terms of a Newtonian mechanics stated in terms of inertial axes different from ours.

I am citing this highly speculative possibility in the hope that it will serve to make the problem of the foundations of geometry more vivid from a physical point of view. It may also serve to indicate some of the elements which must be thought of in a non-static solution of Einstein's gravitational equations. In order to take account of the distribution of matter throughout space in determining the values of the functions g_{ij} at any point, it would seem that we ought to have integral equations to be satisfied by the g's. But whether the partial differential equations given by Einstein are compatible with such integral equations, I do not know.

2. *The Geometry of Paths.* The studies of Riemann geometry which have been stimulated by the relativity theory have revived interest in the foundations of differential geometry. Here one starts with the observation that when arbitrary coordinates x undergo a general analytic transformation the differentials of these coordinates suffer only a linear transformation.

The differentials at a point may be regarded as representing directions emanating from this point. Thus we have at each point a projective, and also an infinity of affine and euclidean geometries to represent the relations among these directions.

But to begin with there is nothing to connect the differentials at any point x with the differentials at a point $x+dx$. Such a connection must be obtained by relations involving the higher differentials of the coordinates. It is accomplished in one manner by the introduction of a Riemann metric dependent on a quadratic differential form (1) which makes the process of taking higher differentials definite. A clear view of the Riemann geometry from this standpoint was first obtained by means of Levi-Civita's conception of infinitesimal parallelism. This amounted to finding a geometrical interpretation for the Christoffel symbols

$$\begin{Bmatrix} ij \\ k \end{Bmatrix}$$

as the coefficients of a transformation which carries the components of a vector at a point x into the components of a uniquely determined vector at any point $x+dx$. It provides a definite way to visualize the limiting process by which we carry over the conceptions of euclidean geometry in a finite domain to the Riemann geometry in an infinitesimal domain.

Levi-Civita's construction was obtained by regarding the Riemann manifold as immersed in a euclidean space. The euclidean space is not essential and Weyl defined infinitesimal parallelism as an intrinsic property of the manifold itself. He also generalized it so that

$$V^i = \Gamma^i_{\alpha\beta} V^\alpha dx^\beta$$

represents an arbitrary affine transformation from x to $x+dx$. He thus arrived at the conception of a general affine connection determined by arbitrary functions $\Gamma^i_{\alpha\beta}$ which are symmetric in the lower indices.

A further step in generalization was taken by Cartan. Let us imagine that with each point x of the manifold which we are talking about there is associated a projective space S, one point of the projective space being identical with x itself. Then if we can specify a definite projective transformation from S to the projective space associated with each point $x+dx$, and do so by means of a set of functions of x, we have a projective connection in the sense of Cartan.

All these methods of connecting up the differentials at one point with the differentials at the points in its neighborhood can be seen very clearly in terms of the geometry of paths. By a geometry of paths I mean the theory of any system of analytic curves in an n-dimensional manifold, the curves being such that any point is joined to any other point in a sufficiently near neighborhood of itself by one and only one curve of the system. Such a system of curves is the natural generalization of the straight lines of euclidean geometry and of the geodesics of Riemann geometry.

In a very general case the paths can be represented as solutions

$$x^i = f^i(s) \tag{5}$$

of a set of differential equations,

$$\frac{d^2x^i}{ds^2} = \Gamma^i\left(x, \frac{dx}{ds}\right) \tag{6}$$

in which the right hand members are functions of the x's and their first derivatives. The invariant theory of these differential equations under transformations of the coordinates x is the affine geometry of paths. This geometry can be developed in some detail in case the functions Γ^i are homogeneous of degree 2 in dx/ds. Its most powerful method consists in transforming (6) to normal coordinates, that is, to coordinates such that

$$y^i = \xi^i s, \tag{7}$$

with ξ^i constant, represent the paths through the origin. Whenever the coordinates x undergo an arbitrary analytic transformation the normal coordinates receive merely a linear transformation with constant coefficients.

If we define a system of straight lines by means of linear equations with respect to normal coordinates, we obtain an associated euclidean space in which the straight lines through the origin coincide with the paths (7) through it. Since an arbitrary transformation of the coordinates x brings about a linear transformation of the normal coordinates it brings about an affine transformation of the associated euclidean space.

The choice of the differential equations (6) to represent the paths has determined the type of the parameter representation (5) to a certain extent and has also introduced a relation between the parameters representing different paths through the same point. If we subject the parameter in (5) to an arbitrary transformation the equations of the paths will, in general, no longer satisfy differential equations of the form (6). But the differential equations of the paths can be written in a form,

$$(8)\qquad \frac{\dfrac{d^2x^i}{ds^2}-\Gamma^i\left(x,\dfrac{dx}{ds}\right)}{\dfrac{dx^i}{ds}}=\frac{\dfrac{d^2x^j}{ds^2}-\Gamma^j\left(x,\dfrac{dx}{ds}\right)}{\dfrac{dx^j}{ds}}$$

which is unaltered by transformations of the parameter s.

A transformation of the parameter brings about, in the euclidean space associated with each point by means of the normal coordinates, a transformation which is not in general affine. It thus requires the introduction of points at infinity in each of these euclidean spaces. In certain cases the adjunction of the points at infinity will convert the euclidean spaces into projective spaces.

If we specialize the functions Γ^i so that (6) takes the form

$$(9)\qquad \frac{d^2x^i}{ds^2}+\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0$$

where $\Gamma^i_{\alpha\beta}$ are functions of x, we have the sort of affine geometry of paths which has been studied by Eisenhart, Thomas, and myself and which is equivalent to the affine geometry of Weyl. For infinitesimal parallelism is simply invariance of the components of a vector in the neighborhood of the origin of normal coordinates and covariant differentiation is merely differentiation with respect to normal coordinates at the origin. Indeed, differentiation of any order with respect to these coordinates at the origin is an invariant operation which enables us to find a complete set of invariants for the affine geometry.

When the parameter s in the equations of a path satisfying (9) undergoes a transformation which is not linear, the functions $f^i(s)$ in the equations of the path (5) are transformed into functions which are solutions of equations

$$\frac{d^2x^i}{ds^2}+\Lambda^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0,$$

in which

$$\Lambda^i_{\alpha\beta}=\Gamma^i_{\alpha\beta}+\delta^i_\alpha\varphi_\beta+\delta^i_\beta\varphi_\alpha,$$

the functions φ_α being components of a vector, and δ^i_j the Kronecker delta. In a very general case it is found that the normal coordinates undergo a linear fractional transformation when the functions Γ are changed in this manner into the functions Λ. We thus find that a set of homogeneous coordinates whose ratios are equal to normal coordinates for the functions containing the indeterminate vector φ are suitable for the determination of projective invariants. These projective invariants can be said to give genuine properties of the paths since they are independent both of the choice of the coordinates x and of the particular parameter representation of the paths.

Certain choices of the vector φ bring about particular transformations of the parameters s in the equations (5) which amount to non-affine projective transformations of the euclidean space mentioned above. By adjoining points at infinity to this space we obtain a projective space which is essentially the same as that considered by Cartan. The projective transformations determined by φ bring about a class of transformations of the parameters of all paths through the origin of normal coordinates. It is an interesting fact that any transformations of the parameter on any path may be obtained in this way.

The successive specializations which we have been introducing in the specification of the system of paths can all be regarded as closer and closer restrictions on the manner in which the paths are simultaneously given a parameter representation. This process can be continued by assuming that the equations (9) possess a first integral, say a homogeneous quadratic first integral

$$g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = \text{constant.}$$

This enables us to define the parameter along any path by means of the definite integral

$$\int\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\,dt$$

and gives the parameter many of the characteristic properties of distance. Finally by requiring that the path actually minimize this integral we specialize down to a Riemann geometry.

In thus shadowing forth something like a set of axioms for the various types of path geometry, I have not given the calculus of variations its due. The problem of determining those systems of paths which can be regarded as extremals of a definite integral is one whose solution will doubtless modify our views on the foundations of differential geometry.

3. *Choice of Undefined Terms.* In studying the Riemann geometry and its generalizations we make use of the euclidean geometry at every turn. Hence our interest in these subjects by no means kills our interest in the foundations of euclidean geometry. But these studies and the revival of the physical point of view necessarily have an influence on our preference among the many ways of formulating the axioms.

The Riemann geometry and its generalizations all start with the assumption that the set of points with which they deal is a portion of an n-dimensional manifold, or an n-cell in the sense of analysis situs. This assumption may be formulated by asserting that there exists a way of labelling the points by means of ordered sets of n numbers

$$(x^1, x^2, \cdots, x^n),$$

one set of numbers for each point and one point for each set of numbers. This is equivalent to assuming that an n-cell is a set of points capable of supporting a euclidean geometry. For if there is a coordinate system there is a system of curves which are linear with respect to it, and which function as the straight lines of a euclidean space; and if the points are in correspondence with a euclidean space, a coordinate system can be set up. Thus any set of postulates for a euclidean space gives rise to a set of postulates for an n-cell.

Of course, a set of postulates for any particular one of the generalized geometries which we are considering would characterize an n-cell just as well as the postulates for euclidean geometry. Any two of these geometries are in a relation which may be stated as follows: If A be an arbitrary point of the space described by one n-dimensional geometry, A is surrounded by an n-dimensional region R_A in which the points are uniquely denoted by coordinates x and in which the relations of the geometry (such as the formula for distance or the equations of the paths) are expressed by analytic functions in terms of these coordinates. If B be an arbitrary point of the space described by any other n-dimensional geometry there is a region R_B having analogous properties for this geometry, and such that between the coordinates y of R_B and the coordinates x of R_A there is an analytic relation

$$y^i = f^i(x)$$

with a unique inverse.

Thus in a certain sense the two geometries are analytically related. It is not that there is an analytic transformation of the one geometry into the other. But the fundamental loci of each geometry determine a class of coordinate systems in terms of which these loci are analytically representable, and there exists an analytic transformation of any coordinate system of the one geometry into any coordinate system of the other.

One of these geometries is that of Euclid. Hence a perfectly sound way of stating the postulates of any of the geometries we are considering is first to give a set of postulates for euclidean geometry and then to postulate that the geometry which we are describing is analytically related to that of Euclid.

It seems clear to me that when they are to be used for the purpose of characterizing an n-cell we can definitely prefer certain sets of postulates of elementary geometry to others. The purpose is to characterize a set of points; therefore there should be as few other elements as possible.

For example, a set of postulates in terms of points, lines, planes and congruence is less desirable than one in terms of points and lines only. Moreover a set of postulates about points and lines in which the lines are regarded as sets of points is to be preferred to one in which the lines are objects of some other sort to which the points are joined by means of an undefined relation.

Between a set of postulates in terms of points and lines and one in terms of points and the betweenness relation, there would doubtless be little to choose. In either case, space is characterized as a class of points which are arranged in sub-classes related among themselves in a particular way.

The fact that physical discussions seem always to treat the line and plane as sets of points would seem to confirm this preference. Also the fact that in its direct application to nature the euclidean geometry is used to give an approximate description of a portion of the universe would seem to establish a presumption in favor of axioms which are stated for only a finite region of space.

I do not mean to suggest that we should try to reduce the number of undefined terms to a minimum. For in a mathematical science which has physical applications there is much to be said for making an undefined term out of each object or relation which has an independent physical definition.

A physical definition is made in terms of human operations. Thus, for example, to define what a point is physically we have to describe a series of acts which another person must perform in order to locate a point. Or in order to say what we mean by the distance between two points we must describe the operation of making the measurements necessary to determine the number we call the distance.

This description is in general very complicated. For example, in the case of distance it would involve nearly all the technique of practical astronomy. Moreover the definition obtained is in general only approximate. For example, we never have an absolutely complete determination of what we mean by a point.

The physical definition of a point is notoriously without effect on the mathematical theory which follows. So far as the mathematical operations are concerned the point is purely and simply an undefined term. The abstract mathematical theory has an independent, if lonely, existence of its own. But when a sufficient number of its terms are given physical definitions, it becomes a part of a vital organism concerning itself at every instant with matters full of human significance. Every theorem can then be given the form "if you do so and so, then such and such will happen".

The places at which this life-blood of human meaning flows into the mathematical theory should, it seems to me, be the undefined terms. The process of making the physical application of the mathematical discipline will then consist simply in specifying the human operations by which we give meaning to the mathematically undefined terms. Each of the undefined terms will separate off a group of theorems in which the term appears, and these theorems will refer particularly to the set of physical operations involved in defining this term.

Thus in particular the question whether the plane as well as the line shall be an undefined term depends on how the plane is thought of physically. When solid objects and flat surfaces were fundamental physical objects, it was natural to think of the plane as a fundamental entity, somehow distinct from the set of points on it. But with particles and their paths playing the role which they do in modern physics it would seem more natural to regard lines and planes as sets of points.

I am not pretending to exhaust this question about the choice of undefined terms—only to indicate one respect in which the physical application seems to indicate a preference. In passing, however, I should like to refer to a remark by the philosopher Geiger* who has made up postulates in terms of points, lines, planes and 3-spaces, of which last he postulates that there exists only one. Geiger's remark is that these are not the only "Elementargebilde", but that circles, parabolas, spheres, ellipsoids, and so on, also have a right to be considered. So "vom konsequenten Standpunkt aus" these should also appear as undefined terms. But for obvious reasons this program has not yet been carried out.

4. *Postulates for Analysis Situs.* The characterizations of an n-cell in terms of coordinates or of its correspondence with euclidean space have the disadvantage that they make use of ideas which are not invariant under the group of continuous transformations which we study in analysis situs. The undefined terms which we should expect to use are point and region, as R. L. Moore has already shown is possible for the two-dimensional case, or point and limit point.

To carry out the latter program successfully one should be able to start with the most general type of abstract set in which there is a definition of limit point, and then by successively excluding various types of sets, narrow down to the n-cell. This ought not to be very far out of the range of possibilities at the present time.

Another question about the foundations of analysis situs which deserves to be studied, and which can, I think, be studied with some hope of success, is the problem of a combinatorial set of axioms. This question has been before us with increasing clearness for some time. The researches of Poincaré made it evident that a large class of the fundamental problems of analysis situs are combinatorial in character. A combinatorial set of postulates was sketched in the Dehn-Heegaard article in the Encyclopädie. My Cambridge Colloquium lectures contain a treatment in which combinatorial elements are separated out, but in which it is necessary to use the definition of an n-cell by means of correspondence with an ordinary geometric figure in order to obtain the proofs of invariance of the combinatorial constants. More recently Weyl has given a treatment from which he has excluded everything which he cannot obtain by combinatorial processes and into which the notions of "circuit" and of "partition" are brought by means of axioms instead of definitions.

---

* M. Geiger, *Systematische Axiomatik der Euklidischen Geometrie*, Augsburg, 1924.

There are several ways in which we can formulate the problem of a combinatorial analysis situs attack on the foundations of geometry. One of them would be this: Our undefined terms are 0-cells, 1-cells, 2-cells, and so on up to n-cells. We have relations of incidence between 0-cells and 1-cells, between 1-cells and 2-cells, and so on. We can in the three-dimensional case—but not yet in the four-dimensional one—state in abstract terms what conditions these incidence relations must satisfy in order that the complex shall be a manifold. Progress is being made —notably by Alexander—toward finding out what further conditions must be satisfied in order that this manifold shall be of a particular type, say a sphere or a 3-cell with its boundary. When we have this result we should be able to combine with the postulates so obtained further assumptions about the subdivision of the cells into complexes of cells and the combination of sets of cells to form single cells. From these we should be able to get the theorems of invariance and thus be able to prove all the theorems of analysis situs which can be formulated in terms of cells.

Such a treatment would put in evidence the properties of space which have to do with continuous deformations and indefinite subdivision, without carrying the subdivision to any limit. We should thus remain within the range of finite sequences of operations and avoid the difficulties which beset the theory of classes of large cardinal number. My impression is that Weyl has referred to this advantage of the combinatorial attack on the foundations of geometry in a letter to me. The letter is unfortunately not at hand. In any case, it cannot be doubted that he has considered the question in connection with the doubts which he and Brouwer entertain as to the validity of the ordinary theory of the continuum.

The matter is also of interest from the physical point of view in that a theory of this sort could doubtless be worked out in terms of n-cells as the sole undefined objects. It would give a kind of n-dimensional continuum without necessarily implying the existence of any points. Some twenty years ago I had a similar project in mind, that of stating the foundations of geometry in terms of overlapping 3-cells or "chunks" of space, and discussed it a great deal with R. L. Moore without getting very far. Since then Huntington and Whitehead have given sets of postulates using solid portions of space as undefined elements. But they both have let the point in as a limiting case, whereas the true solution of the problem I have in mind would be a set of postulates which would lead to the main physical properties of space without providing a limiting process to define points. It would correspond to the true state of our physical knowledge of space, which is vague both for the infinitely large and also for the infinitely small.

5. *The Arithmetical Point of View.* While such an enterprise as that of which we have just been speaking is inspired by physical considerations, it would probably not be of any immediate interest to physicists. For the everyday uses of physics all the methods of approaching the Riemann geometry or the foundations of analysis situs must give way to the straight-forward statement that the points we mean to talk about are objects capable of representation by coordinates. Indeed, the definition of mathematics which will be most readily accepted by physicists is that which has been so cogently set forth

by Study* in various recent publications, that "mathematics includes computation with natural numbers and everything that can be founded upon it, but nothing else". This is a more inclusive account of mathematics than that given by Kronecker because Study allows the Dedekind cut process in building upward from the integers.

Whether we accept the arithmetical definition of mathematics or not, we have to admit that it gives us one way at least of treating the material in each of the branches of mathematics which we now regard as important. Moreover it puts us in touch at once with an indispensable process in each of these branches. Therefore from the point of view of teaching mathematics I think that the arithmetical method is bound to prevail. Most of the people who set out to learn mathematics do so with a view to some use which they are going to make of it. The arithmetical method will give them a maximum of mathematical information and power in a minimum of time. Everything that is irrelevant to this will have to be discarded.

I mean this statement to be as sweeping as possible. The treatment of elementary geometry modelled on that of Euclid is already being discarded. It will be replaced, not by a modern equivalent which stands apart as a logical structure whose chief methods are peculiar to itself, but by a properly developed analytic geometry. Likewise the synthetic treatment of projective geometry, however seductive it may be to some of us individually, will have to be left to one side as a study for specialists and as a theory which may be expounded to students at the stage when it is desirable for them to see what a well-rounded logical structure is like.

But the arithmetical method will be used for the primary exposition of all those theorems which we encounter in the main stream of mathematics. This will happen not because mathematics has been thus defined by any person however brilliant, but simply because life is too short for people who are not primarily mathematicians to learn mathematics otherwise.

---

* *Die Realistische Weltansicht und die Lehre vom Raum*, Braunschweig, 1914, and *Mathematik und Physik*, Braunschweig, 1923. Our quotation is from the latter book.

I am tempted to add a remark that will not be so easily accepted by some of my friends who are devoted to analysis. Namely that much of the present theory of functions of one complex variable (which is really a conformal geometry incapable of generalization) will suffer the same fate as synthetic geometry. Its highly elegant but highly introversive methods will be left to one side for the specialist. The main current of mathematics will flow by, carrying with it only the more important facts of this theory; and these facts will be formulated as we are now learning to formulate them with the aid of analysis situs, in terms of the theory of functions of n real variables.

But while the arithmetical definition of mathematics specifies the actual content of present day mathematics and also corresponds to the most practicable way of expounding the subject to those who use it, it hardly seems adequate when we examine its foundations. If it is true that the basic ideas of geometry, mechanics, thermodynamics, etc., can all be defined in terms of real numbers, and real numbers in terms of natural numbers, it is also true that the natural numbers are mixed up with the basic ideas of formal logic in the most sticky fashion. Therefore the whole arithmetical structure is a portion of the edifice of formal logic which we are not as yet able to separate off. So, for example, the *Principia Mathematica* of Whitehead and Russell begins with the logic of propositions and goes far with the logic of classes before the natural numbers fully emerge. After that their treatment amounts essentially to the process of arithmetization.

But even after we are fully embarked on the arithmetical stage of the theory we are still beset by difficulties as to what we mean by a class of objects and what are the allowable reasoning processes which can be applied to classes. We can neither obtain our definition of irrational number nor prove all the theorems desired, without reasoning about classes which are very different from those accessible to the simple experiments upon which the elementary logic is based. I mean experiments in counting and in identifying the same object at different times. The theory of infinite classes is obviously incomplete and beset with difficulties and paradoxes which have baffled the many excellent mathematicians who have attempted to set it in order. The latest and, I think, the most promising attempt to make it comprehensible is that of Hilbert who seems to propose that we apply our elementary logic to the symbols which are used in writing out the statements made in the logic required for a general theory of classes. It thus becomes a feasible problem to state what sorts of combinations of symbols give self-contradictory statements and then to study the possible combinations of symbols with a view to finding out by means of our elementary logic whether the postulates of the higher logic give rise to self-contradictory statements.

Whether or not this is a correct statement of Hilbert's program, the conclusion seems inescapable that formal logic has to be taken over by mathematicians. The fact is that there does not exist an adequate logic at the present time, and unless the mathematicians create one, no one else is likely to do so. In the process of constructing it we are likely to adopt the Russell view that mathematics is coextensive with formal logic. This, I suppose, is apt to happen whether we adopt the formalist or the intuitionalist point of view.

PRINCETON UNIVERSITY

RICCI'S COEFFICIENTS OF ROTATION*

BY HARRY LEVY

In a Riemann space whose linear element is given by the positive definite quadratic form

$$(1) \qquad ds^2 = g_{ij} dx^i dx^j,$$

let us take n congruences of curves, defined by the equations

$$(2) \qquad \frac{dx^1}{\lambda_h|^1} = \frac{dx^2}{\lambda_h|^2} = \cdots = \frac{dx^n}{\lambda_h|^n} \quad (h = 1, 2, \cdots, n).$$

Let us denote, as usual, the covariant components of congruence $\lambda_{h|}$ by $\lambda_{h|r}$, so that

$$(3) \qquad \lambda_{h|r} = g_{rt} \lambda_h|^t.$$

We assume, further, that these n congruences are mutually orthogonal. We may then write

$$(4) \qquad g_{rt} \lambda_h|^r \lambda_k|^t = \delta_{hk},$$

where δ_{hk} is Kronecker's delta, that is,

$$(5) \qquad \delta_{hk} = \begin{cases} 0, & h \neq k, \\ 1, & h \neq k. \end{cases}$$

Ricci† has found a set of invariants which he calls *coefficients of rotation* which are very important in the physical and geometric applications. They are defined by the equations

$$(6) \qquad \gamma_{hij} = \lambda_{h|r,t} \lambda_i|^r \lambda_j|^t,$$

where $\lambda_{h|r,t}$ is the covariant derivative of $\lambda_{h|r}$,

$$(7) \qquad \lambda_{h|r,t} = \frac{\partial \lambda_{h|r}}{\partial x^t} - \lambda_{h|p} \Gamma_{rt}^{\,p},$$

---

* Presented to the Society, May 3, 1924.

† *Dei sistemi di congruenze ortogonali in una varietà qualunque*, MEMORIE DELLA R. ACCAD. DEI LINCEI, CLASSE DEI SCIENZE (5), vol. 2 (1896). See also *Méthodes de calcul différentiel absolu*, by Ricci and Levi-Civita, MATHEMATISCHE ANNALEN vol. 54 (1901), p. 147 ff.

where Γ_{rt}^{p} is the Christoffel symbol of the second kind of the fundamental form (1).

Lipka* has obtained the geometric interpretation of these functions γ_{hij} and the conditions that a congruence of curves be parallel in the sense of Levi-Civita.† In this paper I shall obtain Lipka's results in another way, and I shall also prove a new theorem.

Take an arbitrary curve, C, of congruence $\lambda_{j|}$ and fix on it some point P. Let $\mu_i|^r$ for $r = 1, 2, \cdots, n$ be the contravariant components of the vector which is parallel to itself along C and which at P coincides with the tangent vector to the curve of congruence $\lambda_{i|}$. Then, by the definition of parallelism, we have

$$\mu_i|^r{}_{,t}\lambda_j|^t = 0 \tag{8}$$

along C, where $\mu_i|^r{}_{,t}$ is the covariant derivative of $\mu_i|^r$. Let ω_{hi} be the angle, at the points of C, between the vectors $\lambda_{h|}$ and $\mu_{i|}$. Then

$$\cos\omega_{hi} = \lambda_{h|r}\mu_i|^r. \tag{9}$$

Differentiating covariantly with respect to x^t, we find

$$\frac{\partial \cos\omega_{hi}}{\partial x^t} = \lambda_{h|r,t}\mu_i|^r + \lambda_{h|r}\mu_i|^r{}_{,t}.$$

Multiplying by $\lambda_j|^t$, summing on t, and taking into account equations (8), we obtain

$$\frac{\partial \cos\omega_{hi}}{\partial s_j} = \lambda_{h|r,t}\mu_i|^r\lambda_j|^t, \tag{10}$$

where s_j is the arc of C. From (6) and from the definition of $\mu_{i|}$ it follows that at P we have

$$\left.\frac{\partial \cos\omega_{hi}}{\partial s_j}\right|_P = \gamma_{hij}.$$

---

[*] *On Ricci's coefficients of rotation*, JOURNAL OF MATHEMATICS AND PHYSICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, vol. 3, 1924.

[†] *Nozione di parallelismo in una varieta qualunque*, RENDICONTI DI PALERMO, vol. 42 (1917). Also Bianchi's paper in RENDICONTI DI NAPOLI, (3a), vol. 27 (1922).

Hence γ_{hij} *is the rate of change along a curve of congruence* $\lambda_{j|}$ *of the cosine of the angle between the curves of* $\lambda_{h|}$ *and the parallels, with respect to* $\lambda_{j|}$, *to the curves of* $\lambda_{i|}$.

Since $\gamma_{hij} = -\gamma_{ihj}$,* the angle between the curves of $\lambda_{h|}$ and the parallels to $\lambda_{i|}$ changes at the same rate as the angle between the curves of $\lambda_{i|}$ and the parallels to $\lambda_{h|}$. In a euclidean space $\partial(\cos\omega_{hi})/\partial s_j$ is exactly what we would call the rotation of the congruence $\lambda_{i|}$ about $\lambda_{h|}$.

Suppose congruence $\lambda_{i|}$ forms a system of parallels with respect to congruence $\lambda_{j|}$, that is, those curves of $\lambda_{i|}$ which intersect one and the same curve of $\lambda_{j|}$ are parallel with respect to that curve, and this holds along all curves of $\lambda_{j|}$; then, $\mu_{i|}$ coincides with $\lambda_{i|}$, and

$$\cos\omega_{hi} = \delta_{hi}, \qquad (h = 1, 2, \cdots, n).$$

Hence

$$(11) \qquad \gamma_{hij} = 0, \qquad (h = 1, 2, \cdots, n),$$

is the necessary condition that the congruence $\lambda_{i|}$ forms a system of parallels with respect to $\lambda_{j|}$. This is also sufficient. For from equations (6) we have

$$\lambda_{h|r,t} = \sum_{i,j=1}^{n} \gamma_{hij}\lambda_{i|r}\lambda_{j|t}.$$

By virtue of (3) and (4), (10) become

$$\frac{\partial\cos\omega_{ih}}{\partial s_j} = \sum_{l=1}^{n} \gamma_{ilj}\lambda_{l|r}\mu_h|^r = -\sum_{l=1}^{n} \gamma_{lij}\lambda_{l|r}\mu_h|^r.$$

If (11) are satisfied, we have

$$\frac{\partial\cos\omega_{ih}}{\partial s_j} = 0, \qquad (h = 1, 2, \cdots, n).$$

Hence ω_{ii} is constant, and therefore $\lambda_{i|}$ is parallel along $\lambda_{j|}$.

The author has shown† that the necessary and sufficient conditions that a congruence $\lambda_{k|}$ have a family of m-dimensional hypersurfaces as orthogonal trajectories is that

---

* Ricci and Levi-Civita, loc. cit., p. 148.

† *Normal congruences of curves*, this BULLETIN, vol. 31, p. 39.

$$\gamma_{hij} = \gamma_{jih} \tag{12}$$

where i takes on a definite set of $n-m$ values of the integers $1, 2, \cdots, n$ including the value k, and h and j take on those m values that i cannot assume. If each of $n-m$ congruences forms a system of parallels with respect to every one of the remaining m congruences equations (11) are satisfied for $h = 1, 2, 3, \cdots, n$; $i = i_1, i_2, \cdots, i_{n-m}$; $j = j_1, \cdots, j_m$; $i \neq j$, (12) is surely satisfied, and we have the following theorem.

THEOREM. *If each of $n-m$ congruences forms a system of parallels with respect to every one of the remaining m congruences, then the former have a family of m-dimensional hypersurfaces as orthogonal trajectories.* When $m = n-1$ this reduces to one of Lipka's theorems.

PRINCETON UNIVERSITY

SUR LES VALEURS ASYMPTOTIQUES DES COEFFICIENTS DE COTES

PAR J. OUSPENSKY

1. Parmi les formules de quadratures pour le calcul approché des intégrales définies la plus simple est, sans contredit, celle de Cotes, qui correspond à la division de l'intervalle d'intégration en parties égales. Supposons l'intervalle d'intégration $(0, 1)$ subdivisé en n parties égales; alors on peut déterminer $n+1$ constantes $A_0, A_1, A_2, \cdots, A_n$, nommées "coefficients de Cotes", de manière que la formule

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f\left(\frac{1}{n}\right) + A_2 f\left(\frac{2}{n}\right) + \cdots + A_n f(1)$$

soit exacte pour toute fonction $f(x)$ se réduisant à un polynôme d'un degré n'excédant pas $n-1$. Dans d'autres cas cette "formule de Cotes" n'est qu'approchée. Comme le degré d'approximation fourni par elle depend des valeurs numériques des coefficients $A_0, A_1, A_2, \cdots, A_n$, la question sur les valeurs asymptotiques de ces coefficients pour n très grand n'est pas, ce me semble, dénuée de quelque interêt.

2. En posant

$$x_0 = 0, \quad x_1 = \frac{1}{n}, \quad x_2 = \frac{2}{n}, \quad \cdots, \quad x_n = 1,$$

et

$$\varphi(x) = x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\cdots\left(x-\frac{n}{n}\right),$$

on a l'expression suivante pour le coefficient A_k, k étant un des nombres $0, 1, 2, \cdots, n$:

$$A_k = \int_0^1 \frac{\varphi(x)\,dx}{(x-x_k)\varphi'(x_k)}.$$

Mais comme l'on sait que

$$\varphi'(x_k) = \frac{k}{n}\cdot\frac{k-1}{n}\cdots\frac{1}{n}\cdot\frac{-1}{n}\cdot\frac{-2}{n}\cdots\frac{-(n-k)}{n}$$
$$= (-1)^{n-k}\frac{\Gamma(k+1)\,\Gamma(n-k+1)}{n^n},$$

on trouve après quelques réductions faciles:

$$A_k = \frac{(-1)^{n-k}}{n\Gamma(k+1)\Gamma(n-k+1)}\int_0^n \frac{x(x-1)\cdots(x-n)}{x-k}dx.$$

Ainsi la question se réduit à trouver l'expression asymptotique pour n très grand de l'intégrale:

$$(1) \qquad J_n = \int_0^n \frac{x(x-1)\cdots(x-n)}{x-k}dx,$$

k étant un nombre quelconque de la suite $0, 1, 2, \cdots, n$. En faisant usage des formules bien connues

$$x(x-1)\cdots(x-n) = (-1)^n\frac{\Gamma(x+1)\,\Gamma(n+1-x)\sin \pi x}{\pi},$$

$$\frac{\Gamma(x+1)\,\Gamma(n+1-x)}{\Gamma(n+2)} = \int_0^1 \xi^x(1-\xi)^{n-x}d\xi,$$

nous allons présenter d'abord J_n sous la forme d'une intégrale double:

$$J_n = \frac{(-1)^n \Gamma(n+2)}{\pi} \int_0^1 (1-\xi)^n d\xi \int_0^n e^{x\log\frac{\xi}{1-\xi}} \frac{\sin \pi x}{x-k} dx.$$

Ensuite, en utilisant la formule évidente

$$\int_0^n e^{x\log\frac{\xi}{1-\xi}} \frac{\sin \pi x}{x-k} dx = (-1)^k \left(\frac{\xi}{1-\xi}\right)^k \times\left[\int_0^{n-k} e^{x\log\frac{\xi}{1-\xi}} \frac{\sin \pi x}{x} dx + \int_0^k e^{x\log\frac{\xi}{1-\xi}} \frac{\sin \pi x}{x} dx\right];$$

et en transformant les intégrales du second membre à l'aide de la formule

$$\int_0^h e^{\alpha x} \frac{\sin \pi x}{x} dx = \frac{\pi}{2} + \arctan\frac{\alpha}{\pi} + \pi(-1)^{h-1} \int_{-\infty}^{\alpha} \frac{e^{hx}\,dx}{\pi^2+x^2},$$

que l'on établit aisément en supposant h un entier quelconque, on parvient à cette expression définitive de J_n:

$$J_n = (-1)^{n-k}\Gamma(k+1)\,\Gamma(n-k+1) + \Gamma(n+2) \times\left[(-1)^{n-1}\int_0^1 \xi^k(1-\xi)^{n-k} d\xi \int_{-\infty}^{\log\frac{1-\xi}{\xi}} \frac{e^{kx}\,dx}{\pi^2+x^2} - \int_0^1 \xi^k(1-\xi)^{n-k} d\xi \int_{-\infty}^{\log\frac{1-\xi}{\xi}} \frac{e^{(n-k)x}\,dx}{\pi^2+x^2}\right].$$

Nous allons faire usage de cette formule en supposant d'abord k différent de 0 et n. On voit de suite qu'en posant

$$(2)\quad \begin{cases} P_1 = \displaystyle\int_0^{\frac{1}{2}} \xi^k(1-\xi)^{n-k} d\xi \int_{-\infty}^{\log\frac{1-\xi}{\xi}} \frac{e^{kx}\,dx}{\pi^2+x^2}; \\ P_2 = \displaystyle\int_0^{\frac{1}{2}} \xi^k(1-\xi)^{n-k} d\xi \int_{-\infty}^{\log\frac{1-\xi}{\xi}} \frac{e^{(n-k)x}}{\pi^2+x^2} dx; \\ P_3 = \displaystyle\int_0^{\frac{1}{2}} \xi^k(1-\xi)^{n-k} d\xi \int_{-\infty}^{\log\frac{\xi}{1-\xi}} \frac{e^{kx}\,dx}{\pi^2+x^2}; \\ P_4 = \displaystyle\int_0^{\frac{1}{2}} \xi^k(1-\xi)^{n-k} d\xi \int_{-\infty}^{\log\frac{\xi}{1-\xi}} \frac{e^{(n-k)x}\,dx}{\pi^2+x^2} \end{cases}$$

on a

$$(3)\qquad J_n = (-1)^{n-k}\Gamma(k+1)\Gamma(n-k+1)+\Gamma(n+2)$$
$$\times[(-1)^{n-1}P_1-P_2+(-1)^{n-1}P_3-P_4].$$

Mais comme

$$\int_{-\infty}^{\log\frac{\xi}{1-\xi}}\frac{e^{kx}\,dx}{\pi^2+x^2}<\int_{-\infty}^{\log\frac{\xi}{1-\xi}}e^{kx}\,dx=\frac{\xi^k}{k(1-\xi)^k},$$

$$\int_{-\infty}^{\log\frac{\xi}{1-\xi}}\frac{e^{(n-k)x}\,dx}{\pi^2+x^2}<\frac{\xi^{n-k}}{(n-k)(1-\xi)^{n-k}},$$

on voit immédiatement que

$$0<P_3<\frac{1}{k}\int_0^{\frac{1}{2}}\xi^n\,d\xi=\frac{1}{k(n+1)2^{n+1}};$$

$$0<P_4<\frac{1}{n-k}\int_0^{\frac{1}{2}}\xi^n\,d\xi=\frac{1}{(n-k)(n+1)2^{n+1}},$$

ce qui permet de présenter la formule (3) de cette manière:

$$(4)\qquad J_n = (-1)^{n-k}\Gamma(k+1)\Gamma(n+1-k)+\Gamma(n+2)$$
$$\times\left[(-1)^{n-1}P_1-P_2+\frac{\Theta}{k(n-k)2^{n+1}}\right],\quad -1<\Theta<1.$$

Toute la question se réduit ainsi à trouver les expressions asymptotiques de P_1 et P_2, ce qui présente un peu plus de difficulté.

3. La solution de cette question dépend en grande partie de la recherche de la valeur asymptotique de l'intégrale

$$K_n=\int_0^h\frac{(1-x)^n\,dx}{\left(\log\frac{1-x}{x}\right)^s}$$

pour n indéfiniment croissant, tandis que h est un nombre fixe, satisfaisant aux inégalités

$$0<h<\frac{1}{2},$$

et s un nombre entier positif donné. Considérons l'intégrale un peu plus générale

$$K_n(\alpha) = \int_0^h \frac{\left(\frac{1-x}{x}\right)^\alpha (1-x)^n}{\left(\log\frac{1-x}{x}\right)^s} dx, \tag{5}$$

qui dépend d'un paramètre α et se réduit à K_n pour $\alpha = 0$. Nous allons supposer dans la suite que

$$-\frac{1}{2} \leqq \alpha \leqq 0.$$

L'inégalité presque évidente

$$0 < \int_0^h \frac{(1-x)^{n-\frac{1}{2}}x^{\frac{1}{2}}dx}{\left(\log\frac{1-x}{x}\right)^k} < \frac{1}{\left(\log\frac{1-h}{h}\right)^k} \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(n+2)},$$

et la formule asymptotique bien connue

$$\lim \frac{\Gamma(n+\beta)}{n^\beta \Gamma(n)} = 1, \text{ pour } n = \infty,$$

font voir que pour chaque valeur fixe de k, l'intégrale

$$\int_0^h \frac{(1-x)^{n-\frac{1}{2}}x^{\frac{1}{2}}dx}{\left(\log\frac{1-x}{x}\right)^k}$$

est de l'ordre $1/n^{3/2}$, c'est à dire qu'il existe une constante L telle que l'intégrale précédente soit $< L/n^{3/2}$.

Il suit immédiatement de cette remarque que les valeurs que prennent K_n (α) et ses dérivées pour $\alpha = -\frac{1}{2}$ se réduisent aux quantités d'ordre $O(1/n)^{3/2}$. Ensuite la dérivée d'ordre s de K_n (α) se present sous la forme

$$\begin{aligned} \frac{d^sK_n(\alpha)}{d\alpha^s} &= \int_0^h (1-x)^{n+\alpha}x^{-\alpha}dx \\ &= \frac{\Gamma(n+\alpha+1)\Gamma(1-\alpha)}{\Gamma(n+2)} - \int_h^1 (1-x)^{n+\alpha}x^{-\alpha}dx. \end{aligned} \tag{6}$$

L'intégrale qui figure dans le second membre de cette formule est positive et plus petite que $(1-h)^{n-1/2}$; de l'autre côté les formules asymptotiques bien connues font voir que la quantité

$$\frac{\Gamma(n+\alpha+1)\,\Gamma(1-\alpha)}{\Gamma(n+2)}$$

peut être présentée ainsi

$$\frac{\Gamma(1-\alpha)}{n^{1-\alpha}}+\frac{\varphi(n,\alpha)}{n^{2-\alpha}},$$

$\varphi(n,\alpha)$ étant une fonction bornée de n et de α pour toutes les valeurs de n et pour toutes les valeurs de α comprises dans l'intervalle $(-\frac{1}{2}, 0)$. Il suit de ces remarques qu'on peut poser

$$\frac{d^s K_n(\alpha)}{d\alpha^s}=\frac{\Gamma(1-\alpha)}{n^{1-\alpha}}+\frac{\psi(n,\alpha)}{n^{2-\alpha}},$$

où la fonction $\psi(n,\alpha)$ est de la même nature que la fonction $\varphi(n,\alpha)$ tout à l'heure mentionnée. On a d'après la formule de Taylor

$$K_n(\alpha)=K_n(-\tfrac{1}{2})+(\alpha+\tfrac{1}{2})K_n'(-\tfrac{1}{2})+\cdots$$
$$+(\alpha+\tfrac{1}{2})^{s-1}\frac{K_n^{(s-1)}(-\frac{1}{2})}{(s-1)!}+\int_{-\frac{1}{2}}^{\alpha}\frac{(\alpha-x)^{s-1}}{(s-1)!}K_n^{(s)}(x)\,dx.$$

Par ce qui précède on a uniformément en α

$$K_n(-\tfrac{1}{2})+(\alpha+\tfrac{1}{2})K_n'(-\tfrac{1}{2})+\cdots$$
$$+(\alpha+\tfrac{1}{2})^{s-1}\frac{K_n^{(s-1)}(-\frac{1}{2})}{(s-1)!}=O(1/n^{3/2}).$$

D'autre part en substituant au lieu de $K_n^{(s)}(\alpha)$ son expression (6) on obtient

$$\int_{-\frac{1}{2}}^{\alpha}\frac{(\alpha-x)^{s-1}}{(s-1)!}\frac{d^sK_n(x)}{dx^s}\,dx=\int_{-\frac{1}{2}}^{\alpha}\frac{(\alpha-x)^{s-1}}{(s-1)!}\frac{\Gamma(1-x)}{n^{1-x}}\,dx$$
$$+\int_{-\frac{1}{2}}^{\alpha}\frac{(\alpha-x)^{s-1}}{(s-1)!}\frac{\psi(n,x)}{n^{2-x}}\,dx.$$

La dernière intégrale est manifestement une quantité d'ordre $O(1/n^2)$ indépendamment de α. Quant à la première, l'application répétée de l'intégration par parties conduit à ce résultat

$$\int_{-\frac{1}{2}}^{\alpha}\frac{(\alpha-x)^{s-1}}{(s-1)!}\,\frac{\Gamma(1-x)}{n^{1-x}}\,dx$$
$$=\frac{\Gamma(1-\alpha)}{n^{1-\alpha}(\log n)^{s}}\left[1+\frac{\lambda(n,\alpha)}{\log n}\right]+O(1/n^{3/2}),$$

$\lambda(n,\alpha)$ désignant une fonction bornée de n et de α. Enfin nous arrivons à cette expression asymptotique de $K_n(\alpha)$:

$$K_n(\alpha)=\frac{\Gamma(1-\alpha)}{n^{1-\alpha}(\log n)^{s}}\left[1+\frac{\Lambda(n,\alpha)}{\log n}\right],$$

la fonction $\Lambda(n,\alpha)$ étant de même nature que $\lambda(n,\alpha)$. En posant $\alpha=0$ nous voyons que

$$\int_{0}^{h}\frac{(1-x)^{n}dx}{\left(\log\frac{1-x}{x}\right)^{s}}\sim\frac{1}{n(\log n)^{s}};$$

ou d'une manière plus précise

$$(7)\qquad\int_{0}^{h}\frac{(1-x)^{n}dx}{\left(\log\frac{1-x}{x}\right)^{s}}=\frac{1}{n(\log n)^{s}}\left[1+\frac{\mu_n}{\log n}\right];$$

μ_n désignant une fonction bornée de n.

4. Considérons maintenant l'intégrale

$$\int_{-\infty}^{\omega}\frac{e^{hx}dx}{\pi^{2}+x^{2}},$$

où h est un nombre entier positif et ω un nombre positif quelconque. L'intégration par parties donne

$$(8)\qquad\int_{-\infty}^{\omega}\frac{e^{hx}dx}{\pi^{2}+x^{2}}=\frac{e^{h\omega}}{h(\pi^{2}+\omega^{2})}+\frac{2}{h}\int_{-\infty}^{\omega}\frac{e^{hx}x\,dx}{(\pi^{2}+x^{2})^{2}}.$$

En intégrant par parties de nouveau, nous avons

$$\int_{-\infty}^{\omega}\frac{e^{hx}x\,dx}{(\pi^{2}+x^{2})^{2}}=\frac{e^{h\omega}\omega}{h(\pi^{2}+\omega^{2})^{2}}+\frac{1}{h}\int_{-\infty}^{\omega}\frac{e^{hx}(3x^{2}-\pi^{2})}{(\pi^{2}+x^{2})^{3}}\,dx,$$

mais comme

$$\text{val. abs.} \int_{-\infty}^{\omega} \frac{e^{hx}(3x^2-\pi^2)}{(\pi^2+x^2)^3}\,dx < 3\int_{-\infty}^{\omega} \frac{e^{hx}dx}{(\pi^2+x^2)^2}$$

$$< \frac{3e^{h\omega}\omega}{(\pi^2+\omega^2)^2} + \frac{3}{h\pi^4} < \frac{3e^{h\omega}}{\omega^3} + \frac{3}{h\pi^4} < \frac{6e^{h\omega}}{\omega^3},$$

nous voyons que

$$\text{val. abs.} \int_{-\infty}^{\omega} \frac{e^{hx}x\,dx}{(\pi^2+x^2)^2} < \frac{7e^{h\omega}}{h\omega^3}.$$

Cette inégalité combinée avec la formule (8) nous conduit à cette formule utile

$$\int_{-\infty}^{\omega} \frac{e^{hx}dx}{\pi^2+x^2} = \frac{e^{h\omega}}{h(\pi^2+\omega^2)} + \frac{\mu e^{h\omega}}{h^2\omega^3}, \tag{9}$$

μ désignant une quantité dont la valeur absolue est toujours < 14.

5. Ces préliminaires établis, désignons par ξ_0 une racine de l'équation

$$\log \frac{1-\xi_0}{\xi_0} = 1,$$

qui est naturellement > 0 et $< {}^1/_2$, et considérons l'intégrale

$$\int_0^{\xi_0} \xi^k(1-\xi)^{n-k}d\xi \int_{-\infty}^{\log\frac{1-\xi}{\xi}} \frac{e^{kx}dx}{\pi^2+x^2}.$$

En utilisant la formule (9) nous aurons

$$\int_0^{\xi_0} \xi^k(1-\xi)^{n-k}d\xi \int_{-\infty}^{\log\frac{1-\xi}{\xi}} \frac{e^{kx}dx}{\pi^2+x^2}$$

$$= \frac{1}{k}\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2} + \frac{M'}{k^2}\int_0^{\xi_0} \frac{(1-\xi)^n}{\left(\log\frac{1-\xi}{\xi}\right)^3}d\xi.$$

Mais

$$\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}$$

$$= \int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\left(\log\frac{1-\xi}{\xi}\right)^2} - \pi^2\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\left(\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2\right)\left(\log\frac{1-\xi}{\xi}\right)^2},$$

et

$$\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\left(\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2\right)\left(\log\frac{1-\xi}{\xi}\right)^2} < \int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\left(\log\frac{1-\xi}{\xi}\right)^4},$$

donc, en tenant compte de la formule asymptotique (7), nous avons d'abord

$$\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2} \sim \frac{1}{n(\log n)^2},$$

et ensuite nous voyons que le rapport de l'intégrale

$$\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\left(\log\frac{1-\xi}{\xi}\right)^3} \quad \text{à} \quad \int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}$$

est asymptotique à

$$\frac{1}{\log n},$$

ce qui nous conduit à la formule

$$\int_0^{\xi_0} \xi^k(1-\xi)^{n-k} d\xi \int_{-\infty}^{\log\frac{1-\xi}{\xi}} \frac{e^{kx}dx}{\pi^2+x^2}$$
$$= \frac{1}{k}\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}\left[1+\frac{N}{k\log n}\right],$$

où N pour des valeurs suffisamment grandes de n et quelque soit k reste au dessous d'une limite assignable, p. e., 14. D'un autre côté, on trouve immédiatement

$$\int_{\xi_0}^{\frac{1}{2}} \xi^k(1-\xi)^{n-k} d\xi \int_{-\infty}^{\log\frac{1-\xi}{\xi}} \frac{e^{kx}dx}{\pi^2+x^2}$$
$$< \frac{1}{k}\int_{\xi_0}^{\frac{1}{2}} (1-\xi)^n d\xi < \frac{(1-\xi_0)^{n+1}}{k(n+1)},$$

et le rapport de cette quantité

$$\frac{(1-\xi_0)^{n+1}}{k(n+1)} \quad \text{à} \quad \frac{1}{k}\int_0^{\xi_0}\frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}$$

étant pour n suffisamment grand plus petit que, p. e.,

$$2(1-\xi_0)^{n+1}(\log n)^2,$$

peut être mis sous la forme

$$N'/(k\log n) \quad \text{où} \quad N'<2n(1-\xi_0)^{n+1}(\log n)^2$$

tend vers zéro n grandissant indéfiniment. Ainsi nous avons enfin

$$\int_0^{\frac{1}{2}}\xi^k(1-\xi)^{n-k}d\xi\int_{-\infty}^{\log\frac{1-\xi}{\xi}}\frac{e^{kx}dx}{\pi^2+x^2}$$

$$=\frac{1}{k}\int_0^{\xi_0}\frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}\left[1+\frac{G}{k\log n}\right],$$

ou bien

$$P_1=\frac{1}{k}\int_0^{\xi_0}\frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}\left[1+\frac{G}{k\log n}\right],$$

G désignant une quantité qui reste au dessous de 15 pour n suffisamment grand quelque soit k. De même

$$P_2=\frac{1}{n-k}\int_0^{\xi_0}\frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}\left[1+\frac{G'}{(n-k)\log n}\right],$$

où G' est aussi <15 pour n assez grand quelque soit k. Il suffit maintenant d'introduire ces expressions dans la formule (4) pour avoir

$$J_n=(-1)^{n-k}\Gamma(k+1)\Gamma(n+1-k)+\Gamma(n+2)$$

$$\times\left\{\int_0^{\xi_0}\frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}\left[\frac{(-1)^{n-1}}{k}-\frac{1}{n-k}\right.\right.$$

$$\left.\left.+\frac{(-1)^{n-1}G}{k^2\log n}-\frac{G'}{(n-k)^2\log n}\right]+\frac{\Theta}{k(n-k)2^{n+1}}\right\}$$

et aussi

$$A_k = \frac{1}{n} + \frac{\Gamma(n+2)}{n\Gamma(k+1)\Gamma(n-k+1)}$$

$$\times \left\{ \int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2 + \left(\log \frac{1-\xi}{\xi}\right)^2} \left[\frac{(-1)^{k-1}}{k} + \frac{(-1)^{n-k-1}}{n-k} \right.\right.$$

$$\left.\left. + \frac{(-1)^{k-1} G}{k^2 \log n} + \frac{(-1)^{n-k-1} G'}{(n-k)^2 \log n} \right] + \frac{\Theta}{k(n-k)2^{n+1}} \right\}.$$

Le rapport de

$$\frac{\Theta}{k(n-k)2^{n+1}} \quad \text{à} \quad \frac{(-1)^{k-1}}{k} + \frac{(-1)^{n-k-1}}{n-k}$$

est plus petit que

$$\frac{1}{2^{n+1}|n-k+(-1)^n k|} < \frac{1}{2^{n+1}},$$

car le nombre entier $n-k+(-1)^n k$ ne devient jamais égal à zéro. De même la valeur absolue du rapport de

$$\frac{(-1)^{k-1} G}{k^2 \log n} + \frac{(-1)^{n-k-1} G'}{(n-k)^2 \log n} \quad \text{à} \quad \frac{(-1)^{k-1}}{k} + \frac{(-1)^{n-k-1}}{n-k}$$

pour n suffisamment grand, est plus petite que

$$\frac{15}{\log n} \cdot \frac{k^2+(n-k)^2}{k(n-k)|n-k+(-1)^n k|},$$

c'est à dire pour n pair

$$\frac{15}{\log n} \cdot \frac{k^2+(n-k)^2}{k(n-k)n},$$

et pour n impair

$$\frac{15}{\log n} \cdot \frac{k^2+(n-k)^2}{k(n-k)|n-2k|}.$$

Or quel que soit $k = 1, 2, 3, \cdots, n-1$,

$$\frac{k^2+(n-k)^2}{k(n-k)n} < 1, \quad n \text{ pair}$$

et

$$\frac{k^2+(n-k)^2}{k(n-k)|n-2k|} < 3, \quad n \text{ impair}$$

de sorte que le rapport précédent est toujours plus petit que

$$\frac{45}{\log n}$$

et tend vers zéro quand n grandit indéfiniment. De tout ce qui précède il est maintenant facile de déduire cette expression asymptotique de A_k:

$$A_k \sim \frac{\Gamma(n+1)}{n(\log n)^2\Gamma(k+1)\Gamma(n-k+1)}\left[\frac{(-1)^{k-1}}{k}+\frac{(-1)^{n-k-1}}{n-k}\right],$$

valable en ce sens que le rapport des deux membres de cette égalité tend uniformément vers 1, quand n croît au delà de toute limite, k ayant une valeur quelconque $1, 2, 3, \cdots, n-1$. Quant aux coefficients A_0 et A_n, l'analyse toute semblable et même un peu plus simple donne

$$A_0 = A_n \sim \frac{1}{n\log n}.$$

Ainsi nous avons

$$A_0 \sim \frac{1}{n\log n}, \quad A_1 \sim \frac{1}{(\log n)^2}, \quad A_2 \sim \frac{-n}{4(\log n)^2} \text{ etc.}$$

et pour $n = 2l$, p. e.,

$$A_l \sim \frac{(-1)^{l-1}4^l}{\sqrt{\pi}\, l^{5/2}(\log 2l)^2}.$$

On voit que les coefficients $A_2, A_3, \cdots, A_{n-2}$ tendent vers l'infini, ce qui fait voir que la formule de Cotes perd, toute valeur pratique tant que le nombre des ordonnées devient considérable.

TORONTO, 25 Août, 1924.

FUNCTIONS WITH AN ESSENTIAL SINGULARITY*

BY PHILIP FRANKLIN

1. *Introduction.* In this note we prove certain properties of functions possessing essential singularities. The results grew out of an attempt to prove that the equation

$$\sin x = x$$

has an infinite number of complex roots. This particular fact can be deduced very simply from Picard's theorem (see Theorem VI below), but it suggests other inferences which are much less immediate. The character of the theorems, which are given explicitly below, may be indicated by observing that they prove the existence of an infinite number of roots of the following typical equations:

$$x^3 e^{x^2} + 3x = 0,$$

$$\cos x - \frac{2x+1}{x^2-4} = 0,$$

$$\sec x + 5x = 0.$$

2. *A Corollary to Picard's Theorem.* One possible formulation of Picard's theorem is the following:†

If the function $f(x)$ has an essential singularity at the point P, and in some deleted neighborhood of P is analytic except for a finite number of poles and has only a finite number of zeros, then the equation

$$f(x) - a = 0$$

has an infinite number of roots in the neighborhood in question for all values of a different from zero. The proposed corollary follows.

THEOREM I. *If the function $f(x)$ has an essential singularity at the point P, and in some deleted neighborhood of P is*

---

* Presented to the Society, May 3, 1924

† Cf., e. g., Osgood, *Funktionentheorie*, 2d ed., vol. 1, p. 709.

analytic except for a finite number of poles and has only a finite number of zeros, while the function $A(x)$, *not identically zero, is analytic except for poles in the complete neighborhood of* P, *then the equation*

$$f(x)-A(x) = 0$$

has an infinite number of roots in this neighborhood.

To prove this, consider the function $f(x)/A(x)$. This function is analytic except for poles in the deleted neighborhood of P, and has an essential singularity at P. Further, it has at most a finite number of zeros and poles in the neighborhood in question, since both $f(x)$ and $A(x)$ have at most a finite number of zeros and poles there. Thus, by the theorem of Picard just stated, the equation

$$f(x)/A(x)-1 = 0$$

must have an infinite number of roots. Moreover, since $A(x)$ has only a finite number of poles, the equation

$$A(x)[f(x)/A(x)-1] \equiv f(x)-A(x) = 0$$

must in consequence have an infinite number of roots, which proves our contention.

3. *Applications to Entire Functions.* We may obtain a simplified statement of our theorem by specializing it somewhat. Let us take the point P at infinity, the deleted neighborhood as the proper plane, and require the function to have no poles. The function is then an entire function, and the theorem takes the following form.

THEOREM II. *If* $E(x)$ *is an entire function (not a polynomial) with only a finite number of zeros, and* $R(x)$ *is a rational function (not identically zero), the equation*

$$E(x)-R(x) = 0$$

always has an infinite number of roots.

If we further restrict $R(x)$ to be a polynomial, and notice that changing a finite number of the coefficients in the Maclaurin's expansion of a function is equivalent to adding a suitably chosen polynomial to it, we obtain the following remarkable theorem.

THEOREM III. *If an entire function has at most a finite number of zeros, any function formed from it by changing a finite number of the coefficients in its Maclaurin's expansion has an infinite number of zeros.*

A statement equivalent to Theorem III is the following one.

THEOREM IIIa. *If an entire function is known to have only a finite number of zeros, and the coefficients of its Maclaurin's expansion from a certain point on are known, the preceding ones are uniquely determined.*

It is interesting to observe that the assumption that there actually exists an expansion with the desired properties is necessary for the truth of the theorem. That is, we can not start with any infinite expansion, and determine the earlier coefficients so as to give a function with only a finite number of zeros, as one might suppose. For example, if the terms following the mth are

$$K(x) = \sum_{m}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

no choice of the earlier terms will give a function which does not have an infinite number of zeros. This follows from Theorem V below, since $K(x)$ is the remainder in the expansion of sin x.

4. *The Generalized Corollary.* There is an extension of our corollary, analogous to the extension of Picard's theorem to the case in which the function has an infinite number of poles and fails to take on two values more than a finite number of times. The extension is as follows.

THEOREM IV. *If the function $f(x)$ has an essential singularity at the point P, which is a cluster point of poles, otherwise being analytic in some neighborhood of P, and takes on two distinct values a and b at most a finite number of times in this neighborhood, while the function $A(x)$ is analytic* *except for poles in the complete neighborhood of P, and is not identically equal to a or b, then the equation*

$$f(x) - A(x) = 0$$

has an infinite number of roots in the neighborhood in question.

To prove this, we first observe that the function

$$\frac{f(x)-a}{f(x)-b} = 1 + \frac{b-a}{f(x)-b}$$

has an essential singularity at P (since a and b are distinct), and that it has at most a finite number of zeros and poles by our hypothesis as to $f(x)$. Likewise the function

$$\frac{A(x)-a}{A(x)-b}$$

is analytic except for poles in the complete neighborhood of P, and is not identically zero. Thus the equation

$$\frac{f(x)-a}{f(x)-b} - \frac{A(x)-a}{A(x)-b} = 0$$

satisfied the hypothesis of Theorem I, and hence has an infinite number of roots.

Next consider

$$\frac{[f(x)-b]\,[A(x)-b]}{a-b}\left[\frac{f(x)-a}{f(x)-b} - \frac{A(x)-a}{A(x)-b}\right] = 0.$$

Any value of x which satisfies the previous equation will satisfy this one, unless it is a pole of one of the outside factors. But such a value would have to be a pole of $f(x)$ or $A(x)$, and from the form of the previous equation, we see that it would have to be a pole of both, if of either. Thus all the roots lost are included in the poles of $A(x)$, and since this function has at most a finite number of poles, the equation last written has an infinite number of roots. Since this equation reduces to

$$f(x) - A(x) = 0.$$

the theorem is proved.

Both Theorems I and IV may be slightly strengthened by replacing the function $A(x)$ by two functions. The revised statement of Theorem IV is as follows:

THEOREM IVa. *If the function* $f(x)$ *has an essential singularity at the point* P, *which is a cluster point of poles, otherwise being analytic in some neighborhood of* P, *and takes on two distinct values* a *and* b *at most a finite number of times in this neighborhood, while the functions* $A(x)$ *and* $B(x)$ *are analytic except for poles in the complete neighborhood of* P, *and* $A(x)/B(x)$ *is not identically equal to* a *or* b, *then the equation*

$$B(x)f(x)-A(x) = 0$$

has an infinite number of roots.

This follows at once from Theorem IV, since that theorem may be applied to the equation $f(x)-A(x)/B(x) = 0$. The analogous strengthened form of Theorem I is somewhat similar to a theorem given by Borel for entire functions of finite genus, where our functions $A(x)$ and $B(x)$ are required to be polynomials.*

5. *Applications to Periodic Functions.* A periodic function which is not a constant must have an essential singularity at infinity. Hence the preceding theorems may be used to derive theorems about periodic functions. In fact, the periodicity enables us to weaken the hypothesis. We have, for example, the following theorem.

THEOREM V. *If* $f(x)$ *is periodic, is not a constant, and has no poles, while* $A(x)$ *is analytic except for poles in some complete neighborhood of infinity, and is not a constant, then the equation* $f(x)-A(x) = 0$ *has an infinite number of roots in the neighborhood in question.*

If the theorem were false, the equation

$$f(x)-A(x) = 0$$

would have only a finite number of roots, and its left member would satisfy all the conditions required of the function $f(x)$ of Theorem I. Likewise the function $A(x+p)-A(x)$, where p is the period of $f(x)$, would satisfy all the conditions required of the function $A(x)$ of that theorem. It is not identically zero since $A(x)$, being analytic or having a pole at infinity, and not being a constant, is not periodic. Thus by Theorem I, the equation

---

* *Leçons sur les Fonctions Entières,* 1921, p. 90.

$$[f(x)-A(x)]-[A(x+p)-A(x)] = 0$$

would have an infinite number of roots. But, since

$$f(x) = f(x+p),$$

this equation is equivalent to

$$f(x+p)-A(x+p) = 0,$$

which accordingly has an infinite number of roots. Since this would necessitate the existence of an infinite number of roots for the equation

$$f(x)-A(x) = 0,$$

it contradicts our assumption, and the theorem is proved.

For one special choice of the function $A(x)$, we may weaken the restriction on $f(x)$ still further, so as to obtain the following theorem.

THEOREM VI. *If $f(x)$ is periodic and not a constant, the equation*

$$f(x)-ax = 0$$

always has an infinite number of roots.

For, if the equation $f(x)-ax = 0$ had only a finite number of roots, so would

$$f(x+np)-a(x+np) = 0,$$

p being the period of $f(x)$, and n being any integer. Since this is equivalent to $f(x)-ax = anp$, we would have a function with an essential singularity which failed to take on all the values anp more than a finite number of times, which contradicts Picard's theorem.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

THE CALCULUS OF VARIATIONS*

BY LEONIDA TONELLI

1. *The Purpose of this Note.* It is proposed to explain briefly here the method followed in my *Fondamenti di Calcolo delle Variazioni*†, of which the first two volumes have already been published. In these the simplest problem of the calculus of variations in the plane and the corresponding isoperimetric problem are considered, for integrands which are functions only of the coordinates and the inclination of the tangent at a point of the curve along which the integrals are taken; the other problems of the calculus of variations will be discussed in later volumes.

2. *Reasons for the Development of a new Method in the Calculus of Variations.* The inconveniences which present themselves in the classical methods of the calculus of variations are well known. Here I emphasize only the fact that, aside from the difficulties inherent in the verification for a given curve of the well known sufficient conditions for an extremum, there is further a serious difficulty which sometimes arrests at the very beginning the application of the classical results. It is that the theory of differential equations offers but few methods of deciding whether or not there exists an extremal joining two points which are not conveniently near to each other. I propose on that account to find a more direct procedure for the solution of the problems of the calculus of variations, independent of the theory of differential equations. At the same time my researches have two other objectives. The first is that of obtaining general results in the domain of absolute extrema, as contrasted with the classical method which is particularly devoted to relative extrema. In the second place it will be interesting to see if the new methods

---

* Translated into English and prepared for publication by G. A. Bliss.

† Bologna, Zanichelli; vol. 1, 1922; vol. 2, 1924.

are applicable to classes of curves which are more general than those which are usually considered in the treatises on the calculus of variations. One is necessarily restricted there to more special classes because the methods which are followed are founded essentially upon the consideration of extremals. The classical problem of Newton, the determination of a solid of revolution of least resistance, is a first example of the necessity of considering more general classes of curves than the ordinary ones.

3. *Fundamental Conceptions of the new Method.* Having thus specified the scope of the proposed investigation a natural method presents itself, that of the functional calculus according to the ideas of Volterra. The integrals which are considered in the calculus of variations, and which it is desired to make a maximum or minimum, depend for the most part upon curves or surfaces and are real valued functions of such curves or surfaces. These belong therefore in the domain of the functional calculus, and it is evident that the principal developments of the calculus of variations can be obtained by constructing, for its functions of curves and surfaces in the domain of the functional calculus, a theory of extrema corresponding to that which is usually developed for functions of one or more variables in the ordinary treatises on analysis. It is precisely this method which is followed here.

A first difficulty presents itself at once. In the ordinary theory of maxima and minima one starts from a fundamental property of the functions there considered, namely, *continuity*. But *continuity* cannot be made the basis of the new theory to be constructed, since the functions of curves and surfaces to be considered are in general discontinuous. For greater precision and in order to remain in the domain of the reasoning of the first two volumes of my *Fondamenti*, let us limit ourselves here to the functions of curves which are integrals whose integrands depend only upon the coordinates and the inclination of the tangent at a variable point of the curve along which the integral is taken. If the property of continuity is required for such an integral, it is found that the only integrals which can be considered are those whose integrands are expressible linearly in terms of the slope of the curve of integration; and in the parametric case, the only integrals with this property are those which depend solely upon the coordinates of the end-points of the curve along which the integral is taken.

It is therefore necessary to discard *continuity*, and to place at the basis of the new theory a conception which is more general and better adapted to the functions which have to be considered.

The analysis of the concept of *continuity*, for the functions of numerical variables of the ordinary differential calculus, has already led Baire to decompose this concept into two others of a more elementary sort called *lower semi-continuity* and *upper semi-continuity*. The simultaneous possession of both these types of *semi-continuity* implies precisely ordinary *continuity*. Baire has also observed that the elementary reasoning which permits one to prove that a continuous function in a bounded closed domain has always a minimum or a maximum, is also sufficient to justify the same conclusion in the case when the function possesses lower or upper semi-continuity, respectively. This observation applied to our functions of curves, permits us to overcome the first serious difficulty and to establish the foundations of the new theory.

If the length of arc is taken as the variable of integration, and if the function to be integrated does not depend upon the derivatives of the coordinates of the points of the curve of integration, that is to say, if it is a function $\varphi(x, y)$ of the coordinates x and y, then an elementary geometric argument furnishes at once the result that when $\varphi(x, y)$ is always positive, its integral possesses lower semi-continuity as a function of the curve of integration. This property has already been noted in the memorable *Thesis* of Lebesgue. But is this semi-continuity a property only of the integrals of the particular form here indicated, or does it belong to a class of integrals so large that it can be made the basis of a general method of the calculus of variations? My researches, begun in 1911, have permitted me to answer the latter part of this question affirmatively.

Among the integrals in the parametric form all those which Hilbert has called *regular* possess semi-continuity, as well as those which I have named *quasi-regular semi-normal,* and also those which are simply *quasi-regular* provided that another condition is added such as, for example, the condition that the function under the integral sign does not change sign. Besides these sufficient conditions for semi-continuity I have investigated also necessary conditions, and conditions which are at the same time necessary and sufficient. Conditions of this last sort for lower semi-continuity are that the integral be quasi-regular positive, and that every so-called *exceptional* point shall be the center of at least one circle such that the value of the integral along an arbitrarily chosen closed arc interior to the circle is always positive or zero. By an *exceptional* point is meant one at which the well known Weierstrassian function F_1 is identically zero in the derivatives x', y'.

For the case of a non-parametric integral a necessary and sufficient condition is simply that the integral be quasi-regular.

The semi-continuity hitherto considered is semi-continuity in the entire field, i. e., at every curve of the field. I have also studied semi-continuity at a particular curve and have arrived here at the conclusion that the property of semi-continuity corresponds to the well known condition of Weierstrass in the calculus of variations, a condition which, as is known, implies also that of Legendre. This conclusion seems to me interesting because it permits one to establish a complete correspondence between the conditions for a relative extremum in the differential calculus and those for the same sort of an extremum in the calculus of variations. In the differential calculus, for a function $f(x)$ which is continuous and has derivatives of the first two orders, a point x_0 in the domain where $f(x)$ is defined furnishes a relative minimum for $f(x)$ if at x_0 the first derivative of the function is zero and the second derivative positive. In the calculus of variations a curve C_0, interior to the field in which the integral to be minimized is considered, furnishes a relative minimum for the integral if the first variation is zero and the second positive, and if furthermore the conditions of Weierstrass and Legendre are satisfied. But these conditions of Weierstrass and Legendre have precisely the effect of assuring us that the integral in question has lower semi-continuity at the curve C_0, a condition which in the case of the functions $f(x)$ is already implicitly contained in the hypothesis of continuity. Consequently to the conditions of the differential calculus relative to the vanishing of the first derivative and the positive sign of the second, correspond exactly in the calculus of variations, and with the same significance, the analogous conditions on the first and second variations. And the correspondence in the two fields between the conditions sufficient for a minimum is also repeated in the case of necessary conditions.

The study thus briefly summarized and the observation, which it may be well to repeat explicitly, that every integral must possess lower semi-continuity on each of its minimizing curves, have led me to construct a new development of the calculus of variations founded essentially on the concept of semi-continuity.

4. *Existence Theorems for Extrema.* A first application of semi-continuity can be made immediately to the demonstration of theorems concerning the existence of an absolute extremum, by following the same method of procedure as that of the differential calculus for extrema of ordinary functions of real variables.

Nevertheless a difficulty presents itself here which has no analogue in the theory of extrema of the differential calculus. To establish the existence of an absolute minimum on a given interval for a function $f(x)$ which possesses lower semi-continuity, one uses, as is well known, a minimizing sequence $x_1, x_2, \cdots, x_n, \cdots$; such a sequence has at least one limit point by the theorem of Bolzano-Weirstrass; and in each one of these limit points the function $f(x)$, in virtue of its lower semi-continuity, assumes its minimum value. In the case of the calculus of variations the analogous minimizing sequence of curves $C_1, C_2, \cdots, C_n, \cdots$ does not always have a limiting curve because for classes of curves the theorem of Bolzano-Weierstrass is not always true. One has here the new circumstance that even though an integral possesses lower semi-continuity in a limited domain it may yet have no minimizing curve. To be able to reach a conclusion like that for the case of a function $f(x)$ it is necessary therefore to introduce some new condition which will assure the existence of a limiting element for the minimizing sequence. This further condition may assume various forms from which result numerous theorems of existence for extrema.

For integrals in parametric form I have begun, in the second volume of my *Fondamenti,* by demonstrating the existence of an extremum for the so-called quasi-regular definite integrals, a special case being the integral of a function $\varphi(x, y)$ which has always the same sign, a case which as I have already remarked was treated by Lebesgue in his *Thesis.* From the quasi-regular definite integrals I then pass to more general cases which require more profound analysis, seeking always to make the theory inclusive of the more important applications of the calculus of variations.

For the integrals in ordinary form the condition which assures the existence of a limiting element for a minimizing sequence takes a form different from that assumed in the parametric case, in that it is related to the behavior of the integrand function $f(x, y, y')$ as y' tends to infinity. Furthermore a variety of existence theorems are here stated designed to satisfy as far as possible the exigencies of the applications, and results are found, as particular cases, which were established in other ways by Hadamard.

The theorems of existence, both for the parametric and ordinary integrals, are stated so as to secure the maximum generality possible with regard to the character of the curves themselves and the character of the class of curves which is to be considered. In the first place the curves are supposed only to be *rectifiable*; and in the second place the functions $y(x)$ representing them are supposed to be absolutely continuous according to the definition of Vitali, and such that along them the integral which it is wished to study exists and is finite. Furthermore, and this is indeed the most important point to set in evidence, the existence theorems are given almost all for the classes of curves which I have named *complete*. In the parametric case a *complete class* is simply a totality of rectifiable curves to which belong all those curves which are at the same time limiting curves of the class and rectifiable. For the ordinary non-parametric case one substitutes for rectifiability in this definition the two properties which I have indicated just above. The theorems of existence assume in this way the maximum of generality; they are completely independent of the use of extremal curves; and they have a form applicable also to problems of extrema which could not be approached in any way by already known methods.

It is important to notice that the method of demonstrating the existence of extrema by the use of semi-continuity permits further the consideration of classes of integrals which are not everywhere semi-continuous. Of this I have given examples in my book, studying those types of integrals which have been the subject of noteworthy researches by C. Carathéodory. Such integrals have in general extremizing curves with corner points. For an integral of this type I have given a demonstration of the existence of an extremum by reducing the problem to one with an integral which possesses everywhere the property of semi-continuity and which is intimately related to the integral originally given. The same procedure permits me then to treat completely also the classical problem of Newton concerning the solid of revolution of minimum resistance, a problem whose solution, indicated by Newton himself, was completely justified only in 1900 by A. Kneser.

5. *Extremaloids and Extremals.* When the existence of an extremum has been proved it becomes necessary to pass to the discussion of the analytic properties of the extremizing curves. It is evident that these properties are intimately related to the character of the class of curves in which the extremum has been sought; and for the study of them the generality of the class considered must necessarily be restricted.

Limiting myself to the classes customarily considered in the calculus of variations, and moreover maintaining with respect to the character of the curves themselves all of the generality which I have indicated above, I am led to introduce the concept of an *extremaloid* which is a generalization of the known concept *extremal.*

In the parametric form, denoting by $F(x, y, x', y')$ the integrand function, I designate as an *extremaloid* every rectifiable curve satisfying the two equations

$$\int_0^s F_x ds - \frac{d}{ds}\int_0^s F_{x'} ds = c_1, \quad \int_0^s F_y ds - \frac{d}{ds}\int_0^s F_{y'} ds = c_2,$$

where s represents the length of arc. These equations are evidently generalizations of those of the extremals; they have the advantage of being in a form which permits them to be satisfied also by a curve which is only rectifiable. Generalizing a classical argument I prove that every extremizing curve, relative to all those of a given neighborhood of it joining the same two end points, is necessarily an *extremaloid* if it is interior to the field considered. I obtain thus the extension of Euler's condition to curves which are rectifiable only. I prove next that, for the *quasi-regular normal* integrals, the extremaloids are necessarily extremals, and I have here the means of treating an important case in which the extremals present singularities arising from the vanishing of the coefficient of the term of the second order in their equation. For the case of the *quasi-regular normal* integrals again, I prove that for fields of a very general type, but with boundaries having no corner points pointing toward the interior of the field itself, all the extremizing curves have everywhere continously turning tangents even on the parts which touch the boundary. For special domains, or for unlimited domains when the behavior of the function F as the point (x, y) tends to infinity is suitably restricted, the extremizing curve turns out to be simply an extremal. *From this I derive theorems of existence for extremals joining two given points not necessarily near to each other, and for periodic extremals.*

I study further the uniqueness of the extremizing curves and of the extremal joining two given points.

It goes without saying that the known conditions for an extremum are re-deduced, those of Weierstrass and Legendre being, as has been already stated, an immediate consequence of the semi-continuity which must be verified on every extremizing curve.

I find moreover the known sufficient conditions for a relative extremum. Here I solve the problem first for a small region, reducing the study of the given integral to that of another integral possessing everywhere semi-continuity, and applying the results already obtained. The same method makes it possible to solve the analogous problem in the general case, which can also be treated by applying the results obtained for a small region.

For the non-parametric integrals greater difficulties in the case of Euler's condition present themselves, depending upon the fact that one can not be sure a priori of the existence of the integrals which occur in the equations of the extremaloids and whose integrands are the derivatives with respect to y and y' of the function $f(x, y, y')$ taken at the points of the extremizing arc. The existence of these integrals seems probable, but I have not succeeded so far either in proving it for all cases or in constructing an example for which they do not exist. Nevertheless I have given in this connection propositions of a very general character. For example I have shown that every extremizing curve, relative to all those in a neighborhood of it joining two given points, is an extremaloid provided that it is interior to the field under consideration and has bounded difference quotients; and I have also shown that the condition relative to the difference quotients can be altogether suppressed for a noteworthy and extensive category of integrals which includes precisely those integrals which for the most part present themselves in the applications of the calculus of variations. For this same category of integrals and also for another still more general, I have moreover proved that every extremaloid is necessarily an extremal. I have studied especially *regular* integrals, and those which are *quasi-regular normal,* seeking to attain results analogous to those for the parametric case. Here the extremizing curves may have first derivatives of their ordinates which become infinite; but I have given conditions sufficient to insure that this shall not occur.

All the arguments made for the parametric case are taken up again for the integrals in non-parametric form; and by the method already indicated one finds here also the known necessary and the known sufficient conditions for a relative extremum. The problem which I have discussed hitherto is that of a non-isoperimetric extremum; but the method described is applied in the second volume of my *Fondamenti* to isoperimetric problems also.

The method of procedure, as is evident from what has been said, does not depend in any way upon the theory of differential equations; indeed it adds to that theory a contribution in the domain of the theorems of existence.

Bologna, Italy

SCHOUTEN ON RICCI-KALKÜL

Der Ricci-Kalkül. By J. A. Schouten. Berlin, Julius Springer, 1924. 312 pages.

This is Volume X of *Die Grundlehren der Mathematischen Wissenschaften,* edited by R. Courant, and is appropriately dedicated to G. Ricci, founder of the absolute differential calculus, in honor of his seventieth birthday. It is the second book on this subject published by Springer within two years. The authors (Struik and Schouten) are closely associated, and therefore the fact that the two books have many things in common is not surprising. The volume under discussion is more extended than Struik's, the additional space being devoted to a fuller discussion of non-Riemannian geometry. The methods are quite similar, but Schouten does not make use of quite so much symbolism, although Struik attributes most of his symbolism to Schouten.

These volumes give an exceedingly complete account of all the developments in differential geometry of the past five or six years, but the reviewer feels that much of the value, as reference books, is lost on account of the symbolism. The reading of many pages is often required for the full understanding of a single theorem, because it is impossible for one to keep these symbols in mind unless constantly using them.

Many of us feel that one of the best things Ricci did was to invent a consistent notation for covariant and contravariant systems. Ricci is always careful that the upper and lower indices shall have a significance. The ultra modern writers, following Weyl, have departed from this. The best example is $\Gamma_{\lambda\mu}^{\nu}$, which, according to the usual notation, should be a mixed system, covariant in λ, μ and contravariant in ν. Such is not the case, however. I, for one, fail to see why this confusion in notation should have been introduced, as it only makes the reading more laborious. One is compelled to investigate to see just what quantities are covariant or contravariant.

Geometers had been accustomed to think that Riemannian geometry was the most general geometry possible, but in the past few years has come the notion of parallel displacement, which leads to geometries quite different from that of Riemann. In the present volume, Schouten gives a very full account of these new geometries. In fact, only about fifty pages are devoted to Riemannian geometry. It does not take long to convince one that this book is entirely modern both as to material and point of view. Nearly every page has a reference to work which has appeared within six or seven years. A modern mathematical theory usually means one that is not more than fifty or seventy-five years old, and this book, therefore, leaves the pleasant sensation of a really modern theory. The parts of the subject which are not modern are labeled so that one can easily recognise them.

The work is divided into seven chapters. The first gives, from a very general point of view, the algebraic part of the absolute calculus. The notions of a vector in a general space and the distinctions between covariant and contravariant vectors is carefully stated. In Chapter II, we begin the analytical part and are introduced at once to the notion of parallel displacement. The displacement is defined by the following properties:

I. The differential of a covariant, contravariant, or mixed quantity, which does not change with change of covariant scale, is a quantity of the same kind.

II. The differential is a linear function of the line elements or

$$\delta \Phi = \Phi_\mu dx^\mu;$$

Φ_μ is a quantity having one more covariant index than Φ and the operation which is represented is symbolized by ∇_μ, hence

$$\Phi_\mu = \nabla_\mu \Phi;$$

$\nabla_\mu \Phi$ is called the covariant derivative of Φ belonging to the displacement.

III. The differential of a sum is the sum of the differentials,

$$\delta(\Phi + \Psi) = \delta\Phi + \delta\Psi,$$

from which follows

$$\nabla_\mu(\Phi + \Psi) = \nabla_\mu \Phi + \Delta_\mu \Psi.$$

IV. The ordinary rules for the differential and therefore of the derivative of a product holds.

V. The differential of a numerical quantity (invariant) is the ordinary differential

$$\delta p = dp, \qquad \nabla_\mu p = \frac{\partial p}{\partial x^\mu}.$$

From these postulates the formulas for differentiation of vectors v^ν, w_λ are easily derived:

$$\nabla_\mu v^\nu = \frac{\partial v^\nu}{\partial x^\mu} + \Gamma^\nu_{\lambda\mu} v^\lambda,$$

$$\nabla_\mu w_\lambda = \frac{\partial w_\lambda}{\partial x^\mu} - \Gamma'^\nu_{\lambda\mu} w_\nu,$$

and similar formulas for quantities of higher order. The character of the displacement then depends on the quantities Γ, Γ', but instead of using them directly the following combinations are used.

Let

$$C_{\mu\lambda}^{\cdot\,\nu} = \Gamma_{\lambda\mu}^{\nu} + \Gamma_{\lambda\mu}^{\prime\nu},$$
$$S_{\lambda\mu}^{\cdot\cdot\nu} = \frac{1}{2}\left(\Gamma_{\lambda\mu}^{\nu} - \Gamma_{\mu\lambda}^{\nu}\right),$$
$$Q_{\mu}^{\cdot\lambda\nu} = \nabla_{\mu} g^{\lambda\nu}.$$

Then it is found that the displacement can be expressed in terms of C, S, Q, g, and the various kinds are classified according to the restrictions of C, S, Q. The affine displacement is characterized by

$$C = 0, \quad S' = \frac{1}{2}\left(\Gamma_{\cdot\cdot\lambda\mu}^{\prime\nu} - \Gamma_{\mu\lambda}^{\prime\nu}\right) = 0, \quad Q' = \nabla_{\mu}\, g_{\lambda\nu} = 0;$$

the Weyl displacement by

$$C = 0, \quad S' = 0, \quad Q = Q_{\mu}\, g_{\lambda\nu},$$

where Q_{μ} represents a vector; the Riemann displacement (defining the ordinary differential geometry) by

$$C = 0, \quad S' = 0, \quad Q' = 0.$$

For the general displacement a geodesic line is defined as a curve whose line elements are moved into line elements of the same curve and the differential equations of such a curve are

$$\frac{d^2x^{\nu}}{dt^2} + \Gamma_{\lambda\nu}^{\nu}\frac{dx^{\lambda}}{dt}\frac{dx^{\mu}}{dt} = \alpha\frac{dx^{\nu}}{dt},$$

where α is a function of the coordinates. These have a marked similarity to the equations of geodesics in Riemannian geometry; and, in case $C = S' = Q' = 0$, they are identical with them.

Curvature is obtained by integrating δv^{ν} around an infinitesimal closed curve. The formula turns out to be very similar to that obtained by Levi-Civita for his parallel displacement. After fully developing these ideas Schouten then specializes the various kinds of displacements, devoting a chapter to each of the three mentioned above.

The whole structure is built up around the general idea of parallel displacement and cannot fail to convince the reader of the far-reaching importance of this notion. No geometer, unfamiliar with this subject, can afford to miss reading this book.

C. L. E. MOORE

SHORTER NOTICES

Vorlesungen über Topologie. I. Flächentopologie. By B. von Kerékjártó. Volume 8 of the Courant Series. Berlin, Julius Springer, 1923. 8+270 pp. and 60 figures.

This production, from the pen of a young Hungarian mathematician, who is beginning to be known for his contributions to analysis situs, is welcome for several reasons. There are altogether too few books on the subject and one more is decidedly in place. It also gives under one cover a fairly complete treatment of the results obtained by Brouwer and his school on two dimensional topology, a useful thing indeed. The many and well chosen examples and figures are another good feature. For the sake of the average reader at least, we wish that the author had better amalgamated his material and introduced greater unity in his presentation. The *Topologie* will be especially useful to the reader familiar with point sets and wishing to learn more about their geometric applications, and also, say, in connection with Veblen's Colloquium Lectures.

The material in the book may be essentially classified into three groups: (a) Topology of the plane and its curves, centering around the Jordan curve theorems and including such questions as invariance of dimensionality and regionality, structure of regions and their boundaries, the general closed curve, etc. (b) Combinatorial analysis situs of two-dimensional manifolds. The treatment of this part is less felicitous than Veblen's in Chapter II of his Lectures. The author takes advantage of the relative simplicity of the theorems on plane Jordan curves, and thus more or less avoids questions of straightness. Unfortunately this method cannot be extended to three or more dimensions, and whereas Veblen's treatment of the two-dimensional case contains in germ, that of the more general one, Dr. Kerékjártó will be obliged to change front in his second volume, which is to be devoted to it. (c) Continuous transformations of surfaces (Brouwer's theorems on fixed points and the like), continuous families of curves on surfaces. This group of questions includes the author's more original and interesting contributions as well as the more recent results obtained by Brouwer and his students. If we mistake not, those on continuous families of curves are largely due to Dr. Kerékjártó himself.

The book concludes with a fair bibliography. The small number of American titles is doubtless to be ascribed to the lack of contact between the scientists of Central Europe and the rest of us—one of the most unpleasant things on the scientific landscape of the last few years.

S. LEFSCHETZ

Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik. By Adolph Kneser. 2d edition. Braunschweig, Vieweg, 1922. 8+292 pp.

Kneser's *Integralgleichungen*, which appeared first in 1911, was a pioneer in its field, and during the time which has elapsed since then, it has remained the only thing of its kind. To be sure, the book of Heywood and Fréchet emphasizes the physicial applications, and the problems of potential theory, in particular, have been methodically developed by means of integral equations. But the present work is, as far as the reviewer knows, unique in its inclusion of and emphasis on the application of the Hilbert-Schmidt theory to important one-dimensional problems.

The first edition was ably reviewed in the BULLETIN* by W. A. Hurwitz. The changes which have been made consist in about eighty pages of new or rewritten material, the remaining pages of the text having been reprinted without other change than an occasional alteration in notation. The most significant feature of the revision is the assembling and rounding out in one chapter (III) of a general theory of the symmetric kernel. The presentation follows the lines of the Schmidt theory, but it is further enhanced by the addition of paragraphs on Mercer's theorem on the convergence of the development of a continuous definite kernel in terms of its characteristic functions, and on Weyl's theorem on the diminishing effect on the positive characteristic numbers of a kernel caused by adding to the kernel a positive definite kernel.

The author finds a physical problem leading to an unsymmetric kernel in the vibrations of an elastic bar, account being taken of thermal effects. For this problem, the development question is treated by means of the theory of residues. Difficulties due to multiple characteristic numbers are not here encountered. There follow indications for a similar treatment of Sturm-Liouville problems in which the coefficients are not always necessarily real. The case in which the fundamental interval extends to infinity is then considered, and Hilb's generalization of the Fourier integral is derived and illustrated in the known cases of the Fourier integral and the corresponding integral in Bessel functions.

Another new topic arises from the treatment of a boundary value problem connected with electric cables, in which a boundary condition depends on the parameter. Here, ordinary orthogonality is replaced by a

---

* Vol. 19 (May, 1913), pp. 406–11. For other analyses and reviews, see *Lacour*, BULLETIN DES SCIENCES MATHÉMATIQUES, vol. 46 (1911), pp. 254–61; Korn, ARCHIV FÜR MATHEMATIK UND PHYSIK, (3), vol. 18 (1911), pp. 82–83; Plancherel, L'ENSEIGNEMENT MATHÉMATIQUE, vol. 13 (1911), pp. 428–29.

"weighted" (belastete) orthogonality: $V_1(1)V_2(1)+\int_0^1 V_1(x)V_2(x)\,dx=0$. Theorems analogous to the usual ones are indicated.

The bibliography in the appendix should not be overlooked. While it lays no claim to completeness, it is particularly valuable in the field of the applications. Its extent has been nearly doubled in the new edition.

The additions have materially enhanced the value of the book, and the chapter on the theory of the symmetric kernel has helped to meet Hurwitz' criticism as to the confusing effect of frequent alternation of general theory and particular examples. But only partially so, for as a rule several pages must be read before one can ascertain the precise conditions for the validity of a theorem, and in some cases the reader must bring an independent judgement to bear on his quest (e. g. on p. 38, line 19; the functions must be continuous and have *continuous* derivatives of first and second orders as well as piecewise continuous derivatives of third and fourth orders if the reasoning indicated is to establish the stated results). The style is further complicated by the habit of deferring the statement of a theorem until after its proof. Thus is imposed upon the attention of the reader the double task of following the reasoning and endeavoring to determine its import. Nor is much help given him by preliminary elucidation as to the general goal or the salient features of the discussion to follow.

The lacuna pointed out by Hurwitz in the proof of the theorem that to a solution of a homogeneous integral equation there always corresponds a solution of the associated equation has been allowed to stand; the section has been reprinted with such fidelity that a confusing typographical error recurs (p. 249, line 16: "Gleichung (7)" should read "Gleichung (3)").

The preparation requisite for a profitable reading of this book includes a knowledge of the rudiments of integral equations, some acquaintance with differential equations, with the theory of functions, and with physics; above all some mathematical maturity is essential. For one so equipped it is highly interesting and suggestive. Certainly no one who has to lecture on integral equations can afford to be unacquainted with its contents.

O. D. KELLOGG

Introduction à la Théorie de la Relativité, Calcul Différentiel Absolu, et Géométrie. By H. Galbrun. Paris, Gauthier-Villars et Cie., 1923. x+457 pp.

This work is a rather complete treatment of the mathematics of the relativity theory. Three chapters, about 100 pages, are devoted to a systematic and detailed exposition of the method of the absolute differential calculus. The differential geometry of n-dimensional space is allotted four chapters including slightly over 150 pages. The remainder of the book is devoted to mechanical and electromagnetic theories, first from the older point of view, then from the standpoint of special relativity. Strangely enough the general theory of relativity is not specifically treated. So we miss the usual denouement of books on relativity—the three crucial tests.

In the very beginning of the exposition of the absolute calculus is an error which should be noted. The author, in his definition of invariant and co- and contravariant systems, has confused the term invariant as used by Ricci with the ordinary sense in which the word is used. With Ricci it is equivalent to the expression "a function which transforms by invariance", which is merely a statement of the transformations by which we are to transform the function and does not imply invariance under the transformations. In a remark following the definitions he seems to imply that for an ensemble of transformations forming a group this preservation of functional form, which his definitions demand in excess of Ricci's, is automatically taken care of and so need not be explicitly stated. As a matter of fact it would seem that in general functions satisfying his definitions do not exist. The upshot of the matter is that with this remark he throws overboard his definitions without once using them, and uses the standard definitions of Ricci throughout the book.

Several things unite to make this treatment perhaps the most valuable book of reference on the absolute calculus that has yet appeared. Very few things in development or proofs are left to the reader's imagination. The author even scorns to forget his signs of summation, as do most of the present generation. Covariant and contravariant derivatives are indicated by Δ's with subscripts and superscripts, and so are distinguishable from other tensors of the same order.

In the part of the book devoted to n-dimensional geometry we again find a valuable work for reference. The notion of a vector, with its co- and contravariant components, is basic in his treatment. Then the notion of parallel displacement of a vector, fundamental to the Weyl geometry, is twice explained in great detail, once for euclidian space, and later for the non-euclidian. Geodesics and curvature are defined in terms of parallel displacement as well as in the usual manner. Thus we are thoroughly prepared for the chapter on Weyl geometry which completes the geometrical part of the book.

Chapter IX is entitled Galilean spaces in rational mechanics and electromagnetic theory, and deals with the older theory but uses the tools of the new, and prepares for Chapter X on Special Relativity, and Chapter XI on the Memoir of Minkowski. Finally come some remarks on the kinematics of relativity, which help much towards clarity of thinking about the conceptions which serve to connect the mathematics and the physics of the relativity theory.

C. E. WILDER

Statique et Résistance des Matériaux. By Paul Montel. Paris, Gauthier-Villars, 1924. vi+273 pp.

The author states in the preface that the book represents a course given in the École des Beaux-Arts on statics and resistance of materials. The methods are almost wholly graphic and confined to problems in the plane. In the first one hundred and sixty-two pages, which are devoted to statics, the various methods for determining the stresses in a frame work are given and also graphical methods for the determination of the center of gravity and moments of inertia. In the chapters on resistance of materials, the main subjects considered are the theory of the beam, the buckling of a long thin rod under thrust, and arches.

The subject is approached in a neat way by calling attention to the difference between the theory of elasticity and the theory of the resistance of materials and the difference between the statically determinate and indeterminate problem is emphasized throughout. Before taking up frame works there is a chapter on reactions of the supports in which the different types of support are considered. The manner in which these things are done leaves the reader with the impression that the subjects treated have been presented more systematically than is generally the case in text books of this kind.

Two or three oversights may be noted. In connection with the discussion of the funicular polygon on page eighteen, the author speaks of two forces forming a couple but a couple is not defined till page thirty-nine. The stress diagram in Figure 64 should have the line df replaced by af. The statement at the bottom of page one hundred and seventy-nine that both theory and practice lead to the value $(2/5)E$ for the shearing modulus is open to question.

PETER FIELD

Binomial Factorizations, giving extensive Congruence-Tables and Factorization-Tables. By Lt.-Col. Allan Cunningham. London, Francis Hodgson. 1923. Vol. I. 96+287 pp.

Binomial Factorizations. By Lt.-Col. Allan Cunningham. London, Francis Hodgson. 1923. Vol. IV. 6+160 pp. Supplement to vol. I.

These two volumes are part of the outcome of thirty years of labor of the veteran computer. There are seven volumes of the work and the greater part is already printed off. The war has delayed the appearance of much of it. The present volumes contain an extensive list of the smallest root of congruences of the type $y^n+1 \equiv 0 \pmod{p^k}$ for various values of p, k and n. Thus there is a table, begun in the first volume and carried on into volume IV giving the smallest values of y satisfying the congruence $y^2+1 \equiv 0 \pmod{p}$ for available prime values of p as high as the limit 99, 989. Similar tables are also given for the smallest values of y satisfying the congruence

$$\frac{y^{3n}+1}{y^n+1} \equiv 0 \pmod{p}$$

for various values of n and various limits of p. Volume IV also contains tables of roots (x_1, x_2) of $x_1^n+x_2^n \equiv 0 \pmod{p}$ for various values of p and n.

The first volume contains also factorization tables giving the factors of a vast number of numbers of special forms. They will be found somewhat difficult to use on account of the new names that confront the reader on every page. One meets here not only "Pellians" with new and strange prefixes, but also "Aurifeullians" of various families. One stands a bit daunted before a "Dimorph-Bin-Aurifuellian". Out of this rather confusing mass of computation emerge, however, some usable tables giving values of y which make y^2+1, $(y^2+1)/2$, $(y^3-1)/(y-1)$, etc. take on prime values. There are also tables giving values of x, y which make x^n+y^n a prime or twice a prime. These tables go much beyond the limits of available factor-tables.

There is no room for doubt that Lt.-Col. Cunningham has undertaken and carried out an immense task, of value in the problem of identifying large primes, and in the breaking down of numbers of special forms into their prime factors.

D. N. LEHMER

The Calculus of Observations. A Treatise on Numerical Mathematics. By E. T. Whittaker and G. Robinson. London, Blackie and Son, Ltd., 1924. 16+395 pp.

"The present volume represents courses of lectures given at different times during the years 1913—1923, by Professor Whittaker to undergraduate and graduate students in the Mathematical Laboratory of the University of Edinburgh, and may be regarded as a manual of the teaching and practice of the Laboratory, complete save for the subject of Descriptive Geometry."

To the teacher of mathematics, in this country at least, a mathematical laboratory will be apt to suggest graphical and nomographical methods. In this book the work is almost entirely arithmetical, and we are told that in the University of Edinburgh graphical methods have almost all been abandoned "as their inferiority has become evident". This result of actual experience covering some ten years will perhaps come as a surprise to many teachers and practical computers to whom such methods appeal especially where great speed is very desirable and only rough approximations are necessary.

The first four chapters deal with the theory of interpolation, and are obtainable in a separate issue entitled *A Short Course in Inter-*

polation, published in 1923. They furnish a practical reference book on interpolation with equal and with unequal intervals, central difference formulas and applications. These four chapters are followed by a short one on Chio's method of computing the numerical value of a determinant.

The chapter on the numerical solution of algebraic and transcendental equations describes, illustrates and compares the more important methods. One might wonder if the value of an up to date calculating machine is fully appreciated at the Laboratory. The computation of the successive values satisfying a difference equation is especially well adapted for machine computation, and the successive values of S_p (the sum of the pth powers of the roots of an equation) can be ground out with uncanny readiness by one who has had a little experience. Then, using Bernoulli's method, the quotient S_{p+1}/S_p can be obtained to as many decimals as may be desired. The computing machine is also invaluable in constructing tables of differences.

There are also chapters on Numerical Integration, Normal Frequency Distributions, Least Squares, Fourier Analysis, Smoothing of Data, Correlation, Search for Periodicity and the Solution of Differential Equations. These subjects are all given with satisfying clearness and detail with plenty of examples, and sample solutions. The whole book presents evidence on every page of sound scholarship and good practical judgement. The authors call attention to the opportunities for research in the subject of numerical mathematics. "There is an evident need for new and improved methods of dealing with many of the problems discussed in the later chapters".

D. N. LEHMER

Les Lieux Géométriques en Mathématiques Spéciales avec Application du Principe de Correspondance et de la Théorie des Charactéristiques à 1,400 Problèmes de Lieux et d'Enveloppes. By T. Lemoyne. Paris, Vuibert, 1923. 146 pp.

This little pamphlet summarizes Chasles' theory of characteristics together with the extensions which the author (with Brocard) developed in volume I of *Courbes Géométriques Remarquables,* and gives 1400 problems by way of illustration. It will be recalled that the characteristics μ, ν of a system of conics refer to the number of conics which pass through an arbitrary point, and are tangent to an arbitrary line, respectively. The characteristics of 170 systems of conics, and of 41 systems of circles are listed. Fundamental theorems give formulas for the order or class of many loci or envelopes connected with these systems in the form $\alpha\mu+\beta\nu$. Chasles gave 32 such fundamental theorems for the values of α and β; Lemoyne adds about 50 more, either with proofs or with references to the *Courbes Géométriques*

Remarquables, an important addition being to certain specializations where Chasles' formulas are inapplicable. The author is then in a position to state, not to be sure 80 times 170 results, for not every fundamental theorem and listed entry of characteristics may be paired, but perhaps 10,000 results (problems). Of these he actually gives about 1400. For two or three theorems he feels compelled to go practically through the 170 entries. Otherwise it is fair to say that he has chosen pretty much at random among those cases where the order or class is low. In some cases the order or class may be extremely large. Thus the locus of the vertices of conics tangent to three given conics and to a given line is a curve of order 1000. But the author does not gloat over these millics.

Lack of system seriously impairs the value of this book. The table of characteristics is extremely awkward for reference; failure to number the entries seems inexcusable. If the reader will number these he will find slips in 100, 102, 133, 134, 135, and 159. Among the problems there are obvious slips in 47, 148-152, 206, 248, 250, 690, 900, 1168, and 1291, while 605, 606, 607 are repetitions of 597, 598, 599, and 1141—except for a non-significant variation in punctuation—of 1121. There are unnecessary and irritating changes in notation, such as the use, in problem 1288, of D to denote both an asymptotic direction and an arbitrary line, while other capital letters stand for points. On the other hand, the enumerative results in these problems have all been checked and have been found correct.

B. H. BROWN

Aufgabensammlung zur Funktionentheorie. By Konrad Knopp. Berlin and Leipzig, Walter de Gruyter. 1923. 135 pp. I. Teil: *Aufgaben zur elementaren Funktionentheorie.* Sammlung Goschen.

This little book contains an excellent collection of problems on the elementary theory of complex functions. They are listed by chapters under six headings, fundamental concepts, sequences of numbers and infinite series, functions of a complex variable, integral theorems, developments in series, conformal mapping. Each of these chapters is further divided into two or three sections.

It has been said that if we could have excellent sets of problems published separately from texts, the task of text-book writers would be greatly simplified and they could concentrate on the presentation of their subject alone. The problems in this book could be a companion to any text which might be written on the subject of the theory of functions of a complex variable, covering the field of a single variable.

H. J. ETTLINGER

NOTES

The third number of volume 26 of the Transactions of this Society (July, 1924) contains the following papers: *The general theory of a class of linear partial q-difference equations,* by C. R. Adams; *The summability of the triple Fourier series at points of discontinuity of the function developed,* by B. M. Eversull; *An unusual type of expansion problem,* by M. H. Stone; *On the independence of principal minors of determinants,* by E. B. Stouffer; *A necessary and sufficient condition that two surfaces be applicable,* by W. C. Graustein and B. O. Koopman; *Extensions of relative tensors,* by O. Veblen and T. Y. Thomas; *Geometries of paths for which the equations of the paths admit a quadratic first integral,* by L. P. Eisenhart; *A general mean-value theorem,* by D. V. Widder.

The concluding number of volume 26 of the Transactions of this Society (October, 1924) contains: *Algebras which do not possess a finite basis,* by J. H. M. Wedderburn; *Determination of all the prime power groups containing only one invariant subgroup of every index which exceeds this prime number*, by H. A. Bender; *Invariants of the linear group modulo* $\pi = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_n^{\lambda_n}$, by C. Gouwens; *A uniqueness theorem for Legendre and Hermite polynomials,* by K. P. Williams; *A new type of class number relations,* by E. T. Bell; *A new method in the equivalence of pairs of bilinear forms,* by R. G. D. Richardson; *Relative extrema of pairs of quadratic and hermitian forms,* by R. G. D. Richardson.

The concluding number of volume 46 of the American Journal of Mathematics (October, 1924) contains: *A general class of problems in approximation,* by D. Jackson; *On a rational plane quintic curve with four real cusps,* by P. Field; *Projective properties of a ruled surface in the neighborhood of a ruling,* by A. F. Carpenter; *Complete characterization of dynamical trajectories in n-space,* by L. M. Kells; *Rods of constant or variable cross section,* by C. A. Garabedian.

The first of the Carus Mathematical Monographs (see this Bulletin, vol. 28, p. 74) has been published recently by the Open Court Publishing Company under the auspices of the Mathematical Association of America. The general purpose of this series is to contribute to the dissemination of mathematical knowledge. The present volume, a book of 189 pages, is by Professor G. A. Bliss on the topic *Calculus of Variations*; it will be reviewed promptly in this Bulletin. Members of the Association may secure it at a reduced price through the

Secretary, Professor W. D. Cairns. It is announced that a second monograph, by Professor D. R. Curtiss on *Functions of a Complex Variable*, will appear in the near future.

The University of London has awarded its Sir John William Lubbock memorial prize in mathematics to Miss E. M. Jones, of Royal Holloway College.

On October 5, 1924, a tablet was unveiled at the birthplace of Charles Hermite, at Dieuze, France. Professors Henri Andoyer and Émile Picard represented the Paris Academy of Sciences at the ceremony.

Professor R. Weitzenböck, of the University of Amsterdam, has been elected a member of the Amsterdam Academy of Sciences.

Professor A. V. Vassilief, formerly of the Universities of Kazan and Petrograd, celebrated on December 12, 1924, the fiftieth anniversary of his entrance into teaching. Professor Vassilief was president of the Physico-Mathematical Society of Kazan at the time of the celebration of the centenary of Lobachevsky and the foundation of the Lobachevsky prize.

On the occasion of the meeting of the British Association for the Advancement of Science and the Mathematical Congress, the University of Toronto conferred honorary degrees on Sir David Bruce, Professor G. Koenigs, Sir Charles Parsons, Sir John Russell, Sir Ernest Rutherford (as announced in an earlier issue of this Bulletin), and Professors F. Severi, V. Stekloff, and C. de la Vallée Poussin.

The one hundreth anniversary of the announcement by Sadi Carnot of the second law of thermodynamics was celebrated in New York City under the auspices of the Engineering Foundation, on December 4, 1924 Addresses were delivered by Professor M. I. Pupin, of Columbia University, and Dr. W. L. Emmet, of the General Electric Company, on *Carnot's principle.*

On the occasion of the centenary of the Franklin Institute, the University of Pennsylvania conferred honorary degrees on Sir W.H.Bragg, Mr. W. C. L. Eglin, Dr. Charles Fabry, Sir Charles Parsons, Dr. E. W. Rice, Jr., and Dr. Pieter Zeeman.

The General Electric Company has appropriated to Union College a fund of $25,000 in memory of Dr. C. P. Steinmetz. The income is to be used for scholarships.

The Rensselaer Polytechnic Institute, of Troy, the oldest school of science and engineering in any English-speaking country, celebrated on October 3-4, 1924, the centenary of its foundation. Dr. G. A. Campbell, of the American Telephone and Telegraph Company, represented the American Mathematical Society at the ceremonies. On this occasion the Institute conferred the honorary degree of doctor of science on Professor A. A. Michelson, president of the National Academy of Sciences.

Professor A. A. Michelson was appointed one of the official representatives of the United States at the third Pan-American Congress, held at Lima, Peru, December 20, 1924—January 6, 1925. The delegates of the American Mathematical Society to this congress were Professors E. V. Huntington and D. N. Lehmer (see this BULLETIN, vol. 30, p. 566), but Professor Lehmer was unable to attend.

Professor A. Einstein will deliver a series of lectures on relativity at Buenos Aires in the spring of 1925, at the invitation of various Argentine scientific bodies.

Professor A. N. Whitehead, of Harvard University, will deliver a series of lectures on *Science and the modern world* under the auspices of the Lowell Institute, of Boston, beginning February 2, 1925.

Professor E. G. Bill, of Dartmouth College, has been appointed chairman of a committee created by the College Entrance Examination Board to consider the desirability of psychological tests to be given by that board.

Dr. Gustav Doetsch, of the University of Halle, has been called to a professorship at the Stuttgart Technical School.

Professor H. A. Wilson, of Rice Institute, has been appointed professor of natural philosophy at Glasgow University.

At St. Olaf College, Northfield, Minnesota, Professor Martin Nordgaard, formerly head of the department of mathematics at Antioch College, has been appointed professor and head of the department of mathematics, and Mr. Arthur Solum of the University of Minnesota has been appointed assistant professor.

At Union University, Schenectady, Professor S. A. Rowland and Mr. L. A. DeRonde have been granted leave of absence and are studying at Chicago and Harvard, respectively.

Mr. G. F. Abrich has been appointed assistant professor of mathematics at Des Moines University.

Dr. G. N. Bauer, has been appointed associate professor of mathematics at the University of New Hampshire.

Professor F. P. Brackett of Pomona College has been granted leave of absence for the first semester of 1924–1925, and is travelling in Europe.

Assistant Professor F. W. Bubb, of Washington University, St. Louis, is on leave of absence for the academic year 1924–1925 and is studying at the University of Chicago as a Research Council fellow in physics.

Miss Mary Campbell, of the University of Texas, has been appointed professor and head of the department of mathematics at South Park Junior College.

Assistant Professor L. J. Comrie of Swarthmore College has accepted a position in the observatory at Northwestern University.

Mr. Victor Doshkess, of Lafayette College, has been promoted to an assistant professorship of mathematics.

Mr. H. K. Fulmer, of the Georgia School of Technology, has been promoted to an assistant professorship of mathematics.

Dr. G. W. Hess has been appointed professor of mathematics and head of the department at Union University, Jackson, Tennessee

At Lehigh University, Professor P. A. Lambert has been made head of the department of mathematics and astronomy.

Mr W. H. Lyon has been appointed assistant professor of mathematics at the Kansas State Agricultural College.

Professor H. B. Mitchell, of Columbia University, has resigned to enter business

Mrs. R. B. Montgomery, of Lynchburg College, has been promoted to an assistant professorship of mathematics.

Miss Inez Morris has been appointed assistant professor of mathematics at the Indiana State Normal School, Terre Haute.

Dr. J. O. Osborn, of Cornell University, has been appointed teacher of mathematics at Harris Teachers College, St. Louis.

Dr. W. P. Ott, of Vanderbilt University, has been appointed head of the department of mathematics at the University of Alabama.

Dr. G. T. Pugh has been appointed professor of mathematics at Winthrop College.

At Catholic University, Associate Professor O. J. Ramler has been promoted to a full professorship of mathematics.

Mr. P. L. Rea, of Marietta College, has been promoted to an assistant professorship of mathematics.

Professor J. E. Rowe, of the College of William and Mary, has been made Director of Extension.

Mr. C. A. Rupp, of Hamline University, has been appointed adjunct professor of mathematics at the University of Texas.

Mr. J. H. Synnerdahl, of Lake Forest College, has been appointed assistant professor of mathematics at Park College, Parkville, Missouri.

Dr. J. S. Taylor, of the Massachusetts Institute of Technology, has been appointed assistant professor of mathematics at the University of Pittsburgh.

Miss Velma Tisdale, of Southwestern University, Georgetown, Texas, has been promoted to an assistant professorship of mathematics.

Mr. W. M. Whyburn, of South Park Junior College, has been appointed assistant professor of mathematics at the Agricultural and Mechanical College of Texas.

Dr. R. L. Wilder, of the University of Texas, has been appointed assistant professor of mathematics at Ohio State University.

Mr. W. L. Williams has been appointed adjunct professor at the University of South Carolina.

Professor Helge von Koch, of the University of Stockholm, died March 11, 1924.

The death is announced of Professor August Foppl, of the Munich Technical School, in his seventy-first year.

Professor Karl Wieghardt, of the Dresden Technical School, died June 11, 1924, at the age of fifty.

Mr. F. H. Bradley, fellow of Merton College, Oxford, and writer on logic and philosophy, died September 18, 1924, at the age of seventy-eight.

Dr. J. E. Campbell, F. R. S., fellow and tutor of Hertford College, Oxford, and mathematical lecturer at University College, Oxford, died October 1, 1924.

Professor C. B. Austin, of Ohio Wesleyan University, died September 9, 1924, at the age of seventy-three.

Professor E. L. Larkin, director of Mount Lowe Observatory, died October 11, 1924, at the age of seventy-seven.

The following have been appointed instructors in mathematics:

Athens College (Georgia), Miss Nancy L. Moorefield;
Brown University, Mr. F. C. Jonah;
University of Buffalo, Mr. H. D. Highet;
Case School of Applied Science, Mr. J. E. Merrill;
Georgia School of Technology, Messrs. G. S. Bruton, G. W. Nicholson, L. K. Patton, H. H. Pixsley, D. P. Richardson;
Hamline University, Mr. E. D. Wells;
University of Kansas, Mr. R. H. Marquis;
Lehigh University, Mr. C. A. Balof;
University of Maine, Mr. E. H. Hadlock;
Mercer University, Messrs. C. H. Berryman and J. L. Talley,
University of Michigan, Messrs. W. C. Green, A. P. Maslow; J. A. Shohat, G. S. Van Fleet;
Purdue University, Mr. J. C. Bennett;
University of South Carolina, Mr. R. L. Jones;
Southwestern University, Georgetown, Texas, Mr. J. C. Chaney;
Swarthmore College, Mr. D. B. McLaughlin;
University of Texas, Miss Elizabeth T. Stafford;
Union University (Schenectady), Messrs. L. W. Hussey and E. E. Steinert;
Washington State College, Miss Eleanor E. Boyd;
Washington University (St. Louis), Messrs. H. R. Grumann and E. H. Lund;
College of William and Mary, Miss Elizabeth Mercer;
University of Wisconsin, Mr. H. S. Pollard;
Wofford College, Mr. W. C. Herbert.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ANDREWS (E. S.) and HEYWOOD (H. B). The calculus for beginners 2d revised edition. London, Scott, Greenwood and Son, 1924. 284 pp.

APPELL (P.). See ZORETTI (L.).

AUBRY (L.). Démonstration du postulat d'Euclide sur les parallèles. Paris, Gauthier-Villars, 1924. 12 pp.

BALDUS (R.). Formalismus und Intuitionismus in der Mathematik. Karlsruhe i. B., G. Braun, 1924. 45 pp.

BERTINI (E.). Einfuhrung in die projektive Geometrie mehrdimensionaler Räume mit einem Anhang uber algebraische Kurven und ihre Singularitäten. Nach der zweiten italienischen Auflage deutsch herausgegeben von A. Duschek. Wien, Seidel, 1924. 22+480 pp.

BUSCO (P.). Les cosmogonies modernes et la théorie de la connaissance. Paris, Alcan, 1924. 436 pp.

CAHEN (E.). Théorie des nombres. Tome II: Le second degré binaire. Paris, Hermann, 1924. 10+736 pp.

CHATELET (A.) et KAMPÉ de FÉRIET (J.). Calcul vectoriel. Théorie. Applications géométriques et cinématiques. Paris, Gauthier-Villars, 1924. 426 pp.

DINGLER (H.). Die Grundlagen der Machschen Philosophie mit Erstveröffentlichungen aus seinen wissenschaftlichen Tagebuchern. Leipzig, Barth, 1924. 106 pp.

DUSCHEK (A.). See BERTINI (E.).

FOUET (E. A.). Leçons de géométrie élémentaire. Paris, Vuibert, 1924. 16+348 pp.

HARDING (A. M.) and MULLINS (G. W.). Analytic geometry. New York, Macmillian, 1924. 8+312 pp.

HARDY (G. H.). Orders of infinity. The "Infinitarcalcul" of Paul du Bois-Reymond. (Cambridge Tracts, No. 12.) Cambridge, University Press, 1924. 6+77 pp.

HEYWOOD (H. B.). See ANDREWS (E. S.).

KAMPÉ DE FÉRIET (J.). See CHATELET (A.).

KING (L. V.). On the direct numerical calculation of elliptic functions and integrals. Cambridge, University Press, 1924. 8+42 pp.

KLEIN (F.). Elementarmathematik vom hoheren Standpunkte aus. 3te Auflage. Band 1. (Die Grundlehren der mathematischen Wissenschaften, Band 14.) Berlin, Springer, 1924. 320 pp.

KNOPP (K.). Theorie und Anwendung der unendlichen Reihen. 2te erweiterte Auflage. (Die Grundlehren der mathematischen Wissenschaften, Band 2.) Berlin, Springer, 1924. 10+526 pp.

LICHTENSTEIN (L.). Neuere Entwicklung der Theorie partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus. (Sonderabdruck aus der Encyklopädie der mathematischen Wissenschaften.) Leipzig, Teubner, 1924. 58 pp.

LIETZMANN (W.). Methodik des mathematischen Unterrichts. Teil 3: Didaktik der angewandten Mathematik. Leipzig, Quelle und Meyer, 1924. 12+234 pp.

MEHMKE (R.). Leitfaden zum graphischen Rechnen. 2te, vermehrte und verbesserte Auflage. Leipzig und Wien, Deuticke, 1924. 8+185 pp.

MOREUX (T.). Pour comprendre le calcul différentiel. Paris, Doin, 1924. 238 pp.

—— Pour comprendre la géométrie dans l'espace. Paris, Doin, 1923. 7+209 pp.

MUIÇA (I.). Théorème de Fermat. 4e édition. Bucarest, Cartea Romaneasca, 1924. 8 pp.

MULLINS (G. W.). See HARDING (A. M.).

NÖRLUND (N. E.). Vorlesungen über Differenzenrechnung. (Die Grundlehren der mathematischen Wissenschaften, Band 13.) Berlin, Springer, 1924. 10+551 pp.

OSGOOD (W. F.). Lehrbuch der Funktionentheorie. 2ter Band, 1te Lieferung. Leipzig, Teubner, 1924. 6+242 pp.

PALMER (C. I.). Practical calculus for home study. London and New York, McGraw-Hill, 1924. 20+443 pp.

POINCARÉ (H.). Des fondements de la géométrie. Paris, Chiron, 1923. 64 pp.

REICHENBACH (H.). Axiomatik der relativistischen Raum-Zeit-Lehre. Braunschweig, Vieweg, 1924. 8+156 pp.

REIFF (M. T.). Bewegungen und Umlegungen in der elliptischen Hermiteschen Geometrie der Ebene. (Dissertation.) Bonn, Rhenania-Druckerei, 1923.

SMITH (D. E.). Historical-mathematical Paris. Paris, Les Presses Universitaires de France, 1924. 48 pp.

TROPFKE (J.). Geschichte der Elementar-Mathematik. 2te, verbesserte und sehr vermehrte Auflage. Band 7: Stereometrie. Verzeichnisse. Berlin, Springer, 1924. 128 pp.

VERITY (E. A.). Mathematics for technical students. London and New York, Longmans, 1924. 12+468 pp.

DE VILLERS (M.). The numeral-words. Their origin, meaning, history and lesson. London, Witherby, 1923. 124 pp.

ZORETTI (L.). Leçons de mathématiques générales. Avec une préface de P. Appell. 2e édition, revue et considérablement augmentée. Paris, Gauthier-Villars, 1925. 16+788 pp.

PART II. APPLIED MATHEMATICS

ASTON (F.W.). Isotopes. 2d edition. London, Arnold, 1924. 21+182 pp.

BARTON (E. H.). Analytical mechanics comprising the kinetics and statics of solids and fluids. 2d edition, revised and enlarged. New York and London, Longmans, 1924. 22+593 pp. $6.20

BERTIN (C.). Marins et aviateurs: Faites comme moi et vos calculs seront faits sans erreurs et sans logarithmes aussi vites que nos observations. Paris, Gauthier-Villars, 1924. 70 pp.

BOGGIO (T.). See BURALI-FORTI (C.).

BOHR (N.). Über den Bau der Atome. Berlin, Springer, 1923. 60 pp.

BOLTON (L.). Time measurement. An introduction to means and ways of reckoning physical and civil time. London, Bell, 1924. 8+166 pp.+8 plates.

BOUASSE (H.). Jets, tubes et canaux. Paris, Delagrave, 1923. 544 pp.

BOWLEY (A. L.). The mathematical groundwork of economics. Oxford, Clarendon Press, and New York, Oxford University Press, 1924. 98 pp. $2.35

BURALI-FORTI (C.) et BOGGIO (T.). Espaces courbes. Critique de la relativité. Torino, Società Tipografico-Editrice Nazionale, 1924. 24+255 pp.

CASPAR (M.). See KEPLER (J.).

COLARDEAU (E.). Traité général de stéréoscopie. Paris, Mendel, 1924. 225 pp.

CRANSTON (J. A.). The structure of matter. London, Blackie, 1924. 16+196 pp.

DAHLMANN (W.). Festigkeit der Schiffe. Berlin, Springer, 1924. 320 pp.

DELLA NOCE (G.). See LAMMEL (R.).

DESSAU (B.). Lehrbuch der Physik. Vom Verfasser aus dem Italienischen übertragen. Band 2: Optik, Elektrizitätslehre. Leipzig, Barth, 1924. 6+932 pp.

DREYER (J L. E.) and TURNER (H. H.), editors. History of the Royal Astronomical Society, 1820–1920. London, Wheldon and Wesley, 1923. 7+258 pp.+12 plates.

HAMILTON (E. H.). Elementary thermodynamics of automobile engines. New York and London, McGraw-Hill, 1923. 11+287 pp.

HAUDIÉ (E.). Cours d'électricité générale de l'Ecole navale. 3 volumes. Paris, Challamel, 1921, 1922, 1924.

HENDERSON (A.), HOBBS (A. W.), and LASLEY (J. W.). The theory of relativity. Studies and contributions. Chapel Hill, University of North Carolina Press, and London, Oxford University Press, 1924. 13+99 pp. $2.50

Henker (O.). Introduction to the theory of spectacles. Translated by R. Kanthack. London, J. W. Atha, 1924. 8 + 336 pp.

Heyl (P. R.). The common sense of the theory of relativity. Baltimore, Williams and Wilkins, 1924. 44 pp.

Hobbs (A. W.). See Henderson (A.).

Kanthack (R.). See Henker (O.).

Kepler (J.). Mysterium cosmographicum. Das Weltgeheimnis. Übersetzt und eingeleitet von Max Caspar. Augsburg, Dr. Benno Filser Verlag, 1923.

Krainsky (N.). Energetik der mechanischen Erscheinungen und die Grundlagen der Energetik, der Elektrophysik und Thermodynamik. Zlatar, V. Suligoj, 1924. 192 pp.

Lamb (H.). The evolution of mathematical physics, being the Rouse Ball Lecture for 1924. Cambridge, University Press, 1924. 48 pp.

Lämmel (R.). I fondamenti della teoria della relatività con introduzione del traduttorre G. Della Noce. Bologna, Zanichelli, 1923. 134 pp.

Langevin (P.). Le principe de la relativité. Paris, Chiron, 1924. 64 pp.

Lasley (J. W.). See Henderson (A.).

Nottage (W. H.). The calculation and measurement of inductance and capacity. London and New York, Wireless Press, 1924. 8 + 224 pp.

Peters (J.). Die mathematischen und physikalischen Grundlagen der Musik. (Mathematisch-physikalische Bibliothek, Nr. 55.) Leipzig, Teubner, 1924.

Poincaré (H.). La mécanique nouvelle. Paris, Gauthier-Villars, 1924. 81 pp.

Prasad (G.). The place of partial differential equations in mathematical physics, being a course of readership lectures delivered at Patna University in 1921. Calcutta, Patna University, 1924. 4 + 49 pp.

Reynaud-Bonin (E.). Appareils et installations téléphoniques. Paris, Baillière, 1924. 487 pp.

Straneo (P.). Teoria della relatività. Saggio di una esposizione secondo il senso fisico. Roma, Libreria di Scienze e Lettere, 1924, 161 pp.

Tummers (J. H.). Die spezielle Relativitätstheorie Einsteins und die Logik. Venlo, 1924. 15 pp.

Turner (H. H.). See Dreyer (J. L. E.).

Versluys (J.). Perspectief. Vijfde druk. Tweede deel. Groningen, Noordhoff, 1924. 174 pp.

Warburg (E.). Über Wärmeleitung und andere ausgleichende Vorgänge. Berlin, Springer, 1924. 10 + 106 pp.

Weyl (H.). Was ist Materie? Zwei Aufsätze zur Naturphilosophie. Berlin, Springer, 1924.

CALCUTTA
LIBR

THE SECOND AWARD OF THE BÔCHER MEMORIAL PRIZE

The Annual Meeting of 1924 was marked by the second award of the Bôcher Memorial Prize. On recommendation of the Committee, this prize, which was offered for a notable research memoir in volumes 19–24 of the TRANSACTIONS, was divided equally between Eric Temple Bell for a memoir entitled *Arithmetical paraphrases,* published in volume 22, and Solomon Lefschetz for a memoir entitled *On certain numerical invariants of algebraic varieties with applications to Abelian varieties,* in the same volume.

The presentation was made at the beginning of the afternoon session on December 30 by President Birkhoff as chairman of the committee. After some personal references to the mathematician whose name this prize commemorates, he gave the following brief resumés of the papers as analyzed by the committee.

"The memoir by Bell is a fundamental contribution to the theory of numbers. That field of mathematics is largely an aggregation of special results and methods. What have been needed are general principles, serving to unify and extend. Such a general principle is given by Bell. This principle may be stated very simply for a special but a typical case. Suppose that by the development of an elliptic or theta function we have found a linear identity between the sines of the angles a_ix+b_iy for $i=1, 2, \cdots$, then if $f(a, b)$ is an arbitrary even function, Bell shows that we have the like linear identity between the $f(a_i, b_i)$. Since we are free to choose the even function, we may deduce as many arithmetical facts as we please from the initial identity. Similarly from a linear identity between cosines we derive a corresponding identity for an arbitrary odd function. These are merely the simplest cases of the endless varieties of paraphrases, each involving arbitrary functions.

"The memoir by Lefschetz constitutes an important contribution to the theory of algebraic varieties. In the first part of an extensive paper, rich in detail and employing with facility the methods of algebra, geometry, analysis, the notion of algebraic cycles plays a central part. It is shown for example that a double integral of the first kind on an algebraic surface has no periods relative to algebraic cycles, and that Picard's number ϱ is equal to the number of two-cycles without periods, for integrals of the first kind. Here also is found the first rigorous proof of Noether's theorem on complete intersections, and the extension stating that a non-singular variety of d dimensions ($d > 3$), which is itself a complete intersection in space of r dimensions ($r > 4$) contains only hypersurfaces which are themselves complete intersections. The second part of the paper deals with Abelian varieties, and in particular it contains an evaluation of ϱ for such varieties, known previously for the case $p = 2$. Many interesting and novel results concerning the relations between the various numerical invariants are found."

Professor Lefschetz, who was present, in acknowledging the honor done him, spoke of the present standing of the American school of mathematics.

The Committee on award appointed by the Council consisted of Professor G. D. Birkhoff, L. E. Dickson and H. S. White.

The third award, to be announced at the Annual Meeting of 1928, is to be made with reference to the volumes of the TRANSACTIONS for the years 1923–1927 inclusive.*

R. G. D. RICHARDSON,
Secretary.

---

* See this BULLETIN, vol. 30 (1924), p. 193.

THE WESTERN CHRISTMAS MEETING OF THE SOCIETY

The twenty-second Western meeting of the Society was held on Friday, December 26, 1924, at the University of Chicago. The first session opened at 10 a. m. in Room 32, Ryerson Laboratory.

The attendance at this meeting numbered about sixty, including the following fifty-two members of the Society:

F. E. Allen, Barnard, Barnett, Bliss, O. E. Brown, Carmichael, Chapelon, H. H. Conwell, H. T. Davis, Dickson, Dostal, Dresden, Edington, Garabedian, Gouwens, G. H. Graves, V. G. Grove, Hildebrandt, Hodge, Hotelling, Louis Ingold, Ingraham, Isaacs, Kempner, Kinney, Kouperman, E. P. Lane, Latimer, Laves, Libman, March, G. A. Miller, Miser, E. H. Moore, E. J. Moulton, C. I. Palmer, Plapp, Rider, Roth, Rowland, Schottenfels, Shaw, Shohat, Simmons, Slaught, E. L. Thompson, J. S. Turner, Van Vleck, Vass, M. B. White, Frederick Wood, F. E. Wood.

The committee appointed to arrange for a symposium at the meeting in April 1925 reported that the symposium lecture, on "Some of the mathematical aspects of cosmology", would be given by Professor W. D. MacMillan.

At the dinner held on Friday evening, at the Del Prado Hotel, 34 persons were present.

Professor Hildebrandt, Vice-President of the Society, presided over the sessions, at which the following papers were presented. The papers of Professors MacNeish, Wahlin, Betz, Dodd and Ettlinger were read by title.

1. Professor F. E. Wood: *Sense relations between the pairs of corresponding triangles of a Desargues configuration.*

In this paper an investigation is made of the relative senses of the triangles in each of the ten pairs of perspective triangles of a Desargues configuration, starting with axioms and theorems developed in the foundations of geometry. A unique canonical lettering for a general Desargues configuration is set up and seven types of Desargues configurations with the canonical lettering are discussed. It is proved that the triangles of each of the ten pairs will have the same sense if and only if each vertex is outside both the triangles corresponding to it, that the senses of the triangles of each of four and only four pairs are the same if and only if there is a vertex inside both its corresponding triangles, while if neither of the above two conditions is satisfied then the senses of the triangles of each of six and only six pairs are the same.

2. Professor E. P. Lane: *Bundles and pencils of nets on a surface.*

A bundle of nets on a surface is the class of nets such that the two component families of curves of each can be taken in such an order that at every surface point its tangents form with the tangents of a fundamental net the same cross ratio. The class of conjugate nets is a bundle for which the fundamental net is the asymptotic net and the constant cross ratio is —1. A pencil of nets in a bundle is a one-parameter family of nets in the bundle such that at every surface point the two tangents of each net form with the tangents of a fundamental net of the bundle a constant cross ratio.

This paper develops a general theory of bundles and pencils of nets. The theory finds application in a critical examination of the existing theory of conjugate nets; it shows what features of this theory are due to the constancy of the cross ratio and what features are due to the size of this constant. In these studies the importance of the ray-conic is again emphasized.

3. Professor V. G. Grove: *A theory of a general net on a surface.*

In this paper, the author sets up a complete system of invariants and covariants of the differential equations arising in the study of a general net. Necessary and sufficient conditions that a net be identically self dual are established. An extended study of Green's relation R between two congruences is made. There exists one and only one pair of congruences in relation R with respect to a general net, whose lines are at the same time reciprocal polars with respect to both quadrics osculating the parametric ruled surfaces of tangents. The associate conjugate net of a general net is discussed. It is shown that many of the properties of the associate conjugate net of a conjugate net hold when the net is general.

4. Mr. J. A. Shohat: *On the polynomial of the best approximation to a given continuous function.*

Consider two functions $f(x)$ and $p(x)$ defined on a finite interval (a, b); $f(x)$ is bounded and integrable; $p(x)$ is integrable and not negative. Assuming the existence of two numbers α, β such that $\int_{x_1}^{x_2} p(x)\,dx > 0$ $(\alpha \leqq x_1 < x_2 \leqq \beta$; $a \leqq \alpha$; $\beta \leqq b)$, the author proves the existence and the uniqueness of a polynomial $U_{n,k}(x)$ of degree $\leqq n$, minimizing the integral $\int_a^b p(x) | f(x) - U_{n,k}(x) |^k dx$, for any $k \geqq 1$.

Furthermore, if $f(x)$ is continuous and $\int_\alpha^\beta p(x)\,dx > 0$ for any α, β belonging to (a, b), then the said minimizing polynomial tends for $k \rightarrow \infty$ uniformly to the polynomial of the nth degree of the best approximation (in Tchebycheff's sense) in (a, b) to the function $f(x)$. Similar results hold for trigonometric approximation.

The results given above are a generalization of those obtained by G. Pólya (COMPTES RENDUS, 1913) and by D. Jackson (see the TRANSACTIONS and BULLETIN of the Society, 1921–1924).

5. Professor P. R. Rider: *The figuratrix in the calculus of variations.*

This paper considers integrals of the form $\int_{t_1}^{t_2} f(x, y, \tau) \sqrt{x'^2 + y'^2}\,dt$, $\int_{t_1}^{t_2} f(x, y, z, \tau, \sigma) \sqrt{x'^2 + y'^2 + z'^2}\,dt$, in which the arguments of f are functions of t. The quantities τ and σ are angles defined by the equations $\tan \tau = y'/x'$, $\sin \sigma = z'/\sqrt{x'^2 + y'^2 + z'^2}$. For the first integral, the figuratrix of the point (x, y) is defined as the envelope of the variable line $p \cos \tau + q \sin \tau = f(\tau)$, p and q being rectangular coordinates with reference to the point $P(x, y)$ as origin; for the second integral, the figuratrix of the point (x, y, z) is defined as the envelope of the two-parameter family of planes $p \cos \tau \cos \sigma + q \sin \tau \cos \sigma + r \sin \sigma = f(\tau, \sigma)$. The figuratrix as thus defined is proved to be identical with the figuratrix as defined by Hadamard (*Leçons sur le Calcul des Variations*, vol. I, pp. 92, 96). Attention is called to a number of the interesting properties of the figuratrix which enable one to interpret geometrically some of the well known functions and theorems of the calculus of variations. The results are generalized to the case of n dimensions.

6. Dr. H. F. MacNeish: *Theorems concerning transversals of the* $(n+1)$*-hedron in* n*-dimensional space.*

The line edges $A_i^0A_j^0 = a_{ij}$ of the $(n+1)$-hedron $A_1^0A_2^0 \cdots A_{n+1}^0$ in n-space will be said to be divided antisymmetrically in the ratio $a_i : a_j$ by the point A_{ij}^0 if the ratio $A_i^0A_{ij}^0 : A_{ij}^0A_j^0 = a_j : a_i$ where a_i, a_j, etc., are constants. In a triangle the 3 lines $A_i^0A_{jk}^0$ concur in the point A_{ijk}^0 which is said to divide the triangle antisymmetrically in the ratio $a_i : a_j : a_k$. The ratio $A_i^0A_{ijk}^0 : A_{ijk}^0A_{jk} = a_j + a_k : a_i$ and $(A_i^0A_{jk}^0)^2 = (a_ja_{ik}^2 + a_ka_{ij}^2)/(a_j + a_k) - a_ja_ka_{jk}^2/(a_j + a_k)^2$. In a tetrahedron the following 7 lines concur in a point which is said to divide the tetrahedron antisymmetrically in a fixed ratio: (a) the 4 lines drawn from the vertices to the antisymmetric points of the opposite faces; (b) the 3 lines joining the antisymmetric points of the pairs of opposite edges. The three results (a) concurrence of transversals, (b) formulas for the lengths of the transversals, (c) the ratio of the segments of the transversals, stated above for 2 dimensions, are generalized to n dimensions.

7. Professor L. E. Dickson: *A new theory of the rational equivalence of linear transformations or pairs of bilinear forms.*

Let a linear transformation S on $\xi_1, \cdots, \xi_n$ with coefficients in any field F, the case of zero determinant not being excluded, replace any linear function x_1 of the ξ's by $x_1' = x_2$, x_2 by $x_2' = x_3, \cdots, x_{a-1}$ by $x_{a-1}' = x_a$, but x_a by a linear function $[x_1, \cdots, x_a]$ of those x's. Choose the leader x_1 so that a shall be the maximum length of all possible chains $x_1, \cdots, x_a$. If $n > a$, let b be the maximum length of a chain whose leader is linearly independent of $x_1, \cdots, x_a$. Continuing in this manner we may reduce S to a canonical form. The characteristic determinant of the partial transformation on any chain is divisible by that of the next chain. Apart from sign, these determinants are in reverse order the invariant factors, other than unity, of S. Two transformations S and T are similar in F (i. e., there exists a non-singular matrix B in F such that $BSB^{-1} = T$) if and only if the λ-matrices of S and T have the same invariant factors. Two pairs of n-rowed square matrices M, N and M_1, N_1 with elements in any field F, N and N_1 non-singular, are equivalent in F if and only if $M - \lambda N$ and $M_1 - \lambda N_1$ have the same invariant factors. A like theorem follows when N and N_1 are singular, but the determinants of $\varrho M+\sigma N$ and $\varrho M_1+\sigma N_1$ are not zero identically in ϱ and σ.

8. Professor Dickson: *Rational theory of pairs of bilinear forms in the singular case.*

The reduction of a pair of bilinear forms φ and ψ in the singular case in which the determinant of $u\varphi+v\psi$ is zero identically in u and v was first accomplished by Kronecker in the BERLINER SITZUNGSBERICHTE, 1890, pp. 1225–37. By a rational preliminary transformation he segregated component bilinear forms φ and ψ belonging to the non-singular case. For the latter, he employed the irrational canonical pair due to Weierstrass. Instead of these, we may employ the rational canonical pair which follows from the author's preceding paper. By purely rational transformation we may therefore reduce the initial φ and ψ to a canonical pair. The conclusion is that, in the singular case, two pairs of bilinear forms in the same $m+n$ variables with coefficients in any field F are equivalent in F if and only if they have the same invariant factors and the same minimal numbers, as defined by Kronecker.

9. Professor G. E. Wahlin: *On the solution of diophantine equations by means of ideals.*

This paper will appear in full in an early number of this BULLETIN.

10. Professor G. A. Miller: *Subgroup composed of the substitutions which omit a letter of a transitive group.*

The main object of this paper is to study transitive substitution groups G from the standpoint of the subgroups G_1 composed of all the substitutions which omit one letter. Among the theorems established are the following: The number of the transitive groups of degree $2k+1$ which have for their G_1 the group of order 2 and of degree $2k$ is the same as the number of the abstract abelian groups of order $2k+1$, k being an arbitrary positive integer. When G_1 is of order 2 and of degree 2^α the number of the possible degrees of the transitive groups which involve G_1 as the subgroup composed of all the substitutions which omit one letter is $\alpha+1$, where α is an arbitrary positive integer. A regular group of order

2^α, $\alpha > 2$, cannot appear as the G_1 of a transitive group of degree $2^\alpha + 2$. If a regular abelian group of order m is the G_1 of a transitive group of degree $m + k$ then $m \leqq 2k$.

11. Professor W. E. Edington: *The transform in the abstract definitions of groups.*

In this paper a study is made of some of the conditions in special cases imposed by the transform $s_2^{-1} s_1 s_2 = s_1^\alpha s_2^\beta$ in the definition of a group whose generators are s_1 and s_2. Also the general relations that must hold between the orders of the generators and the orders of the products of powers of the generators when $s_2^{-1} s_1 s_2 = s_1^\alpha$ are determined and the definitions of certain infinite systems of groups are discussed. Some interesting properties of the transform considered as a substitution are also stated.

12. Professor H. T. Davis: *Asymptotic distribution of characteristic numbers in the problem of the elastic bar.*

The problem of the elastic bar may be generalized by generalizing the Sturmian boundary value problem of fourth order to which it leads. The author has previously derived sufficient conditions that the characteristic numbers of this Sturmian problem shall alternate with the λ-zeros of a solution of a certain Cauchy problem of fifth order. In this paper a study is made of the Cauchy problem and the asymptotic form of the λ-zeros determined in terms of the coefficients of the Sturmian system.

13. Professor H. Betz: *Surface transformations applied to special dynamical problems.*

This paper applies to actual dynamical problems with two degrees of freedom, the surface transformation methods developed by Poincaré and Birkhoff. It has for its particular object the discovery of periodic motions, and the examination of their stability. The "ring of Poincaré" is constructed graphically affording an insight into the nature of the totality of motions. One periodic motion is then singled out for detailed investigation and extensive calculations are made in order to determine its stability.

14. Professor E. L. Dodd: *The frequency law of a function of several variables with given frequency laws.*

The functions considered are characterized by such properties as continuity and limited variation; and formulas are proved for effecting the required integrations, with special attention to justifying changes in the order of integration. The general theory is applied to determine the frequency law for the sum of projections and for the sum of cubes, under stated hypotheses regarding the frequency law for the arguments.

15. Professor L. Ingold: *Note on isogonal trajectories of geodesics.*

In this note conditions are determined on the function $a(u, v)$ in order that the curves $a =$ const. shall be isogonal trajectories of a family of geodesics $c =$ const. This special case of "parallel displacement", so called, is easily treated by vector methods. The resulting formulas are quite similar to the formulas of the more general case.

16. Professor I. A. Barnett: *The parameter group of a continuous group in function space leaving a manifold invariant.*

This paper extends Kowalewski's work on groups in function space to groups involving any number of functions in any number of variables. Three applications of such groups are given, viz., the parameter group of a given group, the adjoint group, and the parameter group of a given group leaving manifolds in function space invariant. Functional invariants and covariants of the manifolds and the forms defining them are also considered.

17. Professor H. J. Ettlinger: *Existence theorems for differential equations.*

In this paper the author proves the existence of solutions of a first order ordinary differential equation by methods analogous to the proof already given for the integrability of continuous functions. Superior and inferior integrals are obtained, which are analogous to the Darboux upper and lower integrals.

18. Professor A. J. Kempner: *Polynomials and their residue systems.*

In a paper bearing the same title (TRANSACTIONS OF THIS SOCIETY, vol. 22) the residue system of a polynomial of a single variable with integral rational coefficients with respect to a modulus m (m a rational integer, prime or composite) was examined. The investigation is now extended to the case of a polynomial of more than one variable.

19. Dr. Harold Hotelling: *Economic problems involving maxima of functionals.*

The assumption is made that an entrepreneur will do all in his power to maximize an integral representing his future profits, discounted according to remoteness in time. A second assumption—usually unjustified in a capitalist society—that a machine will be operated to full capacity, reduces the problem of depreciation of replaceable assets to one in ordinary differential calculus. This is elaborated, with a refinement obtained by solving a Volterra equation, in a paper offered to the JOURNAL OF THE AMERICAN STATISTICAL ASSOCIATION.

The first assumption alone reduces many problems of economics to those involved in maximizing functionals, and in particular to problems of the calculus of variations. The very puzzling problem of mining economics reduces by an immediate first integration of the Euler equations to a system of first-order equations. This solves the problem of depreciation of irreplaceable assets, both under competition and under monopoly. This application is in fact broader than economics, since eudaemonistic ethical theories hold that an integral of future happiness is to be maximized, and so reduce right conduct to a problem in functionals.

20. Professor J. B. Shaw: *On certain linear algebras.*

This paper is concerned with the synthetic construction of algebras from their sub-algebras; particularly those that satisfy a relation $\xi = \Phi(\xi)\sigma$, where ξ is to be any and every number of the sub-algebra, Φ is a linear operator, constant as to ξ, dependent upon σ, and σ is a number not in the sub-algebra. An instance is Dickson's representation of Cayley's octave algebra in the form $q_1 + q_2$, where q_1 and q_2 are quaternions, and $q = \bar{q}\omega$. To be associative, $\Phi(\xi_1)\cdot\Phi(\xi_2) = \Phi(\xi_1\xi_2)$, that is, Φ must leave the sub-algebra automorphic in multiplication. The octave algebra mentioned is not associative. Examples are given from abstract groups considered as algebras.

ARNOLD DRESDEN,
Assistant Secretary.

THE THIRTY-FIRST ANNUAL MEETING OF THE SOCIETY

The thirty-first Annual Meeting of the American Mathematical Society was held at George Washington University, December 29–January 1, inclusive, in conjunction with the meetings of the Mathematical Association of America and the American Association for the Advancement of Science. Monday afternoon and Tuesday morning and afternoon were devoted to the regular scientific papers delivered before the Society. On Wednesday and Thursday mornings there were joint sessions with papers given by request; and on Wednesday and Thursday afternoons the Mathematical Association held its regular sessions. The meeting was brought to a conclusion by the dinner held jointly on Thursday evening with the Mathematical Association and Section A at the headquarters of the mathematicians, the Franklin Square Hotel. President Coolidge of the Mathematical Association presided at this function and some one hundred twenty-five were in attendance.

At the end of the Tuesday morning session it was voted to express to George Washington University and the department of mathematics the thanks of the Society for its cordial hospitality during this meeting, the largest in the history of the Society, and one of the most pleasant and most successful.

The attendance included the following one hundred fifty-five members of the Society:

C. R. Adams, O. S. Adams, R. B. Adams, Alexander, Archibald, G. N. Armstrong, Atchison, Bacon, Barney, W. J. Berry, Birkhoff, Bliss, Borden, E. W. Brown, H. S. Brown, Bruce, J. A. Bullard, W. G. Bullard, Cairns, Cajori, Caldwell, Carmichael, Chapelon, Chittenden, Clements, Clutz, Abraham Cohen, Coolidge, Currier, J. E. Davis, Denton, Dickinson, Dillingham, Dresden, Eiesland, English, Erwin, Everett, Feinler, Finkel, Fite, Foberg, Focke, Fry, Garretson, Gehman, Gerst, Glenn, Grant, Gravatt, L. M. Graves, Guggenbühl, Hancock, J. G. Hardy, G. A. Harter, Hawkesworth, Heal, E. R. Hedrick, Archibald Henderson, Robert Henderson, Hickson, Hildebrandt, Hille, Himwich, Hodgkins, Hughes, Hulbert, Hurwitz, Ingels, Karpinski, Kennelly, Kindle, Kline, W. D. Lambert, Landry, Latimer, Lefschetz, Lehr, F. P. Lewis, Lunn, W. D. MacMillan, Manson, Mathewson, B. I. Miller, J. A. Miller, Mirick, H. H. Mitchell, C. N. Moore, R. L. Moore, Frank Morley, C. C. Morris, Marston Morse, F. R. Moulton, Murnaghan, F. H. Murray, Musselman, C. A. Nelson, Olds, Packer, Pell, F. W. Perkins, Plant, Rainich, Ramler, W. W. Rankin, Ranum, Rawlins, Remick, F. G. Reynolds, J. N. Rice, R. G. D. Richardson, H. L. Rietz, Ritt, Robert, G. E. Robinson, E. D Roe, Roever, Root, Safford, J. B. Scarborough, Schoonmaker, Seely, Seidlin, Sharpe, Shenton, Siceloff, Silverman, T. McN. Simpson. Sinclair, Slaught, C. E. Smith, D. E. Smith, S. E. Smith, W. M. Smith, Synge, J. S. Taylor, J. M. Thomas, B. M. Turner, Tyler, Uhler, Van Orstrand, Veblen, J. L. Walsh, Wernicke, Whited, Widder, Wiener, Wiggin, A. H. Wilson, E. W. Wilson, Woodard, Woolard, Wyant, J. W. Young.

The following organizations were elected to Sustaining Membership:

Connecticut General Life Insurance Company;
Phoenix Mutual Life Insurance Company,
Sargent and Lundy, Construction Engineers.

The Council elected the following twenty-six persons to ordinary membership in the Society:

Mr. William Clarke Arnold, DePauw University;
Mr. Clyde Myron Cramlet, University of Washington;
Mr. Haskell Brooks Curry, Harvard University;
Mr. John H. Evans, Ohio National Life Insurance Company;
Miss Beatrice Alethea Fenner, Palo Alto;
Professor Robert Edwin Gaines, University of Richmond;
Mr. R. L. Grossnickle, University of Minnesota;
Mr. George Pryor Harmount, East High School, Columbus;
Professor Clyde Monroe Hobart, Northern Illinois State Teachers College;
Professor Theodore Jesse Hoover, Stanford University;
Mr. Luther William Hussey, Union College,
Mr. Arthur R. Jerbert, University of Washington;
Miss Myra I. Johnson, New York City;
Mr. Burton Wadsworth Jones, Western Reserve University;
Mr. John Joseph McCarthy, New York University;
Miss Muriel L. Metz, University of Cincinnati;
Professor Allan Benton Morton, Georgia School of Technology;
Professor William Pinkerton Ott, University of Alabama,
Mr. Paul E. Remington, Mountain States Telephone and Telegraph Company;
Dean Jonathan C Rogers, Piedmont College;
Mr. Sergei Alexander Schelkunoff, Western Electric Company;
Professor Walter Francis Shenton, United States Naval Academy;
Mr. George W. Sublette, Minneapolis,
Mr. Walter Phillips Suesman, Providence;
Mr. William Herman Thurow, Technische Hochschule, Charlottenburg;
Miss Mabel Gertrude Whiting, Santa Ana Junior College.

The following twenty-six persons were elected to membership as nominees of sustaining members:

Mr. Horace Richardson Bassford, Metropolitan Life Insurance Company;
Mr. Lewis Warrington Chubb, Westinghouse Company;
Mr. Frank Conrad, Westinghouse Company;
Mr. Walter Dietz, Westinghouse Company;

Mr. George Myron Eaton, Westinghouse Company;
Mr. Robert D. Evans, Westinghouse Company;
Mr. Laurence R. Golladay, Westinghouse Company;
Mr. Clinton Richards Hanna, Westinghouse Company;
Mr. William J. Hudson, Westinghouse Company;
Mr. Robertson G. Hunter, Equitable Life Insurance Company of Iowa;
Mr. Philip C. Irwin, Equitable Life Insurance Company of Iowa;
Mr. Charles Jablow, Westinghouse Company,
Mr. Edwin W. Kopf, Metropolitan Life Insurance Company;
Mr. Reginald Campbell McCankie, Equitable Life Insurance Company of Iowa;
Mr. W. D. MacKinnon, Equitable Life Insurance Company of Iowa;
Mr. Samuel Milligan, Metropolitan Life Insurance Company;
Mr. Henry Scholte Nollen, Equitable Life Insurance Company of Iowa;
Mr. J. Ormondroyd, Westinghouse Company;
Mr. Robert James Piersol, Westinghouse Company;
Mr. Jonathan Gumersol Sharp, Metropolitan Life Insurance Company;
Mr. C. E. Skinner, Westinghouse Company;
Mr. Francis M. Smith, Metropolitan Life Insurance Company;
Mr. Carl Richard Soderberg, Westinghouse Company;
Mr. Hans Christian Specht, Westinghouse Company;
Mr. Dayton Ulrey, Westinghouse Company;
Mr. D. I. Vinogradoff, Westinghouse Company.

The ordinary membership in the Society is now 1477, including 94 nominees of sustaining members and 81 life members. There are also 19 sustaining members the total amount of whose annual dues is $3650. The total attendance of members at all meetings, including sectional meetings, during the past year was 509; the number of papers read was 231. The number of members attending at least one meeting was 365. At the annual election 210 votes were cast.

The reports of the Treasurer and the auditors (Mr. S. A. Joffe and Professor Tomlinson Fort) were received, showing a balance of $11,370.66, exclusive of special funds; of this, $5506.54 is reserved to secure the life memberships, and $2500.54 for publications for 1924 not yet completed. Sales of the Society's publications during the year amounted to $4938.58.

The Committee on Endowment announced that, including the pledges extending over five years, the Endowment Fund stands at approximately $60,000, of which $35,000 is paid in and invested.

The Board of Trustees adopted a budget for 1925, showing estimated expenditures and receipts as $16,072 and $15,400 respectively; the deficit can be met from the cash surplus brought forward from earlier years.

The Librarian reported that the Library of the Society now contains 6733 volumes. This library has more than doubled in size since the last catalogue was published, in 1910; it is proposed to issue a new catalogue in 1925.

At the annual election, which closed on Tuesday morning, the following trustees and officers and other members of the Council were chosen:

Board of Trustees, Professor G. D. Birkhoff, Professor L. P. Eisenhart, Professor W. B. Fite, Mr. Robert Henderson, Professor R. G. D. Richardson.

President, Professor G. D. Birkhoff.

Vice-President, Professor G. C. Evans.

Assistant Secretary, Professor Arnold Dresden.

Member of the Editorial Committee of the Bulletin, Professor E. R. Hedrick.

Member of the Editorial Committee of the Transactions, Professor H. H. Mitchell.

Members of the Council, Dr. G. A. Campbell, Professors E. W. Chittenden, A. J. Kempner, H. E. Slaught, Virgil Snyder.

The tellers appointed by President Veblen to count the ballots were Professor J. F. Ritt, Dr. C. R. Adams, and Dr. Caroline Seely.

At the meeting of the Council, Professor L. P. Eisenhart was named to succeed himself as representative of the Society on the National Research Council. Professors Arnold Dresden and E. B. Stouffer were appointed to represent the Society on the Council of the American Association for the Advancement of Science.

It was voted to hold the Annual Meeting for 1925 in New York City, and that for 1926 in Philadelphia.

Statistics were presented by the Committee on Membership which show that of the approximately twenty-seven hundred teachers of mathematics in the colleges of the United States and Canada only one-third are members of the Society. In the past four years the membership of the Society has been doubled. A resolution was passed thanking Professor Clara E. Smith for her untiring efforts in connection with the membership campaign.

An account of the proceedings connected with the presentation of the Bôcher Memorial Prize appears on page 193 of this issue.

The second Josiah Willard Gibbs Lecture was delivered on Tuesday evening by Mr. Robert Henderson, vice-president of the Equitable Life Assurance Society. This lecture, which appears in full in the present number of this BULLETIN, was entitled *Life insurance as a social service and as a mathematical problem.* Professor James Pierpont has been invited to deliver the next lecture of this series.

At the joint session of the Society, the Mathematical Association of America, and Section A, held on Wednesday morning, the following papers were read:

I. *Remarks on the foundations of geometry,* by Professor Oswald Veblen, retiring President of the Society. This paper appeared in full in the March-April number of this BULLETIN.

II. *The foundations of the theory of algebraic numbers,* by Professor Harris Hancock, retiring Vice-President of Section A.

The program of the joint session held on Thursday morning with the Mathematical Association and Sections A, B, and D was as follows:

I. *Stellar evolution,* by Professor H. N. Russel. (Address delivered at the request of Section D.)

II. *Is the universe finite?* by Professor Archibald Henderson. (Address delivered at the request of the Mathematical Association of America.)

Titles and abstracts of the papers read at the regular sessions of the Society follow below. Retiring President Veblen presided at the sessions of Monday afternoon and Tuesday morning, relieved by Professor Coolidge. President Birkhoff presided at the Tuesday afternoon session. The papers of Adams (second paper), Bennett, Church, Dodd, Franklin, Graustein, Hazlett, Hurwitz (third paper), Kasner, Lubben, MacNeish, R. L. Moore (second paper), Ritt, Stone, Vandiver, Wiener (second paper), Wilder, and Zeldin were read by title; Mr. Merriman's paper was read by Professor C. N. Moore. Professor de la Vallée-Poussin was introduced by the Secretary.

1. Mr. H. S. Vandiver: *On methods for finding factors of large integers.* Second paper.

In the author's first paper with this title (this Bulletin, vol. 30, p. 542) three new methods were described for factoring integers greater than eleven million. The present paper contains further details and examples in connection with the use of two of these methods. They are applied to the factorization of 13,179,643 and 236,364,091.

2. Mr. H. S. Vandiver: *Note on trinomial congruences and the first case of Fermat's last theorem.*

This note contains the proofs of some theorems concerning the relation

$$x^p + y^p + z^p = 0, \tag{1}$$

x, y and z being integers prime to the odd prime p, that depend on the possibility of finding prime integers q such that

$$\xi^p + \eta^p + \zeta^p \equiv 0 \pmod{q} \tag{2}$$

has no solution in integers ξ, η and ζ prime to q. In particular the following result is obtained: If (2) has no solutions under the conditions above mentioned, $q = 1 + mp$, and $m < 10p$, then (1) has no solutions in integers prime to p.

3. Mr. H. S. Vandiver: *Summary of results and proofs concerning Fermat's last theorem.*

This paper contains several results regarding Fermat's last theorem obtained by the author within the last ten years and not hitherto published. The proofs are indicated.

4. Mr. G. Y. Rainich: *Stereographic representation of surfaces.*

Given a fixed plane Π, a fixed point P outside Π and a surface Σ, we represent a point of Σ by the intersection with Π of the bisector of the angle formed at P by the perpendiculars to Π and to the tangent plane of Σ at the point represented. If we take cartesian coordinates of the representing point for parameters of the point represented, the surface can be determined by its second differential form; given the coefficients of this form, which must satisfy the Codazzi equations, we can find the parametric expression of the surface in the shape of three definite integrals. The Weierstrass representation of a minimal surface is a special case of this. The Codazzi equations being linear, we obtain a new solution by adding two known solutions; from the representations of a sphere and a minimal surface we obtain in this way a representation of surfaces of constant mean curvature. The method can be applied to hypersurfaces immersed in a flat space of next higher dimensionality. This is a particular case of the "modified tensor analysis" previously indicated by the author.

5. Mr. G. Y. Rainich: *On integrals in curved space.*

If a tensor field of rank $r+s$ in a flat space is alternating in r of its indices, an r-ple integral can be formed over an r-spread and this integral gives a tensor of rank s; but if the space in which the tensor field is considered is *curved,* the integration cannot be performed remaining in this space. In this case we choose s arbitrary vectors in the containing flat space, and form at each point of the curved space the inner product of the tensor field at this point and the projections on the corresponding tangent bundle of the s vectors of the containing space. In integrating the resulting tensor field of rank r over an r-spread in the curved space we obtain a multilinear function of s arbitrary vectors, i. e., a tensor of rank s, in the containing space. Conditions are discussed under which this tensor vanishes for a closed spread. Application is made of the foregoing in the cases of the indicator, the Riemann tensor, and the contracted Riemann tensor.

6. Miss Marguerite Lehr: *The plane quintic with five cusps.*

A modified form of del Pezzo's equation for this quintic suggests study of the system of conics through four cusps.

This method gives a parametric expression for the curve, involving only a quadratic irrationality, establishes the fact that the curve is unipartite, and that its five cusps and five inflexions occur alternately, and gives the equations of the inflexional tangents. The inflexions, inflexional tangents, cuspidal tangents and a number of points of interest on the curve are obtained from the five given cusps by linear constructions.

7. Professor Anna J. Pell: *A hyperbolic differential equation.*

It is shown by a special example that the spectrum associated with a hyperbolic differential equation with a boundary condition may be continuous.

8. Professor C. N. Moore: *On the application of Borel's method to the summation of Fourier's series.*

In the present paper it is shown that for the summation of Fourier's series Borel's method is less effective than Cesàro's. This is accomplished in two ways, first by a direct study of the kernel of the singular integral which arises from the application of Borel's method to such series, and second by exhibiting examples of continuous functions whose Fourier development cannot be summed by this method. On the other hand it is shown that while the Lebesgue constants corresponding to the application of Borel's method become infinite of the same order as the ordinary Lebesgue constants, it is not true, as one might expect, that the Borel method is no more effective than ordinary convergence in the study of Fourier's series. For it is possible to set up examples of continuous functions whose Fourier development is divergent at a certain point but which can nevertheless be summed there by Borel's method.

9. Mr. G. M. Merriman: *On a theorem concerning the convergence of triple series known to be summable* $(C1)$ *and its application to the triple Fourier series.*

In a paper presented to the Society, December, 1923, the author (1) proved a theorem deriving sufficient conditions for the convergence of double series from known results regarding their summability, and (2) applied this theorem to the double Fourier series, obtaining, as sufficient conditions for its convergence, generalizations of the Dini condition for the convergence of the simple Fourier series.

The present paper contains generalizations to three variables of both of these theorems: (1) a theorem giving sufficient conditions for the convergence of triple series summable $(C1)$, and (2) an application of this theorem to obtain conditions for the convergence of a triple Fourier series. These latter are again generalizations of the Dini condition.

10. Dr. L. M. Graves: *The derivative as independent function in the calculus of variations.*

This paper contains some extensions of the results presented by the author under the same title at the meeting of the Society in Chicago, April, 1924. The integrals I considered are of the form

$$I = \int_{x_1}^{x_2} f\left(x, y_1 + \int_{x_1}^{x} z\,dx, z(x)\right) dx.$$

Admissible functions $z(x)$ are those which are bounded and integrable in the sense of Lebesgue. A necessary condition corresponding to that of Euler that an admissible function furnish a minimum for the integral I is deduced on the basis of very weak restrictions on the function f. Application of this theorem to the second variation yields a necessary condition corresponding to that of Jacobi for an admissible function to minimize I. Sufficient conditions are obtained that a function $z(x)$ continuous almost everywhere furnish a minimum for the integral I relative to the class of all admissible functions. The proofs are made without reference to any values of x outside the interval (x_1, x_2).

11. Dr. L. M. Graves: *On functionals defined implicitly.*

Lamson[*] has obtained a general implicit function theorem, but gives no information about the differentiability of the solutions of the functional equations he considers, which are of the two forms $y = F[y, z]$, and $G[y] = z$. In Part I of the present paper Lamson's classes of functions $[y]$ and $[z]$ are replaced by abstract sets concerning which the necessary properties are postulated, and conditions are given which ensure the existence of a solution $y = Y[z]$ of equations of the general type $G[y, z] = 0$ in the neighborhood of an initial solution (y_0, z_0). Further theorems are obtained concerning the existence of differentials and difference functions of all orders for the solutions $Y[z]$. In Part II a generalization of the imbedding theorems for solutions of systems of ordinary differential equations is obtained from the theorems of Part I by considering sets whose elements are functions on a composite range. In Part III a number of special cases are discussed.

---

[*] American Journal, vol. 42, p. 243.

12. Dr. S. D. Zeldin· *The adjoint group in function space.*

In a number of papers published since 1911, mainly by G. Kowalewski, the Lie theory of continuous groups has been applied to function space. The author extends the notion of the *adjoint group* to problems in function space, and determines its relation to its isomorphic group.

13. Professor J. R. Musselman: *A configuration in space.*

Starting with a triangle in a plane, the author constructs a configuration of 18 points and 36 lines. Connected with it is a dual configuration of 36 points and 18 lines. The points fall naturally into 4 sets of three 3-points each; the whole is invariant under a ternary G_{216}, not the Hesse G_{216}. With a tetrahedron in space we are led to a configuration of 54 lines, 108 points and 108 planes such that 15 points are on each plane and 15 planes on each point. The lines fall into 4 sets of 27 each, such that on each set is a cubic surface of the Eckhardt type where the lines meet by threes in 18 points.

14. Professor H. F. MacNeish: *A nomogram in n-dimensional space for the solution of n linear simultaneous equations.*

In this paper a method is developed for solving n linear equations in n unknowns which is a generalization of the nomogram in the plane for the solution of 2 linear equations in 2 unknowns. This generalization is accomplished by the method of antisymmetric ratios considered in a paper *Theorems concerning transversals of the* $(n+1)$*-hedron in* n*-dimensional space* previously presented to the Society by the author.

15. Professor W. C. Graustein: *Semi-parallel maps of surfaces.*

If a map of two surfaces is called "parallel" it is assumed, in general, that there exist two systems of curves, one on each surface, having the property that their curves correspond and have at corresponding points parallel tangents.

A map is given the name "semi-parallel" if there exist merely two families of curves, one on each surface, which have this property. The general semi-parallel map and important special cases are discussed in this paper.

16. Professor W. C. Graustein: *An invariant of a general transformation of surfaces.*

If two surfaces are in one-to-one correspondence, there exists between the pencils of tangent lines at corresponding points a projective correspondence. The two pencils cut the line of intersection of their planes in two projective ranges of points. The invariant cross ratio of this projectivity is introduced as an invariant of the transformation of the one surface into the other. Applications of it are made to the so-called fundamental or F transformations.

17. Professor O. E. Glenn: *On the residuation of systems of binary forms.*

The two-dimensional system Q_1 of integers generated from an infinite series $S(\alpha)$: $\alpha_1, \alpha_2, \cdots,$ as first column, in which any element is $e = x\gamma + y\delta$, where δ is the element immediately preceding e in the row and γ the one next beneath δ in the column, has, for its general element, the binary form $(\alpha_{v-u+1}, \alpha_{v-u+2}, \cdots, \alpha_v)(x, y)^{u-1}$. This paper is a study of the residue system Q of Q_1, modulo n. Subjects treated are generation of periodic sequences $S(\alpha)$ by means of periodic functions, period parallelograms and drift lines, prime elements. Some of the periodicities considered are manifestations of Fermat's theorem, Euler's function $\varphi(n)$, and the theory of exponents.

18. Mr. C. G. Latimer: *The arithmetics of generalized quaternion algebras.*

The author considers the algebra over the field of rational numbers with the units $e_0 = 1$, e_1, e_2, e_3 and the following multiplication table: $e_ie_0 = e_0e_i = e_i$ $(i = 0, 1, 2, 3)$, $e_1^2 = -\alpha$, $e_2^2 = -\beta$, $e_3^2 = \alpha\beta$, $e_1e_2 = -e_2e_1 = e_3$, $e_1e_3 = -e_3e_1 = \alpha e_2$, $e_2e_3 = -e_3e_2 = \beta e_1$, where α and β are integers $\neq 0$. He finds systems of integral elements, as defined by Dickson[*], when α and β are odd and also in one case when α is even and β odd.

---

[*] See his *Algebras and their Arithmetics*, § 87.

19. Professor F. R. Sharpe: *The irrationality of the general cubic variety in four-dimensional space.*

It has long been considered probable but it has not been definitely proved that the general cubic variety in four-dimensional space is irrational. In the present paper rational cubic varieties are studied by means of the space involution derivable from the hyperplane sections through a point on the variety, and the conclusion is drawn that in all cases the variety is not the general cubic variety. If this is the case, then the general cubic variety must be irrational.

20. Professor Edward Kasner: *Analytic curves for which the chord equals the arc.*

The problem is to find regular analytic curves through a given point O such that the arc OP equals the chord OP for every point P. In complex space there is a large class of such curves besides the straight lines, which are the only real solutions. In an ordinary three-space the curves lie in minimal planes. Generalization to Riemann space of n dimensions is discussed.

21. Professor J. L. Synge: *A generalization of the Riemannian line-element.*

This paper outlines the geometry that results from the assumption that the square of the line-element is a homogeneous function of the second degree of the differentials of the coordinates, without the additional hypothesis of Riemann that this function is quadratic. It is found that the equations of the geodesics may be presented in their usual form,

$$\frac{d^2x^i}{ds^2}+\left\{\begin{matrix} jk \\ i \end{matrix}\right\}\frac{dx^j}{ds}\frac{dx^k}{ds}=0,$$

where the Christoffel symbols contain the coordinates and are homogeneous functions of zero degree in the first derivatives of the coordinates with respect to s. The natural generalization of the Levi-Civita formula for parallel propagation of a vector is found to contain the second derivatives of the coordinates with respect to the parameter as well as their first derivatives. The "angle between two vectors with respect to a given curve" and other geometric concepts are discussed.

22. Professor G. A. Bliss: *Boundary value problems for systems of linear differential equations of the first order.*

In this paper the author studies the problem of determining the solutions of a system of differential equations

$$\frac{du_i}{dx} = \sum_{k=1}^{n}[A_{ik}(x) + \lambda B_{ik}(x)]u_k(x) + g_i(x), \qquad (i = 1, \cdots, n),$$

with boundary conditions of the form

$$s_i(u) = \sum_{k=1}^{n}[M_{ik}u_k(a) + N_{ik}u_k(b)] = h_i, \qquad (i = 1, \cdots, n),$$

where the coefficients M_{ik}, N_{ik} and h_i are constants. A very large number of boundary value problems for ordinary differential equations, hitherto separately studied in the literature, are expressible as special cases of this. It is shown that when the problem is definitely self-adjoint according to a suitable definition the characteristic numbers λ_ν are denumerably infinite in number and real, and each has associated with it a unique set of characteristic functions $u_{i\nu}(x)$ $(i = 1, \cdots, n)$. Every set of functions $f_i(x)$ $(i = 1, \cdots, n)$ with suitable continuity properties which satisfy the boundary conditions $s_i(f) = 0$ $(i = 1, \cdots, n)$ is expressible uniquely in the form $f_i(x) = \sum_{\nu=1}^{\infty} c_\nu u_{i\nu}(x)$ $(i = 1, \cdots, n)$, and the series converges uniformly on the interval $a \leqq x \leqq b$.

23. Professor W. A. Hurwitz: *On de la Vallée Poussin summability.*

The definition of summability given by de la Vallée Poussin has been studied by Plancherel, C. N. Moore, Gronwall, and Kogbetlianz. In the present paper, the author shows its equivalence to other similar definitions, generalizing results of Kogbetlianz, proves that it permits omission and adjunction of terms, establishes an appropriate necessary condition for summability, and considers the relation of the definition to other standard methods of summability.

24. Professor W. A. Hurwitz: *Characteristic parameter values for an integral equation.*

It is well known that the characteristic parameter values for a real symmetric (more generally, complex hermitian) kernel are all real; this result is easily extended to give conditions that the characteristic values lie on any straight line through the origin. In this paper, the author gives like criteria for a straight line not through the origin. A typical result is the following: If $\alpha \neq 0$, and if

$$\bar{\alpha} K(x, y) + \alpha \bar{K}(y, x) = 2 \int K(x, s) \bar{K}(y, s) ds$$
$$= 2 \int \bar{K}(s, x) K(s, y) ds$$

the characteristic parameter values λ for the kernel $K(x, y)$ lie on the straight line $R(\lambda\alpha) = 1$.

25. Professor W. A. Hurwitz: *A trivial Tauberian theorem.*

An example is given of a Tauberian theorem, otherwise unimportant, which exhibits cases of each of the customary Tauberian conditions ($|u_n| < Kf_n$, $u_n < Kf_n$, $u_n > -Kf_n$, $u_n/f_n \rightarrow 0$), and thus throws light on the occurrence of the different types.

26. Professor Marston Morse: *Relations "in the large" between the numbers of extremals of different types joining two fixed points.*

In this paper the author is concerned with the "regular" plane problem in the non-parametric form. He shows that under certain boundary conditions the problem of the determination of the extremals joining two fixed points is equivalent to the problem of determining the critical points of a function, f, of a large but fixed number of variables. On any extremal joining the two given points the number of points conjugate to either end point determines the nature of the corresponding critical point of f. In a paper presented to the Society in December, 1923, the author gave a set of relations between the numbers of critical points of different types of a function f under certain boundary conditions. These relations lead at once to a set of relations between the numbers of extremals of the different types joining the two fixed points.

27. Dr. C. R. Adams: *An existence theorem for a linear partial difference equation of the intermediate type.*

The equation considered is $g(x+1, ry) = a(x, y) g(x, y)$, in which r is a constant, real or complex, not zero and in absolute value different from unity; $a(x, y)$ is a given polynomial in x and y (no added generality is secured by taking it a rational function); and $g(x, y)$ is the function to be determined. There exist in general two formal series solutions, the first in powers of $1/x$ and y and the second in powers of $1/x$ and $1/y$. These series do not in general converge. There exist, however, two solutions which are analytic save for poles over the entire x- and y-planes ($y = 0$ excluded for the second solution) and which, in a certain sense, are asymptotically represented by the corresponding formal series. Roughly speaking, the sense is this: the first solution is represented by the first formal series with respect to x in any half-plane and with respect to y in the complete neighborhood of the origin; the second is represented asymptotically by the second formal series with respect to x in any right half-plane and with respect to y in the complete neighborhood of infinity.

28. Dr. C. R. Adams: *Note on the existence of analytic solutions of non-homogeneous linear q-difference equations, ordinary and partial.*

This paper deals with a system $g_i(qx) = \sum_{j=1}^{n} a_{ij}(x) g_j(x)$ $(i = 1, 2, \cdots, n-1)$, $g_n(qx) = \sum_{j=1}^{n} a_{nj}(x) g_j(x) + b(x)$, no loss in generality being suffered in taking only one equation of the system to be non-homogeneous. If the coefficient functions $a_{ij}(x)$ and $b(x)$ are rational and if $|q|$ is $\neq 1$, the system in general possesses two formal series solutions. These series converge in a large class of cases and so represent, in certain regions, analytic solutions. The region of analyticity of the solutions can be extended by repeated use of the system itself. The relation between the two solutions is found to be expressible in terms of solutions of the associated homogeneous system (obtained by deleting $b(x)$) and certain periodic functions the nature of which can, in important instances, be completely determined. Similar theorems are proved for a system analogous to the above but in which the functions involved depend on more than one variable*.

---

* More precisely, a non-homogeneous system of the class treated by the author in a recent paper *The general theory of a class of linear partial q-difference equations*, TRANSACTIONS OF THIS SOCIETY, vol. 26 (1924), pp. 283–312.

29. Professor E. R. Hedrick: *A generalization of Morera's theorem.*

Generalizations of Morera's theorem have been stated by several authors. According to J. Wolff, who gave one such generalization in a paper entitled *On sufficient conditions for analyticity* at the Toronto Congress, Looman has given generalizations in papers in NIEUW ARCHIEF VOOR WISKUNDE (vol. 14, pp. 234–239). The present paper gives still further generalizations. Let us first assume that the real and the imaginary parts of a function of a complex variable have continuous first derivatives, and let us employ the expression *analytic at a point* in the sense defined by the author and others (LIOUVILLE, (9), vol. 2 (1923), pp. 327–342). Then it is shown in the present paper that a necessary and sufficient condition that a function be analytic at a point is that the limit of the ratio of $f(z)dz$ over a square about that point to the area of the square approach zero with the area, on a given sequence of squares. It follows that a necessary and sufficient condition that the function be analytic at every point is that the same property hold at each point of a set everywhere dense in the region.

30. Professor J. F. Ritt: *A theorem on rational functions analogous to a theorem of Lüroth.*

If $F(z) = \varphi_1[\varphi_2(z)]$, all three functions being rational, $\varphi_2(z)$ will be called a forefactor, and $\varphi_1(z)$ a postfactor of $F(z)$. It is a well known theorem of Lüroth that given any two rational functions, $F_1(z)$ and $F_2(z)$, there exists a rational combination $\varphi(z) = R(F_1, F_2)$ of the two functions, such that $F_1(z)$ and $F_2(z)$ are both rational functions of $\varphi(z)$. This theorem implies immediately, and is almost equivalent to, the following result: Any two rational functions, $F_1(z)$ and $F_2(z)$, have a forefactor $\varphi(z)$ in common such that every common forefactor of $F_1(z)$ and $F_2(z)$ is a forefactor of $\varphi(z)$. The present paper investigates the common postfactors of a pair of functions. A result analogous to the above is obtained, namely that any two rational functions have a common postfactor of which every common postfactor is a postfactor. The writer's proof of this theorem does not resemble that of the theorem of Lüroth. The dissimilarity seems actually to be due to differences in the nature of the theorems.

31. Professor Olive C. Hazlett: *Formal modular covariants as algebraic invariants.*

In this paper it is shown how, as a result of a symbolic theory developed several years ago, it follows that any isobaric formal modular invariant of a binary form f with respect to the Galois field, $GF[p^n]$, of order p^n is congruent, mod p, to an algebraic invariant A of the form f, where A is a rational integral invariant if $p \neq 2$ and is either a rational integral invariant or a square of such if $p = 2$. Another type of formal modular invariant of $f(a; x, y)$ is an algebraic invariant of $f(a^{p^n}; x, y)$ and $f(a; x, y)$. Similarly, the formal modular invariants of $f(a; x, y)$ are in general algebraic invariants of $f(a; x, y)$ and certain related forms. By applying these results, the author attempts to determine a general method of finding the formal modular invariants and covariants of a general form from algebraic invariants, which shall be of general applicability.

32. Professor C. de la Vallée Poussin: *On functions differentiable an infinite number of times.*

This paper discusses (1) general relations between the law of limitation of successive derivatives and that of the Fourier coefficients of a periodic function, and (2) the existence of functions whose successive derivatives take given values at a point.

33. Professor R. L. Moore: *Covering theorems.*

Using the following definition: A point set K is said to be *uniformly relatively well imbedded* in a set of regions G if there exists a positive number d, less than unity, such that each point of K belongs to some region R of G and is at a distance from the boundary of R less than d times the diameter of R, the author proves these theorems: (1) If, for each circle g of a set G, h_g is a definite circle concentric with g and there exists a positive e, less than unity, such that, for every g, the radius of h_g is less than e times that of g, then the set G contains a subset such that, for each g, h_g is within one and only one circle of this subset. (2) If G is a set of circular regions and K is a point set of measure zero, and, for every positive e, K is uniformly relatively well imbedded in a subset of G whose regions are of radius less than e, then, for every positive ε, G contains a subset G^* which covers K, the sum of the areas of the regions of G^* being less than ε. (3) J. Splawa-Neyman's theorem (FUNDAMENTA MATHEMATICAE, vol 6) remains true if the words "à son intérieur" are omitted.

34. Professor R. L. Moore: *Concerning the separation of a continuum which does not separate the plane.*

A subset H of a point set K is said to separate the point A from the point B in K in the weak sense if $K-H$ contains no continuum which contains both A and B. In this paper it is shown that if, in a plane S, A and B are two distinct points of a bounded continuum K no subset of which separates S, and G is a collection of subcontinua no one of which separates A from B, in K, in the weak sense, then, even if there be uncountably many continua in the collection G, their sum cannot separate A from B, in K, in the weak sense.

35. Professor R. L. Moore: *Concerning upper semi-continuous collections.*

Suppose G is an upper semi-continuous collection of mutually exclusive continua which fill up a plane S. In this paper it is shown that, if the continua of the set G are considered as ideal points and the term region is suitably defined, the set of all such ideal points and regions satisfies the set of axioms $\sum_1$ of the author's paper *On the foundations of plane analysis situs,* and thus has all the topological properties of an ordinary plane. Thus we have a theory of arcs, simple closed curves, continuous curves, etc., whose elements are continua of the set G. For example, a simple closed curve of continua of the set G is a closed and bounded set of such continua which is disconnected by the omission of any two continua of G which belong to it.

36. Mr. H. M. Gehman: *A characterization of simple continuous curves.*

It is proved that if M is a closed and connected set of points in a euclidean plane, such that $M-W$ is uniformly connected im kleinen relative to W, where W is any closed and connected subset of M consisting of more than a single point, then M is a simple continuous arc, a simple closed curve, a ray, or an open curve.

37. Professor Norbert Wiener: *Quasi-analytic functions and quadratic forms.*

The author develops an extension of the Denjoy-Carleman theory of quasi-analytic functions, making use of the properties of a certain quadratic form in the Fourier coefficients of a function.

38. Professor Norbert Wiener: *The representation of functions by trigonometric integrals.* Part 2.

The author develops a method for deducing Bohr's relation between the sum of the squares of the coefficients of a quasi-periodic function and the average of the square of the function from the general theory of the trigonometric expansions of bounded functions.

39. Professor Norbert Wiener: *Note on a paper of O. Perron.*

The author points out that the methods of solving the Dirichlet problem in its generalized sense, given by O. Perron in a recent paper in the MATHEMATISCHE ZEITSCHRIFT, enable one to build up a generalization of the Dirichlet problem entirely equivalent to that exhibited by the author in two recent papers.

40. Dr. F. H. Murray: *On certain families of orbits with arbitrary masses in the problem of three bodies.* Second paper.

In this paper a discussion is given of the stability of the straight line and equilateral triangle solutions, respectively, in the problem of three bodies, by means of the results of two preceding papers. It is shown that certain generalized theorems of Bohl can be applied directly to a neighborhood of the straight line solutions, with arbitrary masses, and to a neighborhood of the equilateral triangle solutions, if the masses are such that the characteristic exponents of the generating orbits are not all pure imaginaries; the mutual distances are assumed constant for the generating orbits in both cases.

41. Mr. F. W. Perkins: *On the oscillation of harmonic functions.*

Given any circle C and a function u finite and harmonic within C, let the oscillation of u on C (denoted by $Du|c$) be defined as the difference between the upper and lower bounds of u within C. The most important of the new results in the present paper are contained in the following theorems: Let P_1, P_2 be any two points within a circle C, and draw the two circles C_1, C_2 through P_1 and P_2 which are tangent to C. Denote by A_1, A_2 the angles subtended in these circles by the chord P_1P_2, the vertices being at the points of tangency of C with C_1, C_2 respectively. Let u be any function which is finite and harmonic within C. Then $|u(P_1)-u(P_2)|\leqq[(A_1+A_2)/\pi]Du|c$. Let O be the center and A the radius of a circle C, and let P be a point interior to C, at a distance ϱ from 0. Let u be any function finite and harmonic within C, and denote by $[du/d\xi]_P$ the directional derivative of u at P in an arbitrary direction ξ. Then $|[du/d\xi]_P|\leqq(2/\pi)[A/(A^2-\varrho^2)]Du|c$.

42. Mr. R. M. Foster: *The driving-point impedance of an electric circuit.*

Necessary and sufficient conditions that a rational function $S(\lambda)$ represent the driving-point impedance of an electric circuit consisting of resistances, inductances, and capacities for oscillations proportional to $e^{\lambda t}$ are determined in this paper for all cases in which the impedance function $S(\lambda)$ is of the first or second degree, by determining (1) the domain of values within which the coefficients of S must lie, (2) the corresponding domain of values for the roots and poles of S, and (3) the form of the S curve for real sinusoidal oscillations, that is, for $\lambda = 2\pi f i$, the frequency f increasing from zero to infinity.

43. Professor J. S. Taylor: *A note on the theory of depreciation.*

In a paper previously presented to the Society, the author attacked the problem of distributing the depreciation charges for a machine having a measurable unit output so that the unit cost of production for an average machine would remain constant (or vary in any preassigned way) during the period of use, and of determinig the period of use so that this unit cost would be a minimum. In the present paper, the problem is that of determining the period of use and distributing the depreciation charges so that the profit shall be a maximum. As before, the method is dependent upon the availability of statistical information concerning the behavior of the type of machine in question. Particular attention is called to the dependence of the useful life upon the market value of used machines, with the attendant question of the relative desirability of the purchase of second hand machines instead of new ones.

44. Professor E. W. Chittenden: *A simple proof of the topologic invariance of the sets* $F_{\varrho\delta}$ *of Hausdorff.*

The sets $F_{\varrho\delta}$ of Hausdorff are products of sums of closed sets. S. Masurkiewicz has proved that every set which is homeomorphic with a set $F_{\varrho\delta}$ is also a set $F_{\varrho\delta}$. It is the purpose of the present note to present comparatively simple proofs of this proposition and a related theorem of Sierpinski. This theorem states that the property of being the greatest limit of a sequence of closed sets is also a topologic invariant.

45. Professor A. A Bennett: *Generalized elementary symmetric functions.*

The plan proposed in a previous paper by the author is here carried out. A study is made of the general linear function in a domain of integrity. By combinatorial methods alone, the usual properties of the elementary symmetric functions are established in this more general case. Thus, in particular, Newton's identities, Waring's formula, and related theorems are proved. By introducing further conditions such as the finite existence of $f(1)$, where f is the linear function, a general norm is defined and the usual properties of the determinant are proved in so far as these are independent of the structure of an assumed matrix. Thus all results are equally valid for the determinant of a linear transformation and the norm of an algebraic number. Incidentally, totally new proofs of Newton's identies and related theorems are given which are elementary although frequently not as compact as the usual discussions. The entire treatment is necessarily independent of any appeal to the conjugates of a given element.

46. Professor A. A. Bennett: *Semi-groups.*

The notion of semi-group, or of a set with the "associative group property" is fundamental. Recent widespread interest in linear algebras has developed many concepts foreign to the classical group theory but belonging properly to semi-groups. The author examines the semi-group on its own account. Important concepts are the idempotent elements and absorbent elements. Thus a absorbs b if $ab = ba = a$. Among the theorems proved are the following: In each semi-group there exists a sequence $a_1, a_2, \cdots, a_n$, where each a_i absorbs a_{i+1}, and where the sequence cannot be extended by inserting new terms or in other prescribed ways. In such a complete sequence each non-idempotent element is either immediately preceded or immediately succeeded by an idempotent element. If A and B are semi-groups, and A is a subset of B, such that $AB = BA = A$, A is said to be a *normal* subset of B. It seems possible to accomplish almost or quite as much with this concept of normal subset as with that of invariant subset, while numerous special theorems are shown to hold for normal subsets which are not applicable to the less specific concept of invariant subset.

47. Professor A. A. Bennett· *Five axioms for point and translation in affine geometry.*

This paper appeared in full in the November-December number of this BULLETIN.

48. Professor A. A. Bennett: *Biaffine geometry.*

In this paper the author introduces biaffine geometry as a subject of study on its own account. Geometrically, it may be described as the affine geometry of vectors, and analytically it is the study of homogeneous non-singular linear transformations, without the use of arbitrary multipliers. This geometry enjoys the same duality manifested by projective geometry, while remaining finite. Thus there is parallelism among points as among lines. Parallel coplanar lines fall into two mutually exclusive separated sets. The vector has a dual, the "sector", analogous to an angle in some respects. Conics fall into many types, some self-dual, others not. Points may be added and subtracted except under certain exceptional restrictions. The present study covers points, lines, and conics only.

49. Professor R. L. Wilder: *On connected sets which cut the plane.*

Let M be a connected set (containing more than one point) which lies in a plane S and consists entirely of non-cut points but which is disconnected by the omission of any two of its points. Under these conditions the author shows that M cuts the plane S in the sense that there exist at least two points, A and B, of $S—M$, which do not lie in a subcontinuum of $S—M$. If to the above conditions be added the condition that M is closed, M is, in fact, a simple closed (Jordan) curve.

50. Mr. R. G. Lubben: *On the approximate covering of a point set by a finite set of chains.*

The author shows that if, in space of one dimension, G is a set of intervals covering a point set K, then, for every positive number e, K contains a subset K_e, of exterior measure less than e, such that $K—K_e$ is covered by a finite number of non-overlapping finite simple chains whose links are intervals of the set G.

51. Professor E. L. Dodd: *The frequency law of a function of one variable.*

This paper has appeared in full in the January-February issue of this BULLETIN.

52. Mr. M. H. Stone: *A comparison of the series of Fourier and Birkhoff.*

This paper consists of a study of comparative properties of Fourier's series and of the series defined by Birkhoff in volume 9 of the TRANSACTIONS OF THIS SOCIETY, with respect to any summable function, and an investigation of the term-by-term derived series of Birkhoff's series.

53. Mr. M. H. Stone: *Irregular differential systems of order two and the related expansion problems.*

The problem of the irregular expansions arising from differential systems of order two, as described by Birkhoff in volume 9 of the TRANSACTIONS OF THIS SOCIETY, is here discussed by methods which are an extension of those employed in the paper cited.

54. Dr. Philip Franklin: *Weierstrass's approximation theorem.*

Lebesgue has given a simple proof of the Weierstrass approximation theorem for continuous functions of one variable, but the extensions to more variables involve somewhat more complexity. In this note a simple proof is given for the functions of several variables.

55. Mr. Alonzo Church: *On irredundant sets of postulates.*

A set of postulates is irredundant if the postulates are independent and no one of them can be replaced in the set by a weaker postulate in such a way that none of the implications of the set are lost. A necessary and sufficient condition that a set of postulates be irredundant is that the negatives of every two be contradictory. This paper contains an irredundant set of postulates for the system of positive and negative integers, and this set is shown to be categorical. The general problem is proposed to find irredundant sets of postulates of not too complicated form.

56. Professor Edward Kasner: *Null surfaces.*

The author means by a null surface one whose ds^2 vanishes identically. Such surfaces do not exist in three-space, but do exist in space of four or more dimensions. Geometry in a null plane or surface must be distinguished from geometry in a minimal plane or developable. Null m-spreads in a given n-spread are discussed.

R. G. D. RICHARDSON,
Secretary.

A CORRECTION

BY D. N. LEHMER

On page 401 of volume 8 of this BULLETIN (June, 1902), there is a list of errors in Legendre's tables of linear divisors of quadratic forms. The correction stated under the heading III is wrong. The form $172\,x + 147$ is correct for the form $t^2 - 43u^2$.

LIFE INSURANCE AS A SOCIAL SERVICE AND AS A MATHEMATICAL PROBLEM*

BY ROBERT HENDERSON

The prophet Ezekiel in describing the wonderful vision which appeared to him by the river Chebar uses an expression which has become proverbial. He describes four living creatures and speaks of each as accompanied by a wheel. Regarding these wheels he says "their appearance and their work was, as it were, a wheel within a wheel". I hope that if you get from this talk of mine the impression that the business of life insurance is like the vision of Ezekiel "wheels within wheels", I shall be able to make clear to you that it is like that vision also in the further respect that as Ezekiel also says "the spirit of life was in the wheels". If there is, or ever was, a business which is the embodiment and outgrowth of a vital principle it is that of life insurance.

That life insurance performs a very important social service is shown by the remarkable vitality through the ages of institutions attempting in a small way to furnish benefits which were essentially life insurance and by its remarkable growth in modern times after the correct scientific basis had been discovered.

The recorded history of the world has largely to do with states, their rulers and their wars, and it is only by the chance survival of documents that we learn of those social forces and institutions which did not come to the attention of the rulers as subjects of regulation. It is probably for this reason that we first hear of insurance benefits in connection with voluntary associations or collegia because, while voluntary associations appear to have been comparatively free in the days of the Roman republic they were the subjects of very rigid regulation under the emperors and only those whose objects met with the approval of those in authority were permitted to exist legally.

---

* Delivered before the Society and their guests, as the Josiah Willard Gibbs Lecture, at Washington D. C., December 30, 1924.

The particular associations which come to our attention as embodying life insurance benefits appear to have been in the first place religious societies devoted to the worship of particular deities. In view of the religious element involved in funeral rites a provision for the performance of those rites at the death of members of the society was, therefore, a natural development and might be considered as merely incidental were it not for the fact that this feature appears to have had such vitality that associations came into existence which, so far as we know, had no other object. Some of these societies had their own burial place or columbarium, according to the mode adopted for disposing of the body, while others seem to have merely paid a cash sum at death for the purpose of defraying the necessary expenses.

The membership of these societies of which we are speaking was composed of slaves and other very poor people, so that the benefits were extremely small. For example, in one case the entrance fee was equivalent to three dollars and a half and an amphora of good wine and the monthly contribution to about four and one-half cents. The funeral benefit paid by this society was equivalent to about ten and one-half dollars.

These societies, called Collegia Tenuiorum on account of the grade of society from which their membership was drawn and also called Collegia Funeraticia or burial societies, where, as was usually the case, the death benefit was the predominant feature, were not the only societies of ancient days which involved the life insurance feature. There appear to have also been societies among the soldiers with similar benefits on a somewhat larger scale but associated with other benefits more expensive. For example, one such society of which there is record paid about seventy dollars in case of death on service. The same amount was also paid in case of promotion to a higher rank, transfer to another legion or retirement. The consideration for all of this seems to have been an admission fee of one hundred and five dollars, possibly payable in instalments.

From the nature of the case these associations were not organized on a scientific basis nor carried on in a systematic manner so that they sometimes failed just as institutions similarly organized on a larger scale have failed in modern times.

The principle of cooperation and mutual support, in the face of contingencies that are unforeseen or overpowering, which underlies all forms of insurance is very strongly illustrated also by the gilds which were so important a feature of the middle ages commencing with the Frith Gilds of early times, which had many analogies to the vigilance committees of our frontier days, being voluntary associations of neighbors for that mutual protection and support which the government of the day could not be depended upon to give, and developing through the merchant gilds, the craft gilds and the social and religious gilds. Even in the case of the latter, however, the relief given in case of death was not for death as such but was conditioned on resulting destitution and did not, therefore, exhibit the principle of life insurance as distinctly as did the Roman burial societies. They represented, in fact, insurance against destitution of self or family instead of insurance against any of the contingencies which might cause financial loss.

While, however, the Roman *Collegia* and the mediaeval gilds illustrate the fundamental social value of insurance and especially of life insurance, it is to a different organization that modern life insurance companies are directly linked. The loss sustained by the merchant and shipowner, the two being frequently if not usually in early days one person, must always have been a staggering one. It is only natural, therefore, that marine insurance should have been the first form of insurance to be practiced on an extensive scale. In the early days it appears to have been entirely, as it still si largely, undertaken by individuals not organized into a partnership or company but each separately assuming the amount of risk set opposite their signatures subscribed to the policy of insurance. Hence the origin of the name "underwriters" for those assuming insurance risks.

Out of marine insurance there grew a practice of also insuring the life of the merchant or of the master of the ship for the voyage. These early life insurance policies were probably looked upon as merely marine insurance policies, the perils of the sea being those principally in mind in connection with them and they were underwritten by individuals in the same way as the regular marine policies.

The earliest life insurance policy of which we have a record was dated June 18th, 1583. We have a record of this policy because it was the subject of a trial in the English courts. It was made at the Office of Insurance "within the Royal Exchange" in London and insured the life of one William Gybbons in favor of Richard Martin for a period of twelve months from its date at a premium rate of eight percent. The insured died May 29, 1584 and claim was duly made for the amount insured but the underwriters denied liability on the ground that the insured had survived for more than twelve lunar months of twenty-eight days each. The Commissioners of the City of London, the Clerk of the Office of Assurance and finally the Judges of the Court of Admiralty all ruled against the underwriters and they were required to pay.

In passing it may be as well to note that the wording of this policy shows that life insurance transacted in this manner had for some time been a recognized practice. It says (modernizing some of the language) "It is to be understood that this present writing is and shall be of as much force, strength and effect as the best and most surest policy or writing of assurance which hath been ever heretofore used to be made upon the life of any person in Lombard Street, or now within the Royal Exchange in London" and further on refers to "the use and custom of the said street or Royal Exchange".

The next decisive step towards the modern system of life insurance was taken when, in 1699, there was founded in London the first mutual contribution life office. It was called "The Society of Assurance for Widows and Orphans". The plan was to have a society of 2000 members each of whom paid 5 shillings toward each death claim and an additional 5 shillings once for all on admission and one shilling quarterly. The amount paid in case of death was the 5 shillings from each member, making £500 in all if the number of 2000 were complete, less a deduction of 3 percent for expenses. The provision for running expenses was thus the admission fees, the quarterly dues of one shilling from each member and 3 percent of the claim contributions. Immediately upon the payment of a death claim the 5 shilling contribution for the next claim was collected and it was also collected from each new member in addition to the admission fee.

Two other similar institutions were established in 1700 and 1704 respectively but all three and many others which had been floated in the meantime were swept out of existence when the South Sea Bubble burst in 1720.

An institution was, however, established on slightly different lines in 1706 which was able to survive the disastrous year 1720. It was called the Amicable Society for a Perpetual Assurance. The contributions of the members were fixed at £6.4s per annum and an admission fee of 10s. The number of members was fixed at 2000 which number appears to have been attained in the second year. The amount paid as a death claim depended upon the number of deaths in the calendar year. The provision was that in the first year of the Society one sixth of the total contributions was to be divided among the death claims of the year, in the second year, on the assumption of a complete membership of 2000, £4000 was to be divided, in the third year £6000, in the fourth year £8000 and thereafter £10,000 each year. As the full contribution from the complete roll of members, after deducting 4 shillings per annum from each which went to the registrar, amounted to £12,000 this left a liberal margin for accumulation of surplus so that even though it suffered some financial reverses in its early years it was able to survive. In 1807 its business was placed on a more scientific basis and in 1866 after 160 years of existence it was amalgamated with another company which is still in existence. In order to complete this historical review of prescientific life insurance it may be added that in 1721 two corporations, the London Assurance Corporation and the Royal Exchange Assurance Corporation, which had been already authorized to transact marine insurance, received the additional power to accept life insurance risks. They transacted the business on the old lines which had been followed by individual underwriters. The standard premium seems to have been 5 percent for all accepted ages.

We have thus traced the history of life insurance up to the time when the science of life contingencies as an application of mathematics was developed. It is an interesting fact, however, that although, up to the present time, life insurance has been the most important business to which that science has been applied it was not in connection with life insurance that it was first developed but rather in connection with life annuities. It must be remembered that although life annuities are affected by the same contingencies of life or death as is life insurance and while at the present time the great bulk of the business of granting life annuities is carried on by life insurance companies they serve a fundamentally different purpose and had a radically different origin.

With respect to life annuities our search for origins leads us back, as in the case of life insurance, to ancient Roman days but instead of associations of slaves we have in this case the testamentary dispositions of the wealthy who directed that certain legatees should receive out of their estate a fixed income for life. Out of this practice a problem arose when a law was enacted that at least one quarter of any estate should be distributed among the legal heirs. For the purpose of determining if this law was complied with it was necessary to have some rule for finding the commuted value of these obligations to pay an income for life and a schedule of values graded according to quinquennial groups of ages of the life tenant was actually drawn up and used for that purpose. We have no record, however, of the basis upon which the figures in that table were arrived at nor of any investigations which would furnish appropriate data, so that as far as we can judge it was purely empirical or hypothetical.

There are probably instances in all subsequent ages of such life interests or annuities arising out of wills but it was from a different source that the great impetus to the business of life annuities come.

In early days no distinction was made between excessive and moderate rates of interest for loans. Possibly there were no instances of a moderate rate but in any event all interest or usury as it was called was condemned. This condemnation was at first a purely religious one but at various times laws were enacted by the civil authorities in various countries prohibiting usury. In some cases this prohibition was general but in others it applied only to the Christian subjects of the state so that non-Christians and aliens were permitted to practice it. The economic tendency which these church edicts and civil laws were endeavoring to combat was, however, so strong that even the clergy seem to have in many cases engaged in the practice. Capital, however, in the hands of men of ordinary prudence prefers a legal investment if it can be found. A legal way of satisfying economic needs was accordingly sought and found in the purchase and sale of rent charges and life interests. These were at first always charged against specific parcels of real estate the owner of which, desiring to raise an immediate sum of money for some purpose, would agree to a definite annual charge against the income from the property either in perpetuity or for the lifetime of one or more nominees. Later these annuities were sometimes issued on the personal credit of the granter and in such cases were almost necessarily either for life or for a fixed term of years.

The sale of annuities was not confined to individual property owners. As early as the eighth century monasteries and other religious bodies sold life leases of land or rents for life. This means of raising money became popular with public bodies and from about the end of the thirteenth century it was largely monopolized by them, both town authorities and governments of states taking part in it. Many towns, in fact, forbade the sale of annuities by private persons and while in this step they may have been actuated by the motive of removing competition there were probably also actual complaints of abuses in the private sale.

Annuities were issued on two or more lives payable as long as any of the nominees survived. The income might be payable to the purchaser or his heirs the lives nominated not being financially interest in the contract or it might be payable to the nominees in a definite order of succession or in, the third place might be equally divided among the survivors or their respective nominators. This latter plan with a large number of nominees develops into the form of annuity plan associated with the name of Tonti and therefore called tontines.

It is evident that it such an annuity is established on say ten thousand lives of sufficient amount to give each ten dollars a year at the start it will give very large incomes to each surviver near the end. This form of contract has for the individual purchaser a very considerable gambling element and for the seller is very nearly equivalent to a perpetuity.

This form of contract was quite popular in the seventeenth century in Europe and is of interest to us from the fact that a feature developed in connection with it which, I believe, helped to prepare the way for the scientific study of life contingencies. Taking a glance forward to the deferred dividend or tontine dividend system of profit distribution formerly so popular in this country and comparing it with this tontine annuity scheme it appears that with anything of the tontine nature estimates of results are almost inevitably associated. The promoters of tontine annuity schemes sometimes put out such estimates showing out of a given number of annuity purchasers the estimated number surviving after various intervals and the consequent income received by each.

Unfortunately, however, for historical completeness there is, so far as I know, no record of any of them explaining the basis of the estimates. They did, however, undoubtedly accustom people to the idea of the possibility of estimating how many out of a given group of people now formed would remain alive at any given date in the future and how many would die in each year.

Having thus outlined the steps by which the force of circumstances prepared the problem of life contingencies for the attention of mathematicians let us now turn our attention briefly to the other side of the screen and observe how mathematical science was being prepared to attack the problem. This leads us to a consideration of the foundation of the theory of probabilities and although any remarks of mine on this subject before this audience may possibly be likened to "carrying coals to Newcastle", I shall, for the sake of completeness, ask you to bear with me for a few minutes.

Although there are slight hints of the subject in the writings of Cardan, Kepler and Galileo the definite origin of the science is usually, and with justice, attributed to Pascal. With his name is associated that of Fermat who was consulted by Pascal and who worked out an independent solution of the problem under discussion. It is interesting to note that at the date, namely 1654, when they discussed this problem and so laid the foundations of the theory, Pascal and Fermat were the most distinguished mathematicians of Europe, Wallis in England being practically their only rival. Descartes had died in 1650, Newton was only twelve years old, and Leibnitz eight. Another interesting fact is that both of these founders of the science might be classed as amateur mathematicians, by which I mean that mathematics was to each of them an avocation or hobby and not a source of income. Pascal had just the year before been called from his religious retreat at Port Royal to administer his father's estate and was at the time occupied with that business. He returned to the retreat in November of 1654. Fermat was and had been for twenty-three years councillor for the local parliament at Toulouse. He was at the time about fifty-three years of age and Pascal was thirty-one.

A third interesting fact is that the problem which brought the subject to Pascal's attention was itself of comparatively little importance and arose out of a game of chance. This may perhaps be considered natural because the subject matter of the science is chance and games of chance, at that time at least, received more attention than any other chance events.

The problem was proposed to Pascal by the Chevalier de Meré, who seems to have been addicted to games of chance, and was in substance as follows: Two players want each a given number of points in order to win. If they separate without playing out the game, how should the stakes be divided between them? It will be seen that this is equivalent to determining their respective chances of winning the game. This problem was solved on the assumption that the two players had equal chances of winning any particular point and that this was true regardless of who may have won any other point.

The methods followed by the two mathematicians in attacking the problem differed but their fundamental assumptions were the same and the results consequently the same.

Pascal applied the method to some allied problems and intended to write a treatise on the subject but the accident which was the occasion of his return to the Port Royal retreat occurred shortly after and he does not appear to have ever carried out his intention. He died in 1662 at the age of 39. Todhunter in his *History of the Theory of Probability* says

"It might have been anticipated that a subject interesting in itself and discussed by the two most distinguished mathematicians of the time would have attracted rapid and general attention; but such does not appear to have been the case. The two great men themselves seem to have been indifferent to any extensive publication of their investigations; it was sufficient for each to gain the approbation of the other. Pascal finally withdrew from science and the world; Fermat devoted to mathematics only the leisure of a laborious life, and died in 1665".

The work of systematic presentation was taken up by Christian Huygens who wrote in 1657 a small treatise on the theory of probabilities founded on the correspondence of Pascal and Fermat. It was first published in Latin at the end of a collection of mathematical exercises by Frans van Schooten. This little treatise is notable because for a period of more than fifty years it formed the best account of the subject and it was during this fifty year period that the foundation of actuarial science was laid down. It is interesting to note that van Schooten's collection was one of the books read by Newton as an undergraduate at Cambridge. It would seem probable that Halley was also familiar with it or at least with the treatise by Huygens in view of the fact that a translation of it was included in an anonymous essay edited by Motte and printed in England in 1692. It was in this treatise of Huygens that the term *expectation* in the sense in which it is used in probabilities was first introduced.

Although, contrary to the usual opinion, the *expectation of life* is not a fundamental element of actuarial computations and is in fact merely a convenient basis of general comparison between different mortality tables the idea of *expectation* as developed in the theory of probability is fundamental. Its explicit introduction in this treatise is, therefore, worthy of note. The idea itself was, of course, implicit in the original problem proposed to Pascal. The quantities sought in that problem were the respective expectations of the players if play had been continued.

The calculus of probabilities, young and neglected as it was, had now reached a stage where it was ready for application to the problems of life contingencies and although the birth of actuarial science is usually referred to the first known calculations involving a combination of the theories of compound interest and of probabilities it may be well in passing to mention a development which immediately preceded that event. I refer to the publication in 1661 by Captain John Graunt, a London merchant, of a book entitled *Natural and Political Observations on the Bills of Mortality.* In this book, besides giving valuable statistics of deaths analyzed by cause and by age, an effort was made to draw conclusions regarding population and rates of mortality, including a table showing how many in each hundred die within the first six years of life and how many in each decade thereafter. This table was used twenty-one years later in connection with a tontine scheme as furnishing an outline mortality table. Considerations which must be taken into account if a mortality table anywhere near correct is to be obtained were apparently ignored and the table has no claim to accuracy as a representation of the rates of mortality. It will be interesting to compare the percentages of those entering each age interval who die in that interval according to this table with those according to the Carlisle Table which was the first based upon a comparison of deaths analyzed by age with the corresponding population. They are as follows:

Age Interval	Percentage of Death Graunt	Carlisle
0–6	36.0	33.2
6–16	37.5	6.2
16–26	37.5	6.8
26–36	36.0	9.1
36–46	37.5	12.2
46–56	40.0	14 1
56–66	50 0	27.7
66–76	66.7	47.7
76–86	100.0	75.8

It will be noted that in Graunt's table these percentages are substantially the same for the five decennial intervals from age six to age fifty-six and correspond with an annual rate of about forty-six per thousand, a rate not reached in the Carlisle table until age sixty-eight.

While this book is clearly entitled to first place in the history of vital statistics it is difficult to define its relation to our subject because we cannot trace any definite connection between it and the preceding work of Pascal, Fermat and Huygens in the calculus of probabilities on the one hand nor between it and the subsequent actuarial developments on the other.

We now come to the definite starting point of actuarial science. It is a curious fact, however, that in one sense this starting point is not definite because the first piece of actuarial work on record has apparently no connection with the subsequent development of the science, which, in fact, followed continuously from an independent origin. This was partly due to special circumstances but the whole incident clearly demonstrates the value of scientific societies for the preservation and dissemination of the results of original research.

The importance of these incidents in connection with our subject will perhaps excuse some reference to the individuals concerned, as that will give a clearer idea of the circumstances. The author of the first actuarial report was John deWitt, Grand Pensionary of Holland from 1653 to 1672. He was born at Dort in 1625, his father being a prominent leader of the party which was opposed to the extension of the executive power then vested in the Prince of Orange as elective stadtholder. The position of stadtholder of the United Provinces was in practice hereditary in the house of Orange but a formal election was necessary at each vacancy and when in 1650 the stadtholder, William II, died, leaving only a posthumous son born one week after his death, the post of stadtholder was left in abeyance and the chief executive authority devolved on the Grand Pensionary of the Province of Holland. Young John deWitt was elected to this position in 1653 at the age of twenty-eight and being re-elected at five year intervals he continued as the head of the government until 1672. Then when the young prince, William III of Orange, afterwards King William III of England, had attained his majority and the country was engaged in war against France and England a violent reaction not only drove John deWitt from power and installed William III as stadtholder but resulted in the murder of John deWitt and his brother Cornelius by a mob on August 20, 1672.

Before his election as Grand Pensionary he had already in 1650, displayed his interest in mathematics by writing a treatise on curved lines and it was natural, therefore, that he should take a personal interest in the questions of a mathematical nature which came up during his administration.

Reference has already been made to the custom of municipalities and other governments raising money by the sale of life annuities, the purchaser being allowed to nominate the person on whose life the annuity was to depend. The ordinary terms appear to have been that a life annuity was considered as worth one half of the value of a perpetuity of the same annual amount so that, for example, on a four percent basis, a perpetuity being worth twenty-five years' purchase, a life annuity would be worth twelve and a half years' purchase. In the case, however, of the annuities sold by the government of Holland this seems in deWitt's time to have already been modified so as to require fourteen years' purchase instead of twelve and a half.

The object of deWitt's report was to show that a life annuity at fourteen years' purchase yielded a better return to the purchaser than a redeemable annuity or perpetuity at twenty-five years' purchase. This he does by calculating the value at age three, on a four percent interest basis, of a life annuity of 20,000,000 stuyers per annum payable half yearly. He assumes that the same number die in each half year for fifty years from age three to age fifty-three, then two-thirds as many each half year for the next ten years, one half as many for the next ten years and one third as many for seven years and that all are dead at age eighty. His method of calculation is to multiply together the number assumed to die in each half year and the present value of the annuity they will each have drawn and to divide the sum of these products by the total number. It will be seen that this is a direct calculation on the principles of the theory of probability of the expectation of the purchaser.

The report itself does not attempt to justify the assumptions made regarding the distribution of deaths but in subsequent correspondence with John Hudde, Burgomaster of Amsterdam, and also of some note as a mathematician, he indicates that they were based in some way on observations. He states that from further observations he believes that the uniform decrements should extend up to age seventy-five.

The value which he had arrived at for a life annuity at age three was just over sixteen years' purchase and in a supplementary report he indicates a practical check which he had applied. He says that he had extracted from the records the data regarding several thousand purchasers of annuities and had calculated the average value of the annuities drawn in each of considerably more than one hundred classes of about one hundred lives each and that in no case had he found the average value less than sixteen years' purchase but they were frequently much higher, even up to eighteen years' purchase. Only a few copies of this report were printed and, possibly due in part to the troubles of the following year, when deWitt was driven from power, all trace of the report was lost although the fact was known that it had been made and it was not recovered until the middle of the last century, when, through the researches of Frederick Hendriks, a copy was found in the archives of the Netherlands.

The next to venture in this field, while not so prominent politically as de Witt, was much more prominent scientifically. I refer to the famous astronomer Edmund Halley. It is unnecessary to recite to this audience the fact that he was the first to calculate the periodic time of a comet nor the great service to mathematical and physical science that he rendered by inducing Newton to write his *Principia.* The whole cost of its publication was borne by Halley who also corrected the proofs.

If the results which flowed from Halley's work illustrated as we have already pointed out the value of scientific societies for the preservation and publication of original work his actually undertaking it illustrated their value in actually stimulating such work. He was at the time actively interested in the work of the Royal Society and in order to insure the continuation of the publication of its TRANSACTIONS he undertook at a meeting of the council in December, 1692, to furnish five sheets in twenty himself. It was probably under the spur of this promise that he took up the analysis of the data regarding births and deaths in the city of Breslau which had just recently been contributed to the Society and the result was a paper entitled *An estimate of the degree of mortality of mankind, drawn from curious tables of the births and funerals at the city of Breslau; with an attempt to ascertain the price of* *annuities upon lives* which was presented early in 1693. It is notable as containing the first known example of a complete mortality table. This table was constructed from the deaths during the five years 1687 to 1691 inclusive on the assumption that the population was stationary and that consequently the number attaining any age would equal the number of deaths above that age. It was not quite in the usual modern form because the number shown opposite each age was not the number attaining that precise age but the number living at any given moment in that year of age, for example, opposite age 25 we find the population at age 25 next birthday. These figures can of course be used in the calculation of annuity values just like the usual number living (l_x) column of a mortality table provided we remember that the value so arrived at corresponds to age next birthday and not to exact age.

Among the uses for this table suggested by the author are the determination of the premiums for what we now call one year term insurance and the valuation of annuities on one life, on two lives and on three lives. His formula for the value of a life annuity is the one still used in explaining the elementary theory of the subject and represents the sum for positive integral values of n of the products of the present value of a unit due n years hence and the probability that the life or lives on whose survival the payment was conditioned would survive the n years. For an annuity on one life this was mathematically equivalent to the formula used by de Witt but was simpler in practical operation and more easily applied to two or more lives. In a subsequent communication he points out the great advantage of using logarithms for these calculations. The more practical methods proposed by Halley and this later suggestion regarding logarithms shows the effect of his experience with arithmetical calculations in connection with his work in astronomy.

The work of these two independent pioneers brings out the fact that the work of actuarial science like that of every other branch of applied mathematics falls into two main divisions. The first is the ascertainment of those facts of nature which can be used as premises for the mathematical calculations, in this case primarily the facts regarding human mortality, although the rate of interest appropriate to the particular problem in hand and the rate of expense necessary to be provided for are also factors. The second is the development of the mathematical processes best suited to the problems arising and in this division economy of labor is second in importance only to substantial accuracy of result. It may be of interest to outline the main problems which come up for consideration in each of these divisions, merely prefacing the whole with the remark that there is a very considerable mathematical element in the first division as well as in the second.

It might appear that the straightforward way of constructing a mortality table from actual experience was to take note of a large number of babies born about the same time and have them all observed until the last one died, noting the number dying in each year of age. The mere statement of the process, however, shows that it is impracticable. Some other process is therefore necessary and it was perhaps fortunate for the business of life insurance that the methods first followed erred in the direction of too high rates of mortality so that the premiums charged were higher than proved ultimately necessary. Graunt's figures were apparently a simple analysis by age of the deaths reported in London during a few years. The basis of deWitt's calculations was stated as a pure hypothesis without any attempt at justification in detail. Halley apparently recognized the disturbing effect of migrations and of a rapidly changing population. Owing, however, to the fact that the recorded births in Breslau only slightly exceeded the recorded deaths he assumed that the population might be considered stationary. He, therefore, graduated by inspection the average number dying per annum at the various ages and assumed that this gave the decrements of the table, the number living being given by summing from the oldest age downwards. The next important landmark is from one point of view a backward step, appearing to substitute hypothesis for observation, but from another point of view it might be considered as the first attempt to graduate a mortality table by a mathematical law. I refer to deMoivre's hypothesis that the number dying was the same at all ages with eighty-six as the limiting age. I refer to it as a graduation because the smooth series of the hypothesis agreed in a general way, except in infancy and extreme old age, with Halley's and later tables constructed on the same principle that were current at the time he propounded it. This brings out the point that the first division of subject consists of two parts, the ascertaining of the crude facts observed and the adjustment of the results. In fact, Halley in constructing his table had not used the schedule of deaths exactly as reported but had substituted a smoother series.

With regard to the ascertaining of the crude facts the greatest advance was made with the construction of the Carlisle table by Milne because he was able to put into effect the principle that in the construction of a mortality table the fundamental functions are the rates of mortality at the successive ages and not the deaths at the various ages. These rates of mortality he was able to calculate because he had as data not only the deaths in the parishes of St. Mary and St. Cuthbert, Carlisle, for the nine years 1779 to 1787 classified by groups of ages but also the population January 1st, 1780, classified in substantially the same way and an estimate at least of the increase in population up to December 31st, 1787. I speak of it as an estimate at least because the figures used indicate an increase of exactly 1000. It is interesting to compare these various tables and hypotheses including the Northampton table constructed by Dr. Price from the deaths in the parish of All Saints, Northampton, during the forty-six years from 1735 to 1780 inclusive. I have chosen for the purpose of comparison the probability of a child aged six years dying within thirty years and the similar probability at age forty-six. The comparison is as follows:

	Percentage dying within 30 years	
	Age 6	*Age* 46
Graunt	75.0	90.0
deWitt	49.2	92.7
Halley	32.0	80.9
DeMoivre	37.5	75.0
Northampton	35.1	73.8
Carlisle	20.5	67.5

The marked excess of the early estimates of mortality at young ages is very noticeable.

The problems involved in the tabulation of the unadjusted rates of mortality derived from returns relating to the general population differ in many particulars from those arising in connection with insurance experience. On the former the returns since the day of systematic periodic censuses and fairly accurate death registration furnish ample data and the special problems arose at first from the fact that the age classification was only made for groups of ages and the principles of finite differences had to be applied to obtain the rates for individual ages. In later years when returns were made for individual ages we are confronted by the fact that people have a bias in favor of round figures of various kinds in stating their ages so that a careful regrouping is necessary to minimize the error from this source. Also experience indicates a deficiency in the enumerated population in the first two years of age and the question of an approximate correction of the deficiency is an interesting one. In insurance experience the ages are more accurately known and the primary problem is the practical treatment of those withdrawing from the field of observation otherwise than by death. As the requirements of the business, however, progress we find that a simple classification by attained age is not sufficient. We must investigate special classes of risk on account of sex, residence, occupation, physical impairment or other special features and in doing so must take account of the fact that the rate of mortality depends also on the time elapsed since the life was accepted for insurance. In other words for insurance purposes a man is not simply a man of a given age but we know that other ascertainable features affect the probabilities of death.

The graduation or smoothing of the crude rates of mortality is justified by the fact that according to the theory of probability the actually observed ratios of death are not likely to agree exactly with the true probability and indeed cannot unless the product of the probability and the number at risk happens to be an integer. Also a fluctuation in one direction at a given age is as likely as not to be followed by a fluctuation of opposite sign at the next age. An actually observed irregular series is therefore not inconsistent with the natural assumption that the series of true probabilities is a smooth one. Various methods have been used to graduate mortality tables. Halley's table and the Northampton table were graduated roughly by simple inspection. Milne used a graphic method to distribute the population and deaths into individual years in such a way as to preserve the totals of each group of ages unchanged. We have already mentioned deMoivre's hypothesis as possibly entitled to be considered a sort of graduation by mathematical law but the law most extensively applied to the graduation of mortality tables was Makeham's modification of Gompertz' law. Gompertz originally suggested that the force of mortality or instantaneous death rate increased in geometrical progression with the age. It was, however, found impossible to use a single rate of progression throughout even adult life so that it was necessary to break the table into two parts using one rate for one part and another for the other. To overcome this difficulty Makeham proposed breaking the force of mortality into two parts, one of which remained constant and the other increased in geometrical progression. With this modification it was found possible to reproduce with a fair degree of accuracy the mortality indicated by a number of different tables from an age in the twenties to the limit of life. This law possesses advantages in connection with joint life calculations so that it is desirable to use it if possible but recent experiences have been rather intractable. In the case of the experience on American insured lives recently published by the Actuarial Society it was necessary to introduce an additional term into the expression for the force of mortality in order to fit the table from age 30 up. This term was negative in simple proportion to the age.

Other formulas have been used, some of them very elaborate, but no particular one has been used generally and none of them possess the practical advantage of Makeham's.

A third general method of graduation is based on the principle of moving average. As first applied it consisted in substituting for each term of the series the average of five consecutive terms of which it was the central and then repeating the process a second and third time. This, however, would introduce an error into a regular second difference series so a correction is introduced to eliminate this error so that a regular third difference series would be reproduced. Many variations have been suggested and the investigation of the smoothing power and residual error of the various formulas is an interesting problem. A fourth plan which has been used is to apply the principles of finite differences to determine adjusted values for the exposed and deaths at quinquennial ages and then having calculated from these the rates of mortality at those ages to fill in the remainder by interpolation.

A new method of graduation has recently been put forward by E. T. Whittaker of Edinburgh which consists in determining a graduated series of values such that the sum of the squares of the third differences of those values plus the sum of the squares of the differences between the corresponding graduated and ungraduated terms multiplied by a selected constant shall be a minimum. This requirement gives a linear difference equation of the sixth order with end conditions sufficient to determine uniquely a particular solution which is the graduated series required. It is evident that the graduating power depends on the constant selected. The smaller the value of the constant the smoother the resulting series and the greater the sum of the squares of the departures of the graduated from the ungraduated values. For all values of the constant the algebraic sum of the departures and of those departures multiplied by the respective ages or by the squares of the ages will vanish.

Taking up now the other division of our science, namely, the development of the mathematical processes suited to its problems, it will be recalled that the methods of Halley brought out the element so frequent in actuarial calculations, of the determination of the value of an expectation. Where the benefit as in the case of a life annuity payable annually consists of payments to be made on the anniversaries of the date of calculation provided specified contingent events have happened this is theoretically simple. The only difficulties arise from the complicated probabilities sometimes met with. The case of life insurance is ordinarily brought under this heading by assuming that the sum insured is payable at the end of the year in which death occurs. As the possible future life time extends over a considerable number of years the preparation of a table of values if calculated independently is rather laborious. DeMoivre accordingly showed how the value of a life annuity at any given age may be derived by a short calculation from that at the next older age and a similar principle may be applied to other values. His formula is still of theoretical value and of occasional practical use but for general purposes has been superseded by the device of commutation columns, the fundamental idea of which is the discounting back not to the present time but to the date of birth of the total payments called for if a contract of the kind being valued were in force on every life represented by the number living according to the mortality table used. The functions, therefore, which enter into valuation at one age enter largely into the value at another age and certain functions are found of very wide availability thus yielding a great economy of labor.

The principles of finite differences or of the infinitesimal calculus come into play in cases of two kinds. In the first case periodical or other payments may be payable otherwise than annually on the anniversary and it may be desired to derive formulas as accurate as possible for the present values not necessarily for actual use but in order to know the limits of error in the simpler formulas actually used. For such cases formulas have been derived for the correction to pass from the annual case to the fractional. Again the probabilities involved in a particular problem may be so complicated that their calculation for every year of possible future duration of the contract would be too laborious. In such cases they are calculated only at intervals and the principles of finite differences applied to obtain approximate values on the assumption that the probabilities vary regularly in the intervals.

In this connection it is interesting to note that the development of the theory of osculatory interpolation has come almost entirely from the needs of life insurance calculations. The curves used for interpolating in the successive intervals are so adjusted as to pass smoothly into one another without using too high an order of differences. Such a method was more necessary in life insurance than in astronomical work because the basic values which must be used upon which to interpolate are not quite as regular as the calculated values occurring in astronomy.

I have thus outlined briefly and, let us hope, not too tiresomely the nature of the mathematical processes required for actuarial calculations. Let us review a few of the results that have followed the development of the business.

Life insurance still serves in increasing measure the special purposes which called it into existence and as it was the demand for what we would now call business insurance which had sufficient financial importance to call for the establishment of life insurance corporations so it is appropriate that to-day the same demand should call for the largest policies and furnish the most interesting news items. Large amounts of insurance are called for by wealthy people to cover inheritance taxes or to provide ready money to meet the obligations likely to mature at their death. Partnerships ask for it to provide ready money to settle with the estate of a dead partner without sacrificing the business. Businesses which are largely built about the skill, technical knowledge or executive capacity of some individual demand insurance to compensate in a measure for their loss. The purpose, however, of life insurance which is of the greatest social importance is to continue in part and for a time at least to the dependents of the insured the income and support which would otherwise cease at death. To this has recently been added the furnishing of a similar support to the insured himself and his family should he suffer from total and permanent disablement which is, financially at least, worse than death. It has also developed the service which it offers freely to the beneficiaries of life insurance policies of holding the proceeds, investing them for the beneficiaries and paying principal and interest in such convenient manner as may have been arranged thus relieving a beneficiary, probably inexpert and certainly unprepared, from the trouble and risk of seeking investments.

An additional important social service performed by life insurance arises from the necessity of finding investment for the reserve funds which the companies must accumulate and upon which they must earn interest. I might dwell at length on the way in which life insurance funds have aided in the development of our farms and railways and in the building of our cities but this feature of its service has been put so much better than I could do it, by the late Mr. Peter McArthur, that I conclude by quoting some extracts from a pamphlet which he published a few years ago. He says:—

"In this time of discontent it should be made clear that no matter what accusations are brought against wealth, insurance money is unselfish money. This has not been clear in the past. Possibly it has not always been true. But insurance funds have now reached such proportions that their real nature is becoming evident. The inherent laws that govern insurance money are being released and their power is becoming apparent. In spite of what any man or group of men may do these great accumulations of the savings of the people are bound by their very nature to be on the side of the people and to work for them. As they go on increasing they will do more and more to safeguard the rights of humanity. In short, I have no hesitation in saying that insurance is now the most steadying force in our civilization.

"It is estimated that in the United States the established fortunes that are devoted to investment rather than to active enterprise amount to something over $15,000,000,000. The insurance funds already equal fully one-half of this amount and at their present ratio of increase will probably exceed it in a few years. In Canada the conditions are much the same. While no man can predict what will be the entire result of this competition between the great fortunes and the insurance companies for possession of the most desirable securities, the power of individual wealth will certainly be curtailed and the safety of the people will be increased. This is the first time in history that anything of this kind has occurred. In the gamble of life the percentage was always in favor of those who had reserves of wealth. Now, however, life insurance is putting the percentage in favor of the ordinary people."

EQUITABLE LIFE ASSURANCE SOCIETY

SIMPLIFICATIONS RELATING TO A PROOF OF SYLOW'S THEOREM

BY G. A. MILLER

Sylow's theorem is usually developed very early in a course relating to the theory of groups of finite order. Hence simplifications in its proof are the more desirable, especially when they enable us to avoid the use of a number theory formula and therefore to confine the proof more closely to group theoretic considerations. Since the simplifications which will be developed in what follows involve a property of double cosets which is not directly used in the proof to which they relate* we shall first exhibit some of the fundamental properties of double cosets, assuming as known the fact that if H_1 and H_2 are any two subgroups of a group G then all the operators of G may be uniquely represented in the following form. $G = H_1 s_1 H_2 + \cdots + H_1 s_\lambda H_2$ if we assume that in each double co-set only the distinct operators are considered.

The number of distinct operators in each double co-set is evidently divisible by h_1 and by h_2, h_1 and h_2 being the orders of H_1 and H_2 respectively. A necessary and sufficient condition that each of the given double co-sets contains the same number of distinct operators is that each of the conjugates of H_1 under G has the same number of operators in common with H_2. This is equivalent to saying that each of these conjugates of H_2 has the same number of operators in common with H_1. When this number of common operators is k then the number of distinct operators in each of these co-sets is $h_1 h_2/k$, and vice versa. In particular, each of these double co-sets must involve the same number of distinct operators whenever at least one of the two subgroups H_1, H_2 is invariant under G, and each of the double co-sets with respect to H_1 and H_2 involves exactly h_1h_2 distinct operators. whenever h_1 and h_2 are relatively prime. When H_1 and H_2 belong to the same set of conjugates under G then the number of these double co-sets which involve exactly h_1 distinct operators is evidently equal to the index of H_1 under the largest subgroup of G which transforms H_1 into itself, and all the other double co-sets involve a multiple of h_1 distinct operators. All of these multiples must be sub-multiples of h_1^2. When H_1 and H_2 are identical it is obvious that they may be regarded as conjugate, and hence the theorem which has just been stated applies directly to the double co-sets with respect to a single sub-group.

---

* Miller, Blichfeldt, and Dickson, *Finite Groups*, 1916, p. 27.

Suppose now that $h_1 = p^\alpha$, where p is a prime number, and assume that the index of H_1 under the largest subgroup of G which transforms H_1 into itself is k, k being prime to p. If the operators of G are represented as double co-sets with respect to H_1, i. e., if H_2 is assumed to be identical with H_1, it results that the order g of G can be represented as follows: $g = kp^\alpha + lp^{\alpha+1}$. Hence p^α must be the highest power of p which divides g. If c represents the number of conjugates of H under G it results that $kp^\alpha = g/c$, and hence $(c-1)g = lp^{\alpha+1}$. That is, $c \equiv 1 \bmod p$. It has therefore been proved by means of double co-sets that if H_1 is of index prime to p under the largest subgroup of G which transforms H_1 into itself then H_1 is a Sylow subgroup of G and the number of its conjugates under G is congruent to 1 mod p.

From the preceding paragraph it results that a necessary and sufficient condition that a subgroup H of order p^α is a Sylow subgroup of G is that the index of H under the group K formed by all the operators of G which transform H into itself is prime to p, for if this index were not prime to p the quotient group K/H would have an order which is divisible by p. Since the order of this quotient group is less than g it may be assumed that this group contains a subgroup whose order is a power of p and hence G contains a subgroup of order $p^{\alpha+1}$. This suggests that a proof of Sylow's theorem as regards the symmetric group S of degree p^m can be based upon a proof of the fact that S involves a subgroup P of order p^α which is of index prime to p under the group formed by all the substitutions of S which transform P into itself. This fact can easily be proved by mathematical induction.

When $m = 1$ the theorem is evident. If we assume that the theorem is true for the symmetric group of degree p^{m-1} it is not difficult to prove it true for S. In fact, S obviously contains a substitution s of order p and of degree p^m. The p^{m-1} cycles of this substitution are evidently transformed according to the symmetric group of degree p^{m-1} by the largest subgroup of S which transforms s into itself. The substitutions of S which transform the cycles of s according to a Sylow subgroup in the symmetric group of degree p^{m-1} generate a group of order p^α which contains no invariant substitution besides those generated by s. Hence all the substitutions of S which transform this group of order p^α into itself also transform s into itself, and therefore this group of order p^α is of index prime to p under the group formed by all the substitutions of S which transform it into itself. This group must therefore be a Sylow subgroup of S. From the fact that S contains a Sylow subgroup of order p^α it is very easy to deduce that every group whose order is divisible by p contains at least one Sylow subgroup whose order is a power of p.

The considerations which precede prove incidentally that the Sylow subgroups of order p^α contained in S are of index $(p-1)^m$ under S, since the index of these Sylow subgroups is obviously $p-1$ times the index of the corresponding Sylow subgroups under the symmetric group of degree p^{m-1}. Hence we not only know that this index is prime to p but we also know its exact value. The said considerations also prove that the number of the Sylow subgroups of order p^α contained in S which have in common a given invariant subgroup of order p is equal to the number of the Sylow subgroups of order a power of p in the symmetric group of degree p^{m-1}. Moreover, the number of the Sylow groups of order p^α contained in S is this number times the number of the possible subgroups of order p and of degree p^m contained in S.

From the method of proof here employed it also results directly that if the largest possible subgroup common to two Sylow subgroups of order p^α in any group G is of order p^β then the number of these Sylow subgroups is $\equiv 1 \bmod p^{\alpha-\beta}$. For instance, it is well known that no two subgroups of order 4 contained in the simple group of order 60 have any operator in common besides the identity. Hence the number of these subgroups must be $\equiv 1 \bmod 4$. As a matter of fact it is 5. It should be added that the simplifications suggested above do not apply to the well and favorably known proof of Sylow's theorem due to G. Frobenius. They apply to the older proof based on some properties of the symmetric group, especially to the form in which this proof is developed in the work to which reference was made in the first paragraph.

In closing we wish to refer to a variation in the proof of another fundamental theorem in the theory of groups of finite order since this variation may possibly simplify the proof for some readers. This variation relates to a proof that every subgroup of index 2 of any group G is invariant under G. To prove this theorem it may first be noted that if s represents any operator of such a subgroup H while t represents any operator of G which is not contained in H then st cannot be in H as otherwise the equation $xy = z$ would have more than one solution in G when two of its symbols are replaced by operators of G. For the same reason the product of two operators of G, neither of which is in H, must be in H. As $t^{-1}st$ is such a product, since neither st nor t^{-1} is in H, $t^{-1}st$ is such a product and must be in H. That is, H must be an invariant subgroup of G as a result of the fact that it is composed of half the operators of G.

THE UNIVERSITY OF ILLINOIS

REDUCTION OF EULER'S EQUATIONS TO A CANONICAL FORM*

BY J. H TAYLOR†

1. *Introduction.* As a by-product of the preparation of an earlier paper by the author, *A generalization of Levi-Civita's parallelism and the Frenet formulas.* Dissertation, University of Chicago, 1924,‡ two useful methods of solving for the second derivatives in Euler's equations associated with the problem of minimizing an integral were discovered. In this paper these two methods are presented in detail. Whereas it appears at first that the assumptions required to effect the solution in the two instances are quite different, it is here shown that in each of the two cases the assumptions which are made may be replaced by the supposition that the F_1 function of the calculus of variations does not vanish.

2. *The Euler Equations.* Suppose that

$$x^\alpha = x^\alpha(u), \quad u_1 \leqq u \leqq u_2, \qquad (\alpha = 1, \cdots, n),$$

are the equations of one of a class of curves joining two fixed points in an n-space, and let us consider the problem of selecting that curve of the class which gives to the integral

$$I = \int_{u_1}^{u_2} F(x, x')du$$

its minimum value. Here x and $x' = dx/du$ stand for the sets $x^1, \cdots, x^n$ and $x'^1, \cdots, x'^n$ respectively. It will be assumed that F satisfies the homogeneity condition

$$(1) \qquad F(x, \varkappa x') = \varkappa F(x, x'), \qquad \varkappa > 0.$$

---

* Presented to the Society, October 25, 1924.

† National Research Fellow in Mathematics.

‡ Transactions of this Society, vol. 27 (1925).

As a consequence of this it follows that

$$(2) \qquad F_\alpha x'^\alpha = F, \quad F_{\alpha\beta} x'^\alpha = 0, \qquad (\beta = 1, \cdots, n),$$

where the notation is $F_\alpha = \partial F / \partial x'^\alpha$, $F_{\alpha\beta} = \partial^2 F / \partial x'^\alpha \partial x'^\beta$, and where we employ the convention that each index letter which appears twice in the same term is understood to indicate a summation with respect to that letter from 1 to n as is customary in tensor analysis. From the second of equations (2) it follows that unless $x' \equiv 0$, the determinant $|F_{\alpha\beta}| = 0$.

The Euler differential equations defining the extremals for the integral I are

$$(3) \qquad \frac{d}{du} F_\beta - \frac{\partial F}{\partial x^\beta} = F_{\beta\alpha} x''^\alpha + \frac{\partial F_\beta}{\partial x^\alpha} x'^\alpha - \frac{\partial F}{\partial x^\beta} = 0,$$
$$(\beta = 1, \cdots, n).$$

These equations are not independent but satisfy the relation

$$(4) \qquad \left(F_{\beta\alpha} x''^\alpha + \frac{\partial F_\beta}{\partial x^\alpha} x'^\alpha - \frac{\partial F}{\partial x^\beta} \right) x'^\beta = 0,$$

which is easily verified by differentiating the identity $F_\beta x'^\beta = F$ with respect to u and applying (2). Let us suppose now that the parameter u has been so chosen that along our solutions of equations (3) we have $F(x, x') = 1$. Clearly such a selection is always possible if $F \neq 0$ along the solution, as we shall suppose, with the aid of the homogeneity condition (1). Since the determinant of coefficients $|F_{\alpha\beta}|$ vanishes we cannot solve (3) for the second derivatives by the usual process. Consider then the solution of the following system of equations

$$(5) \qquad \left\{ \begin{array}{lllll} F_{\beta\alpha} x''^\alpha & + F_\beta w & + \dfrac{\partial F_\beta}{\partial x^\alpha} x'^\alpha & - \dfrac{\partial F}{\partial x^\beta} & = 0 \\ F_\alpha x''^\alpha & & + \dfrac{\partial F}{\partial x^\alpha} x'^\alpha & & = 0. \end{array} \right.$$

It will be noticed that these differ from (3) in the introduction of an auxiliary variable w, and the adjoining of an equation obtained by differentiating $F = 1$. We shall assume that the determinant of coefficients

$$(6)\qquad \begin{vmatrix} F_{11} & F_{12} & \cdots & F_{1n} & F_1 \\ F_{21} & F_{22} & \cdots & F_{2n} & F_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & \cdots & F_{nn} & F_n \\ F_1 & F_2 & \cdots & F_n & 0 \end{vmatrix}$$

is not zero, and hence that the equations (5) may be solved for x''^{α}, w as functions of (x, x'). From (4) it follows at once that $w \equiv 0$, and therefore that the solutions x''^{α} of (5) are also solutions of (3).*

3. *The F_1 Function of the Calculus of Variations.* If we suppose the rank of $|F_{\alpha\beta}|$ to be $n-1$, we have from the second of (2)

$$\frac{x'^1}{F^{\beta 1}} = \frac{x'^2}{F^{\beta 2}} = \cdots = \frac{x'^n}{F^{\beta n}}, \quad (\beta = 1, \cdots, n),$$

where $F^{\beta\alpha}$ denotes the cofactor of $F_{\beta\alpha}$ in $|F_{\mu\nu}|$. On multiplying these ratios by x'^{β} and making use of the symmetry

$$F^{\alpha\beta} = F^{\beta\alpha}$$

it is seen that we may define a function F_1^* by the equations

$$\frac{x'^{\beta}x'^{\alpha}}{F^{\beta\alpha}} = \frac{1}{F_1^*}, \quad (\alpha, \beta \text{ not summed})$$

or

$$(7)\qquad x'^{\alpha}x'^{\beta}F_1^* = F^{\alpha\beta}.$$

The usual notation for this function is F_1, but F_1^* is used here to avoid confusion with $F_1 = \partial F/\partial x'^1$. From (7) it follows that

$$F_{\alpha}F_{\beta}x'^{\alpha}x'^{\beta}F_1^* = F_{\alpha}F_{\beta}F^{\alpha\beta},$$

---

[*] This method of solution was suggested by an analogous scheme used by Mason and Bliss, *The properties of curves in space which minimize a definite integral*, TRANSACTIONS OF THIS SOCIETY, vol. 9 (1908), p. 443.

and, by (2), this becomes

$$(F)^2F_1^* = F_\alpha F_\beta F^{\alpha\beta}.$$

Hence

$$(8)\qquad F_1^* = \frac{-1}{(F)^2}\begin{vmatrix} F_{11} & F_{12} & \cdots & F_{1n} & F_1 \\ F_{21} & F_{22} & \cdots & F_{2n} & F_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & \cdots & F_{nn} & F_n \\ F_1 & F_2 & \cdots & F_n & 0 \end{vmatrix}.$$

That is, the determinant (6) differs from the F_1^* function by a non-zero factor.

4. *A Second Method of Solution.* Let a function $f(x, x')$ be introduced by

$$(9)\qquad f(x, x') = \frac{1}{2}F^2(x, x').$$

The homogeneity condition is now expressed by

$$(10)\qquad f(x, \varkappa x') = \varkappa^2 f(x, x'), \qquad \varkappa > 0.$$

From (9) and (10) the following relations are readily deduced, the subscripts denoting as before the partial derivatives with respect to the corresponding x' variables:

$$(11)\quad \begin{cases} f_\alpha x'^\alpha = 2f, & f_{\alpha\beta}x'^\alpha = f_\beta, \\[4pt] \dfrac{\partial f_\alpha}{\partial x^\beta}x'^\alpha = 2\dfrac{\partial f}{\partial x^\beta}, & \dfrac{\partial f_{\alpha\beta}}{\partial x^\gamma}x'^\alpha = \dfrac{\partial f_\beta}{\partial x^\gamma}, \\[8pt] f_\alpha = FF_\alpha, & F_\alpha = \dfrac{f_\alpha}{F}, \\[8pt] f_{\alpha\beta} = F_\alpha F_\beta + FF_{\alpha\beta}, & F_{\alpha\beta} = \dfrac{f_{\alpha\beta}}{F} - \dfrac{f_\alpha f_\beta}{F^3}, \\[8pt] \dfrac{\partial f_\alpha}{\partial x^\beta} = F_\alpha\dfrac{\partial F}{\partial x^\beta} + F\dfrac{\partial F_\alpha}{\partial x^\beta}, & \dfrac{\partial F_\alpha}{\partial x^\beta} = \dfrac{\dfrac{\partial f_\alpha}{\partial x^\beta}}{F} - \dfrac{f_\alpha}{F^3}\dfrac{\partial f}{\partial x_\beta}. \end{cases}$$

The Euler equations are

$$F_{\beta\alpha}x''^\alpha + \frac{\partial F_\beta}{\partial x^\alpha}x'^\alpha - \frac{\partial F}{\partial x^\beta} = 0,$$

or

$$\frac{f_{\beta\alpha}x''^{\alpha}+\frac{\partial f_{\beta}}{\partial x^{\alpha}}x'^{\alpha}-\frac{\partial f}{\partial x^{\beta}}}{F} = \frac{f_{\beta}}{F^{3}}\left(f_{\alpha}x''^{\alpha}+\frac{\partial f}{\partial x^{\alpha}}x'^{\alpha}\right).$$

If again we suppose the independent variable to be so selected that $F=1$, the right member reduces to zero and the equation becomes

$$f_{\beta\alpha}x''^{\alpha}+\frac{\partial f_{\beta}}{\partial x^{\alpha}}x'^{\alpha}-\frac{\partial f}{\partial x^{\beta}} = 0.$$

By (11), this may be written in the form

$$(12)\qquad f_{i\alpha}x''^{\alpha}+\left(\frac{\partial f_{i\beta}}{\partial x^{\alpha}}-\frac{1}{2}\frac{\partial f_{\alpha\beta}}{\partial x^{i}}\right)x'^{\alpha}x'^{\beta} = 0.$$

The question of the solution of these equations for the second derivatives forces our attention upon the determinant $|f_{\alpha\beta}|$.

The determinant in question is

$$\begin{vmatrix} F_1F_1+FF_{11}, & F_1F_2+FF_{12}, & \cdots, & F_1F_n+FF_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ F_nF_1+FF_{n1}, & F_nF_2+FF_{n2}, & \cdots, & F_nF_n+FF_{nn} \end{vmatrix}.$$

Since

$$|F_{\alpha\beta}| = 0,$$

the determinant reduces to

$$\begin{vmatrix} F_1F_1, & FF_{12}, & \cdots, & FF_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ F_nF_1, & FF_{n2}, & \cdots, & FF_{nn} \end{vmatrix} + \begin{vmatrix} FF_{11}, & F_1F_2, & FF_{13}, & \cdots, & FF_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ FF_{n1}, & F_nF_2, & FF_{n3}, & \cdots, & FF_{nn} \end{vmatrix}$$

$$+\cdots+\begin{vmatrix} FF_{11}, & \cdots, & FF_{1n-1}, & F_1F_n \\ \cdots & \cdots & \cdots & \cdots \\ FF_{n1}, & \cdots, & FF_{nn-1}, & F_nF_n \end{vmatrix}$$

$$= F^{n-1}[F_1F_{\alpha}F^{\alpha 1}+F_2F_{\alpha}F^{\alpha 2}+\cdots+F_nF_{\alpha}F^{\alpha n}]$$

$$= F^{n-1}F_{\alpha}F_{\beta}F^{\alpha\beta}.$$

Hence

$$|f_{\alpha\beta}| = -F^{n-1}\begin{vmatrix} F_{11} & F_{12} & \cdots & F_{1n} & F_1 \\ F_{21} & F_{22} & \cdots & F_{2n} & F_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & \cdots & F_{nn} & F_n \\ F_1 & F_2 & \cdots & F_n & 0 \end{vmatrix},$$

which is the same as the F_1^* function except for a non-zero factor. If now we assume as before that the F_1^* function is different from zero, and therefore $|f_{\alpha\beta}| \neq 0$, the solutions of (12) may be obtained in the form

$$x''^{\lambda} = -\Gamma^{\lambda}_{\alpha\beta}x'^{\alpha}x'^{\beta}, \tag{13}$$

where $\Gamma^{\lambda}_{\alpha\beta}(x, x')$ is a function formed from $f_{\alpha\beta}(x, x')$ in precisely the same manner as the Christoffel symbol of the second kind $\{\alpha\beta, \lambda\}$ is formed from the quadratic form representing the square of the length element of a Riemann geometry. The explicit formulas for these coefficients are $\Gamma^{\lambda}_{\alpha\beta}(x, x') = f^{\lambda\mu}[\alpha\beta, \mu]$, where $f^{\lambda\mu}$ is the element of the reciprocal matrix of $f_{\alpha\beta}$ corresponding to the term $f_{\lambda\mu}$, and

$$[\alpha\beta, \mu] = \frac{1}{2}\left(\frac{\partial f_{\alpha\mu}}{\partial x^{\beta}} + \frac{\partial f_{\beta\mu}}{\partial x^{\alpha}} - \frac{\partial f_{\alpha\beta}}{\partial x^{\mu}}\right).$$

The method of obtaining the solution of (12) in the form (13) is formally exactly the same as that used in obtaining the equations of the geodesics in a Riemann space.*

It may be remarked, in terms of the concepts of *parallelism* and *curvature*, as developed in the author's thesis mentioned in § 1, that along a geodesic the tangent to the geodesic remains parallel to itself, or in other words that a geodesic is a curve of constant direction. A geodesic is also characterized as a curve of zero first curvature.

PRINCETON UNIVERSITY

---

* See F. D. Murnaghan, *Vector Analysis and the Theory of Relativity*, 1922, pp. 89–90.

INTEGRO-DIFFERENTIAL EQUATIONS*

BY I. A. BARNETT

1. *Introduction.* It is proposed in this note to prove that if, in the integro-differential equation

$$\frac{\partial\varphi(x,t)}{\partial t}=\int_0^1\varkappa(x,y)\varphi(y,t)dy, \tag{1}$$

the real and continuous kernel $\varkappa(x,y)$ is also symmetric or skew-symmetric then there can be no solutions of the form

$$e^{\lambda t}\sum_{k=0}^{p}\alpha_k(x)t^k \tag{2}$$

unless all $\alpha_k(x)$, $k\geqq 1$ are identically zero.† It will then be pointed out that a similar property holds for systems of linear differential equations

$$\frac{d\varphi_i}{dt}=\sum_{j=1}^{n}\varkappa_{ij}\varphi_j; \tag{3}$$

viz., if the $\varkappa_{ij}$ are real and constant (with respect to t), then in case the coefficient system is either symmetric or skew-symmetric, no real set of solutions can contain polynomials in t as factors. This fact seems, to the best of the writer's knowledge, to have escaped attention in the literature.‡

---

* Presented to the Society, December 29, 1923.

† See I. A. Barnett, *Integro-differential equations with constant limits of integration*, this BULLETIN, vol. 26, pp. 193–203.

‡ I have recently had the privilege of examining, through the courtesy of Professor Brand of the University of Cincinnati, a course of lectures on linear differential equations that Professor Bôcher gave, in which the result for the symmetric case is given without proof. Professor Bôcher states there that Weierstrass worked out this case in 1858, ten years before he introduced elementary divisors. The writer has examined the collected works of Weierstrass but he was unable to find this result.

2. *Kernel Symmetric.* Consider first the integro-differential equation (2) where $\varkappa(x, y) = \varkappa(y, x)$. Suppose there were a solution of the form (2). Substituting (2) in (1), we find

$$\sum_{k=0}^{p-1} tk \int_0^1 \varkappa(x, y)\,\alpha_k(y)\,dy + t^p \int_0^1 \varkappa(x, y)\,\alpha_p(y)\,dy$$
$$= \sum_{k=0}^{p-1} (k+1)\,\alpha_{k+1}(x)\,t^k + \sum_{k=0}^{p-1} \lambda\alpha_k(x)\,t^k + \lambda\alpha_p(x)\,t^p.$$

Writing $\mu = 1/\lambda$ one sees that this is equivalent to the following simultaneous system of integral equations

$$(4)\quad \begin{cases} \alpha_p(x) = \mu \displaystyle\int_0^1 \varkappa(x, y)\,\alpha_p(y)\,dy, \\ \alpha_k(x) = \mu \displaystyle\int_0^1 \varkappa(x, y)\,\alpha_k(y)\,dy - \mu(k+1)\,\alpha_{k+1}(x), \\ \qquad\qquad (k = p-1, p-2, \ldots, 2, 1, 0). \end{cases}$$

Suppose the integral equation

$$\alpha_p(x) = \mu \int_0^1 \varkappa(x, y)\,\alpha_p(y)\,dy$$

has a characteristic number μ_1 of index q_1 with the corresponding characteristic functions $\Phi_1^{(p)}, \Phi_2^{(p)}, \cdots, \Phi_{q_1}^{(p)}$ which may be supposed normed and orthogonalized. Then the solution $\alpha_p(x)$ is expressible as a linear combination of these solutions. Consider now the second equation in (4),

$$(5)\qquad \alpha_{p-1}(x) = \mu \int_0^1 \varkappa(x, y)\,\alpha_{p-1}(y)\,dy - \mu_p\alpha_p(x),$$

for the same characteristic number μ_1. Necessary conditions that this last non-homogeneous integral equation have solutions are that all the associated characteristic functions of the kernel $\varkappa(x, y)$ be orthogonal to the function $\alpha_p(x)$. But, since the kernel is, by hypothesis, symmetric, it follows that the set of associated characteristic functions is the same as the set of characteristic functions of $\varkappa$; viz., the set $\Phi_1^{(p)}, \cdots, \Phi_{q_1}^{(p)}$. This is impossible unless $\alpha_p(x) \equiv 0$ since $\int_0^1 \Phi_i^{(p)}(x)\,\Phi_i^{(p)}(x)\,dx \neq 0$, $i = 1, 2, \cdots, q_1$. Reasoning in the same way, one can prove that $\alpha_{p-1} \equiv \cdots \equiv \alpha_1 \equiv 0$.

3. *Kernel Skew-Symmetric.* Suppose now that the kernel $\varkappa(x,y)$ is skew-symmetric, i. e., $\varkappa(x,y) = -\varkappa(y,x)$. Equations (4) also hold in this case. It is well known in this case that the characteristic numbers are all of the form λi and that if $(\lambda_1 i, \Phi_1(x))$ are a characteristic number and function, respectively, of the kernel $\varkappa(x,y)$ then $\overline{\Phi}_i(x)$ is the associated characteristic function for the same characteristic number $\lambda_1 i$ where $\overline{\Phi}_1(x)$ is the conjugate of $\Phi_1(x)$. Hence, necessary conditions for the existence of a solution of equation (5) would be $\int_0^1 \alpha_p(x)\overline{\Phi}_j(x)dx = 0$, $j = 1, \cdots, q_1$. If then the q_1 characteristic functions $\Phi_1^{(p)}, \cdots, \Phi_{q_1}^{(p)}$ and the q_1 associated characteristic functions $\overline{\Phi}_1^{(p)}, \cdots, \overline{\Phi}_{q_1}^{(p)}$ are supposed normed and biorthogonalized, it is clear that not all of these necessary conditions can be satisfied since at least one of them reduces to $\int_0^1 \Phi_j^{(p)}(x)\overline{\Phi}_j^{(p)}(x)dx$ which equals unity.

4. *Analogy for Differential Equations.* Consider now the system of differential equations

$$\frac{d\varphi_i}{dt} = \sum_{j=1}^{n} \varkappa_{ij}\varphi_j, \tag{3}$$

and a set of solutions $\{\varphi_1(t), \cdots, \varphi_n(t)\}$. It is well known that if $\{\psi_1(t), \cdots, \psi_n(t)\}$ is any set of solutions of the system of differential equations adjoint to (3), one has the relation

$$\varphi_1(t)\psi_1(t) + \cdots + \varphi_n(t)\psi_n(t) = \text{constant}.$$

But in case the coefficient system $\varkappa_{ij}$ is skew-symmetric, the set $\{\psi_i(t)\}$ coincides with the set $\{\varphi_i(t)\}$* and consequently one must have $\varphi_1^2(t) + \cdots + \varphi_n^2(t) = \text{const}$. In other words the set $\{\varphi_i(t)\}$ cannot contain polynomials in t as factors. In the symmetric case a solution of a given system with t replaced by $-t$ is a solution of the adjoint system and so again we have the same result.†

THE UNIVERSITY OF CINCINNATI

---

* Goursat-Hedrick, *Mathematical Analysis*, vol. 2, part. II, p. 157.

† The writer is indebted to Professor O. D. Kellogg for suggesting this proof by the method of adjoint systems.

A NEW TYPE OF DOUBLE SEXTETTE CLOSED UNDER A BINARY (3, 3) CORRESPONDENCE*

BY LOUISE D. CUMMINGS

1. *Introduction.* In connection with the Poncelet theorem that if a polygon of n sides can be inscribed in one conic and circumscribed to a second conic then an infinite number of these polygons exist, much investigation, both by elliptic functions and by algebraic methods, has been effected concerning the closure property for the (2,2) correspondence of the double binary forms. For the (3,3) correspondence the direct algebraic attack upon even the point sets of low orders has been, until rather recently, somewhat neglected.

Franz Meyer, among others, studied the (3, 3) correspondence of four points and four planes and obtained the surprising result that if there is a first tetrahedron inscribed in one cubic curve and circumscribed to a second cubic curve there may not be a second tetrahedron, but if there is a second tetrahedron then an infinity of these tetrahedrons occur. The existence of one particular closed set of seven points and seven planes in a (3,3) correspondence, with the poristic property like the Poncelet polygons, was established in 1915 by White† and confirmed by Coble‡ who has investigated the general (m, n) correspondence. While Coble has not attempted an exhaustive classification, he has listed fourteen types, old and new, which are poristic configurations of double binary forms, among them one closed set of five points in a (3,3) correspondence. For periodic sets of six points in a (3,3) correspondence particular results are lacking. In this investigation of the (3,3) correspondence for $n = 6$, two non-congruent types of sextettes have been discovered by a simple algebraic method which is applicable to higher values of n.

---

* Presented to the Society, October 25, 1924.

† Proceedings of the National Academy, vol. 1 (1915), p. 464.

‡ American Journal, vol. 43, No. 1 (Jan., 1921).

A polyhedron of n faces, including the tetrahedron as the special case for $n = 4$, is suggested as a possible geometrical interpretation of the (3,3) correspondence for all values of n.

2. *The Special Problem.* The (3,3) correspondence

$$(a_0x^3+b_0x^2+c_0x+d_0)y^3+(a_1x^3+b_1x^2+c_1x+d_1)y^2+$$
$$(a_2x^3+b_2x^2+c_2x+d_2)y+(a_3x^3+b_3x^2+c_3x+d_3) = 0$$

contains fifteen constants. If the binaries x and y vary from 0 to ∞ and are independent projectively, this gives in general a non-closed correspondence. In an earlier paper* the author established the existence of one type of closed double sextette in a special (3,3) correspondence, possessing the poristic property like the Poncelet polygons in the (2,2) correspondence. This paper exhibits the nature of a second type of poristic sextette, and shows that only two non-congruent types exist. Two related gauche cubic curves are used for exemplification in order that this investigation may be compared readily with Meyer's study of the double quartette in which some of the generalized theorems appear to be erroneous. The points x lying on a twisted cubic K are represented by the six parameters a, b, c, d, e, f; the planes y osculating a second twisted cubic K' are represented by the parameters 1, 2, 3, 4, 5, 6 used as symbols; the double sextette is to be subject to the cyclic substitution $S \equiv (a\,b\,c\,d\,e\,f)(1\,2\,3\,4\,5\,6)$ and the correspondence is to be invariant under S and its powers. The relations between the points x and the planes y are shown in the two following mutual arrays I and I′, where the point $x = a$ lies in the triad of planes 1, 2, 4 or the plane $y = 1$ passes through the triad of points a, d, f.

---

* To appear in PROCEEDINGS INTERNATIONAL MATHEMATICAL CONGRESS, Toronto, 1924.

I

x	a	b	c	d	e	f
y	1	2	3	4	5	6
	2	3	4	5	6	1
	4	5	6	1	2	3

I′

y	1	2	3	4	5	6
x	a	b	c	d	e	f
	d	e	f	a	b	c
	f	a	b	c	d	e

3. *Determination of a Normal Form.* For the (3,3) correspondence with a closed cycle of six points, investigation shows that the form with fifteen constants may, without loss of generality, be replaced by a determinantal form which by the aid of the binary identities is reducible to an expression in four terms. Therefore the following trial form is set up initially,

$$\begin{aligned} F(x,y) \equiv{} & A\cdot xb\cdot xc\cdot xd\cdot y1\cdot y2\cdot y4 + B\cdot xc\cdot xd\cdot xa\cdot y2\cdot y3\cdot y5 \\ & + C\cdot xd\cdot xa\cdot xb\cdot y3\cdot y4\cdot y6 + D\cdot xa\cdot xb\cdot xc\cdot y4\cdot y5\cdot y1 = 0, \end{aligned}$$

where for brevity xa represents the determinant of order 2 or the simple difference $x - a$.

This form visibly satisfies twelve of the conditions required by the array I, all on a, b, c, d; the application of the remaining six conditions furnished by the pairs $x = e$, $y = 5, 6, 2$ and $x = f$, $y = 6, 1, 3$ gives six equations linear in A, B, C, D such as

$$\begin{aligned} A\cdot eb\cdot ec\cdot ed\cdot 61\cdot 62\cdot 64 &+ B\cdot ec\cdot ed\cdot ea\cdot 62\cdot 63\cdot 65 \\ &+ D\cdot ea\cdot eb\cdot ec\cdot 64\cdot 65\cdot 61 = 0. \end{aligned}$$

Three of these equations determine the constants B/A, C/A, D/A, and with the binary identities allow the reduction of the form to any desired normal form such as the following:

$$\begin{aligned} F(x,y) \equiv{} & 12\cdot 35\cdot 56\cdot ae\cdot cf\cdot xb\cdot xc\cdot xd\cdot y1\cdot y2\cdot y4 \\ & + 61\cdot 25\cdot 14\cdot bf\cdot ce\cdot xa\cdot xc\cdot xd\cdot y2\cdot y3\cdot y5 \\ & + 12\cdot 15\cdot 25\cdot ce\cdot cf\cdot xa\cdot xb\cdot xd\cdot y3\cdot y4\cdot y6 \\ & + 15\cdot 23\cdot 26\cdot cf\cdot de\cdot xa\cdot xb\cdot xc\cdot y4\cdot y5\cdot y1 = 0. \end{aligned}$$

The three remaining equations, which might conceivably impose redundant conditions upon the twelve parameters, give for the special case of the double sextette three independent relations E_1, E_2, E_3, equivalent to an invariant set of fifteen, where

$$E_1 \equiv \frac{ab\cdot cd}{ac\cdot bd} - \frac{23\cdot 45}{24\cdot 35} = 0; \quad E_2 \equiv \frac{ab\cdot ce}{ac\cdot be} - \frac{23\cdot 46}{24\cdot 36} = 0;$$

$$E_3 \equiv \frac{ab\cdot cf}{ac\cdot bf} - \frac{23\cdot 41}{24\cdot 31} = 0.$$

These fifteen invariant relations $E_1 = 0$, etc. express a unique projectivity of the x range to the y range, namely, that

$$\text{is projective to} \quad \begin{matrix} a & b & c & d & e & f \\ \\ 2 & 3 & 4 & 5 & 6 & 1. \end{matrix}$$

The form $F(x, y)$ thus admits by its construction one configuration of period 6; is this an isolated closed sextette, or does the form admit an infinitude of these configurations and so possess the poristic property?

4. *Conditions for the Porism of the Form.* To ascertain if the sextette is necessarily poristic, the twelve parameters which express the correspondence are replaced by the neighboring set $a+da$, $b+db$, $\cdots$, $6+d6$, which must determine the same correspondence by the same formal equation. That is, the first differential must be a doubly cubic form identical with the original form $F(x, y)$, so that

$$F(x, y) + \frac{\partial F}{\partial a} da + \cdots + \frac{\partial F}{\partial 6} d6 \equiv M \cdot F(x, y).$$

A comparison of the coefficients of these two forms may be established by making use again of the eighteen pairs of points (x, y) given by the array I. These furnish eighteen equations of condition whose explicit form is derivable either by an algebraic process or by the following simple geometric method, in which for brevity da is replaced by a'. The array I$'$ shows that the form $F(x, y)$ regarded as a plane sextic is cut by the line $y = 1$ in the three points a, d, f and in a triple point at infinity, while the neighboring line $y = 1 + 1'$ cuts the curve in the points

$a+a'$, $d+d'$, $f+f'$ and in the same triple point at infinity. The slopes at the points $(a, 1)$, $(d, 1)$, $(f, 1)$ furnish the three equations

$$1' = \left(\frac{dy}{dx}\right)_{a,1} a' = \left(\frac{dy}{dx}\right)_{d,1} d' = \left(\frac{dy}{dx}\right)_{f,1} f'.$$

Replacing $\left(\frac{dy}{dx}\right)_{a,1}$ by $\left(\frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\right)_{a,1}$ and making similar substitutions for the slopes at $(d, 1)$ and $(f, 1)$, we have the three equations

$$\left(\frac{\partial F}{\partial x}\right)_{a,1} a' + \left(\frac{\partial F}{\partial y}\right)_{a,1} 1' = 0; \quad \left(\frac{\partial F}{\partial x}\right)_{d,1} d' + \left(\frac{\partial F}{\partial 1}\right)_{d,1} 1' = 0;$$
$$\left(\frac{\partial F}{\partial x}\right)_{f,1} f' + \left(\frac{\partial F}{\partial y}\right)_{f,1} 1' = 0.$$

The substitution S applied to each of these three equations gives three cycles of six equations or eighteen equations of condition to be satisfied by the twelve differentials a', b', $\cdots$, $5'$, $6'$. To establish the consistency of these eighteen equations and the uniqueness of the solution their number must be reduced to eleven. This can be done if these equations satisfy the two following identities.

$$(1) \quad \frac{1'}{2'} \cdot \frac{2'}{3'} \cdot \frac{3'}{1'} \equiv 1; \qquad (2) \quad \frac{1'}{2'} \cdot \frac{2'}{3'} \cdot \frac{3'}{4'} \cdot \frac{4'}{5'} \cdot \frac{5'}{6'} \cdot \frac{6'}{1'} \equiv 1,$$

where

$$\frac{1'}{2'} = \frac{13 \cdot 16 \cdot ad \cdot af \cdot bc \cdot ce}{23 \cdot 26 \cdot ae \cdot cf \cdot ab \cdot cd},$$

and the other ratios have values similar in form. Now both of these identities are found to be satisfied if

$$\frac{ac \cdot bd \cdot fe}{ae \cdot bc \cdot fd} = 1,$$

that is, if the three pairs of points f, c; a, d; b, e are in a quadric involution.

The identity (1) enables us, with the aid of the cyclic property, to discard six of the eighteen equations, while the identity (2) disposes of one more, leaving eleven linear equations for the determination of the eleven ratios

$$\frac{a'}{1'}, \frac{b'}{1'}, \ldots, \frac{6'}{1'}.$$

Therefore the eighteen equations become consistent and determine uniquely the ratios of the twelve differentials. This gives then the differential operator which generates in the Lie sense a singly infinite group of transformations of the curves K and K' into themselves, or, as we prefer to regard it, which transforms the sextette of points on the curve into a simple infinity of sextettes similarly related to the curves K and K'. Hence we have the following theorem.

THEOREM. *Six points selected arbitrarily on a twisted cubic K can form in a* (3,3) *correspondence an isolated closed system of the type here sought, but if the six points are specialized as three pairs in a quadric involution on the cubic then the sextette is poristic and slides along the cubic K while the six associated planes continue to osculate the same second curve K' of class* 3.

5. *The Secant-Axes of the Double Sextette.* In our closed set of six points and six planes the array I$'$ shows a line *ad* joining two points of K and lying in two planes 1 and 4 of K'—a line which is a secant of K and an axis of K'. So also two other secant-axes *be* and *cf* are obviously present. Now a well known theorem of Cremona fixes the number of secants of one twisted cubic, which are at the same times axes of another twisted cubic, at six, and while three of these secant-axes are in evidence the whereabouts of the remaining three requires further investigation.

6. *Meyer's Erroneous Statement concerning the Hurwitz Relation.* A study by this method of investigation of four

points and four planes in a (3,3) correspondence presents no difficulty and gives results in entire agreement with those obtained by Meyer for the double quartette. But a discrepancy arises from Meyer's further assertion that whenever the number of secant-axes is infinite, then the curves K and K' are necessarily in the Hurwitz relation, that is, these lines can be grouped in sets of six to form *the edges of infinitely many tetrahedrons,* all inscribed in K and circumscribed to K'. The Hurwitz relation exists for the double quartette but Meyer's assertion that it obtains in every poristic system is erroneous, as this double sextette shows. The secants through a are ad, ac, af and of these ad alone is a secant-axis. Now the Hurwitz relation requires at each point of the curve K three secant-axes, *whereas the sextette possesses one and no more*—other essential conditions also are not fulfilled.

This discrepancy in results can be harmonized only if for $F(x, y) = 0$, we find $A \equiv 0$, $B \equiv 0$, $C \equiv 0$, $D \equiv 0$, but as each of these coefficients is a single product of differences, this cannot occur unless points are assumed coincident.

7. *The Non-Congruent Types of Double Sextettes.* Since this correspondence is to be invariant under the cyclic substitution $S \equiv (abcdef)(123456)$ only three non-congruent types can occur, namely a with the triad 1, 3, 5; a with the triad 1, 2, 4; and a with the triad 1, 2, 3. The type a with 1, 3, 5 is not properly sextic and so is omitted; a with 1, 2, 4 is discussed in this paper. The double sextette in which a is associated with the planes 1, 2, 3 was investigated in an earlier paper* and was found to be poristic for six points specialized as elements in a quadric involution, precisely similar to the specialization required for the double sextette here discussed. In the type a; 1, 2, 3 the six secant-axes are all in evidence but the conditions essential for the Hurwitz relation are not fulfilled.

---

* Loc. cit.

8. *The Poristic Double Quintette.* This method of investigation has been applied to five points and five planes, and gives results in agreement with Coble's theorem concerning the existence of a double quintette in a (3,3) correspondence possessing the poristic property.

Under the substitution $S \equiv (a\,b\,c\,d\,e)(1\,2\,3\,4\,5)$ all possible types are congruent to the type a with the triad of planes 1, 2, 3. A normal form is

$$\begin{aligned} F(x,y) \equiv 45 \cdot ae \cdot xb \cdot xc \cdot xd \cdot y1 \cdot y2 \cdot y3 \\ + 51 \cdot be \cdot xc \cdot xd \cdot xa \cdot y2 \cdot y3 \cdot y4 \\ + 12 \cdot ce \cdot xd \cdot xa \cdot xb \cdot y3 \cdot y4 \cdot y5 \\ + 23 \cdot de \cdot xa \cdot xb \cdot xc \cdot y4 \cdot y5 \cdot y1 = 0. \end{aligned}$$

Conditions are imposed upon the ten differentials a', b', $\cdots$, $4'$, $5'$ by fifteen equations similar in form to those which occur for the double sextette. The fifteen equations are reducible to nine consistent equations, which determine uniquely the nine ratios

$$\frac{a'}{1'}, \frac{b'}{1'}, \cdots, \frac{5'}{1'},$$

if the set $a\ b\ c\ d\ e$ *is projective to* 2 3 4 5 1. No other conditions being imposed upon the parameters, a correspondence of period 5 can be determined with five arbitrary points for a fundamental set. The quintette is poristic; five secant-axes are ab, bc, cd, de, ea but the secant-axes do not fulfil the Hurwitz relation, so that here again Meyer's assertion is found incorrect.

9. *A Geometric Interpretation.* Since the Hurwitz relation does not hold for $n = 5$ nor for $n = 6$, there is for these cases no tetrahedron which slides along the curve K while its faces continue to osculate the second cubic K', as in the case of $n = 4$. However, one geometric figure applicable to all cases may be visualized in the following manner. From a polyhedron with $n-1$ lateral faces, such that no four of these lateral faces pass through a common point, cut a wedge whose upper and lower

bases intersect one of the lateral faces in the same straight line. The n planes which form the faces of the solid intersect in

$$\frac{n(n-1)(n-2)}{6}$$

vertices, and n of these vertices slide along the curve K while the n planes which form the faces continue to osculate the curve K'. This solid obviously becomes the tetrahedron for $n = 4$.

10. *Conclusion.* The method here employed is applicable to n points and n planes in a (3,3) correspondence. The determination of a normal form disposes of fifteen of the conditions imposed by the n-by-3 array I, leaving $3(n-5)$ conditions to be imposed upon the $2n$ parameters. The difficulty of eliminating from these $3(n-5)$ conditions those which are redundant increases rapidly with the increase in n, due to the increasing complexity in the coefficients of the normal form. The $3n$ equations arising in the comparison of $F(x, y)$ and its first derivative can be reduced to $2n-1$ equations, just sufficient for the determination of the $(2n-1)$ ratios $\frac{a'}{1'}, \frac{b'}{1'}, \cdots$, by two test identities similar to (1) and (2) used above. While the increase in n is accompanied always by an increase in the number of determinants in $\frac{a'}{1'}, \cdots$, nevertheless a further exploration of the field, by this method, does not present insurmountable difficulties for point sets of low orders.

The exhibition of these poristic point sets is one concrete step towards the realization of a suggestion offered by Mr. White in a paper on Poncelet polygons—"That all Poncelet systems are associated with linear involutions upon rational curves and that in this feature, possibly, lies even more promise of generalizations and discoveries than in Jacobi's brilliant and beautiful depiction by the aid of periodic functions".

VASSAR COLLEGE

SHORTER NOTICES

An Elementary Treatise on Frequency Curves and their Application in the Analysis of Death Curves and Life Tables. By Arne Fisher. Translated from the Danish by E. A. Vigfusson. With an introduction by Raymond Pearl. American Edition. New York, Macmillan, 1922. xv + 244 pp.

This work is divided into two chapters. The first, "An Introduction to the Theory of Frequency Curves", is practically the same as Chapters XIV–XVIII of *Mathematical Theory of Probabilities*, by the same author. This chapter takes up the treatment of frequency curves associated with the names of Laplace, Poisson, Charlier, Thiele and Gram, and serves as the theoretical introduction to the second chapter entitled "The Human Death Curve." In this chapter the author presents a method of constructing mortality curves from death lists by age and cause without knowledge of the number exposed to risk. In the United States this method has been received with various degrees of enthusiasm—from that of the extremely eulogistic introduction to the book down to utter condemnation by several well known actuaries whose views carry much weight. It is an axiom in actuarial science, often repeated and emphasized, that a mortality curve cannot be constructed from death lists alone. Some criticisms seem to imply that Fisher had done just this thing and a man of straw is created and warmly pelted. But he does not claim to get a mortality curve from death lists alone. It is true that he does not have the exposed to risk, but he has a substitute in something biologists and physicians have told him about the frequency distribution of deaths according to age from various classes of causes.

Dispensing with actuarial technicalities, the problem in the language of the theory of probabilities is as follows:

A bag contains a large number of black and white balls. Each ball has a number stamped upon it which may be any integer from 0 to 105. The black balls are also stamped with one of the letters a, b, c, d, e, f, g, h. Balls are drawn from the bag one at a time and replaced. A record is kept of the numbers and letters on the black balls, but no attention is paid to the numbers on the white balls. Later it is found desirable to construct a table of d_x/l_x, where d_x is the number of black balls, marked with the number x, and l_x is the total number of observed balls so marked. It is of course impossible to make such a table from the black ball records. But there is available information about the distribution of the balls in other bags or of balls in the pile from which bags are filled with justifies one in making an assumption as to the laws of the frequencies in the bag in question. A table constructed upon this hypothesis may be tested by experience and found correct. The whole matter depends upon the validity of the assumption.

The method of the book assumes a given list of deaths according to sex, age and cause, and to this is added the biological hypothesis: "The frequency distribution of deaths according to age from certain groups of causes of death among the survivors in a mortality table tend to cluster around certain ages in such a manner that the frequency distribution can be represented by either a Laplacean-Charlier or a Poisson-Charlier frequency curve."

With this goes a classification of causes of death according to some seven or eight groups. The mortality curve is considered as a compound curve made up from component frequency curves corresponding to the groups. The characteristics of these components are supposed known a priori.

Upon this foundation, Fisher is able to build up a mortality table. In the book he goes through the actual calculation of several tables that turn out to be satisfactory. In the mind of the reviewer the book marks a step forward in actuarial science in that it points out a method of attack on problems connected with that great mass of data in which the exposed to risk is difficult or impossible to find. There is no claim made that the method supersedes the old conventional method depending upon the number exposed to risk and losses among such exposures. However, it is possible that further experience with the method by the author and by others will show limitations in its applications. Before it is accepted by the conservative actuary, he will require much more evidence in the nature of satisfactory tables computed by many actuaries.

A. R. CRATHORNE

Analytical Mechanics. Second edition. By Edwin H. Barton. London, Longmans, Green and Company, 1924. 593 pp.

The first edition of this book was published in 1911. The new material in the second edition consists mainly of alternative proofs and more examples. About three hundred examples chosen from London examination questions set in the years 1911–23 have been added, bringing the total number to more than a thousand. To these a complete set of answers is now given separately at the end.

The text, divided into six parts, covers the field of mechanics quite thoroughly. Besides a preliminary survey of the scope of mechanics, the introductory part contains a collection of formulas from algebra, geometry, trigonometry, and the calculus. The second part gives a very complete account of the kinematics of a point and of a rigid body, including mechanisms and strains. The treatment of kinetics is preceded by a discussion of Newton's principles and criticisms of these principles by Mach and others. Statics follows kinetics and includes a good chapter on attractions and potential. Part V contains a chapter on hydrostatics and one on hydrokinetics. The sixth part consists of a short chapter on statics of elastic solids.

The author expects that most users of the book will bring to it some previous knowledge of the subject, and the elementary parts are briefly outlined to serve as a reference and for logical completeness. The general scope and treatment is designed to meet the needs of degree candidates of London and other universities. Very few, if any, American universities offer courses for which this book would be suitable as a text. But as a reference book for the teacher it would be very valuable because of the large number of examples and the very complete index.

W. R. LONGLEY

Clerk Maxwell's Electromagnetic Theory (The Rede Lecture for 1923). By H. A. Lorentz Cambridge University Press, 1923. 35 pp.

The Theory of Relativity, Studies and Contributions. By Archibald Henderson, A. W. Hobbs, and J. L. Lasley. University of North Carolina Press, Chapel Hill, N. C., 1924. xiii+99 pp.

Some men can put into a brief exposition a revelation of such insight and appreciation of the scientific significance of their topics as they may have won after years of detailed study. Among authors who have written on the theory of relativity, A. S. Eddington has written expositions of the type which I have in mind, his Romanes Lecture (delivered in 1922 at the Sheldonian Theater in Oxford) being an excellent example. The more recent lecture by Lorentz, mentioned above, is in the same class, conveying a nontechnical appreciation of the place of Maxwell's theory in the modern development of physics.

On the other hand the booklet by Professor Henderson and his colleagues is a disappointment to those who have read Professor Henderson's magazine articles on Bernard Shaw. One might have expected a mathematician of comparatively broad interests to bring to the subject of relativity a more novel point of view and a more significant insight into the physical meaning of the subject. Instead he has hurriedly and carelessly written a formal outline of the theory. As an example of carelessness I will refer to page 66 where the reader is suddenly confronted with the statement that the mass of the sun is 1.47 kilometers, with no explanation of the use of a linear unit for mass. Eddington's Report to the Physical Society of London is a better and less expensive work of approximately the same scope.

C. N. REYNOLDS, JR

Einleitung in die Mengenlehre. 2d edition. By A. Fraenkel. Berlin, Julius Springer, 1923. viii + 251 pp.

In this edition this book appears as volume 9 of the series, *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen.* The author in writing this edition tried to produce a book which would fulfill the general requirements of that series and also the purpose of the first edition*, i. e. to give an exposition of the elements of Mengenlehre which would be clearly comprehensible to one of little or no mathematical training. The prolixity of exposition necessary in realizing the latter aim hinders the realization of the former. The elegance and, in particular, the conciseness which the initiated desires is missing. However, the author seems to have come as near as possible to fulfilling with a single book both purposes.

The first 151 pages of the new edition contain almost verbatim all of the material of the first 129 pages of the old edition and in addition the introduction of certain concepts, more abstruse than those of the earlier book, and the details of certain proofs which previously were give only in outline. The material of chapter 12, pp. 129–151, of the old edition is elaborated in 90 pages, pp. 151–241, of the 2d edition. These pages are devoted to the foundations of the subject. About half of this space is taken up with a historical sketch of the critical examination of the theory, including an exposition of the paradoxes which gave rise to this critique and a brief description of the various procedures devised to give a logically consistent theory. Here are briefly explained and contrasted intuitionalism, which recently has been the subject of considerable work by Brouwer and Weyl, the method of logicizing as developed by Russell and Whitehead and J. Königs, and the axiomatic method as employed by Zermelo and Hilbert. The remaining half is devoted to a detailed exposition of the axiomatic developement of Mengenlehre which Zermelo produced in the period from 1904–1908 and to a sketch of Hilbert's recent work on the question of a proof of the consistency of a system of postulates. In connection with the Zermelo axiomatic the author gives in fine print an exposition of his contribution to the clarifying of the notion "eine definite Aussage" as used by Zermelo.

The exposition of the Zermelo-Hilbert way of setting up a theory of sets is well done but the brevity with which the other methods, logicization and intuitionalism, are presented makes it impossible to give an exposition of the latter methods on a par with that of the former. This is to be regretted. However, in these 90 pages we have the most readily available means of obtaining an acquaintance with the present state of the theory of the foundations of Mengenlehre

---

[*] See this BULLETIN, vol. 27 (1921), pp. 333–334.

and, indeed, of mathematics. Although none of the various theories, the logical theories of Russell and Whitehead and of J. Kónigs, the intuitional theory, suggested by Kronecker and developed by Brouwer and Weyl, the axiomatic theory brought forth by Zermelo and contributed to by Fraenkel and others and Hilbert's theory of proof, which are described here, is in a finished form, they all seem to lead to viewing mathematics and logic as a game in which the pieces and the rules of operation are defined as concretely as possible.

The book closes with a chapter of a few pages which contains a discussion of the connection between Mengenlehre and other branches of mathematics and a bibliography of the subject. These last pages are nearly identical with the corresponding ones of the first edition.

No errors in definitions or proofs have been noted, and the typography is excellent, as in previous volumes of the Springer series.

G. A. Pfeiffer

Theorie der Differentialgleichungen. Vorlesungen aus dem Gesamtgebiet der gewöhnlichen und der partiellen Differentialgleichungen. By Ludwig Bieberbach. Berlin, Julius Springer. 1923. viii + 317 pp.

"To write a text book on differential equations which shall be a suitable textbook, and at the same time set forth the spirit, methods and results of the theory in all of their aspects, so that the student shall be prepared to read original papers intelligently—that appears to be an impossibility." With these words Bieberbach opens the preface to the book under review, which is intended only as an introduction to the subject. Its scope is best brought out by giving the titles of the four sections into which the book is divided.

Section I. Ordinary differential equations of the first order.

Section II. Ordinary differential equations of the second order.

Section III. Partial differential equations of the first order.

Section IV. Partial differential equations of the second order.

It is limited therefore to equations of the first and second order. The author is much more interested in bringing out the meaning and the interpretation of differential equations than he is in setting exercises for the student; in fact, there are no exercises at all. Existence proofs are given, and discussions of the nature of the integral curves. He does not limit himself to real values of the variables, for in each of the first two sections he has a chapter dealing with the the complex values of the variable and the nature of singularities.

While the book is elementary in character, its spirit is scholarly, and the book is well worth while to the student who has not gone deeply into the subject of differential equations.

W. D. MacMillan

NOTES

The opening number of volume 47 of the AMERICAN JOURNAL OF MATHEMATICS (January, 1925) contains the following papers: *L'expression la plus générale de la "distance" sur une droite*, by Maurice Fréchet; *Second paper on tensor analysis*, by G. Y. Rainich; *Generalization of certain theorems of Bohl*, by F. H. Murray; *Expansion problems in connection with the hypergeometric differential equation*, by B. P. Reinsch.

The third number of volume 25, series 2, of the ANNALS OF MATHEMATICS (March, 1924) contains: *On the method of least mth powers for a set of simultaneous equations*, by Dunham Jackson; *The derivative of the general integral*, by P. J. Daniell; *On a class of transformations in function space*, by I. A. Barnett; *Congruences of circles studied with reference to the surface of centers*, by J. M. Thomas; *On Neumann's existence proof*, by W. F. Osgood; *On the autormorphic functions of the group* (0,3; 2, 4, 6), by H. H. Dalaker; *Note on Dirichlet's series with complex exponents*, by Einar Hille; *On the types of monoidal involutions*, by Virgil Snyder; *An inequality for the roots of an algebraic equation*, by J. L. Walsh. The fourth number (June, 1924) contains: *Differential equations from the group standpoint*, by L. E. Dickson.

At the annual meeting of the Mathematical Association of America in Washington, the following officers were elected: Professor J. L. Coolidge, president; Professors A. A. Bennett and Dunham Jackson, vice-presidents; Professors R. C. Archibald, L. P. Eisenhart, E. V. Huntington, and H. L. Rietz, members of the Board of Trustees.

Among the officers of the American Association for the Advancement of Science elected at the Annual Meeting of 1924, held at Washington, are the following: President, Professor M. I. Pupin, of Columbia University; vice-president, Section A (mathematics), Professor W. H. Roever, of Washington University; secretary, Section A, Professor R. C. Archibald; member of the executive committee, Professor E. B. Wilson, Harvard School of Public Health. Professors Arnold Dresden and E. B. Stouffer have been appointed representatives of the American Mathematical Society on the Council of the Association for 1925. The next Annual Meeting will be held at Kansas City.

The following have been elected officers of the London Mathematical Society: President, A. L. Dixon; vice-presidents. L. N. G. Filon, H. Hilton, and W. H. Young; treasurer, A. E. Western; secretaries, G. H. Hardy and G. N. Watson.

The next meeting of the International Research Council will be held at Brussels, beginning July 7, 1925. The International Physics Union will also meet in Brussels in July. The International Astronomical Union will meet at Cambridge, England, July 14–22.

A journal devoted largely to mathematics has recently been founded at Kief, the BULLETIN DE LA CLASSE DES SCIENCES PHYSIQUES ET MATHÉMATIQUES DE L'ACADÉMIE DES SCIENCES DE L'OUKRAINE; Professor N. M. Kryloff is editor. The Academy, which was founded in November, 1918, has also begun publication of its MEMOIRES, CLASSE DES SCIENCES.

The Hungarian Francis-Joseph University of Szeged (formerly in Kolozsvar, Transylvania) has recently founded a periodical of general character, the ACTA LITERARUM AC SCIENTIARUM, which publishes articles in English, French, and German. In the first volume, the section devoted to mathematics contains articles by F. Riesz, Kurschák, Fejér, Haar, M. Riesz, Kerékjárto, and others. This section will form annually a volume of about 250 pages, containing original articles and also some notes on bibliography. The subscription price is three dollars per volume. Subscriptions should be sent to the Editor, Professor F. Riesz, Dugonicster 13, Szeged, Hungary.

A new series of monographs on mathematical topics has been established, under the patronage of the Société Mathématique de France and of the scientific academies of Paris, Servia, Belgium, Bucarest, Poland, Ukraine, Spain, Prague, Rome (dei Lincei), and Sweden, and will be known as MÉMORIAL DES SCIENCES MATHÉMATIQUES. It will be issued by the firm of Gauthier-Villars, under the general direction of H. Villat, editor of LIOUVILLE'S JOURNAL. The first of these monographs, by P. Appell, entitled *Sur une Forme Générale des Équations de la Dynamique,* has just appeared. Six others are in press, and no less than eighty-nine others are in preparation. The purpose is to present in very brief form the fundamental ideas in an extensive range of mathematical topics.

There has been established in Moscow a Joint Information Bureau, whose purpose is to improve the contact between scientists of Russia and those of other countries, and to facilitate an exchange of scientific periodical and non-periodical publications. Personal exchange of their works between scientists may be arranged, as may also the publication of articles. Those interested should address the Russian Information Bureau, 2819 Connecticut Ave., N. W., Washington, D. C.

Cambridge University has awarded its John Bernard Seely prize in aeronautics to W. A. Johnson, of Peterhouse, for an essay on *Dynamical similarity and scale effects.*

The Institut de France has awarded its Osiris prize to Professor Charles Fabry, for his interferometer.

The Paris Academy of Sciences announces the following awards: the Poncelet prize to Professor Ernest Vessiot, of the University of Paris, for his mathematical work, the Francœur prize to Commandant Ernest Malo, for his researches in algebra and arithmetic; the Fourneyron prize (founded in honor of Benoit Fourneyron, inventor of the turbine) to Marcel Crozet-Fourneyron, for his inventions of apparatus for regulating hydraulic motors and for his history of the invention of the turbine; the Henri de Parville prize to Commandant Paul Bloch, for his *Essai de balistique aérienne*; the La Caze prize to Professor Paul Langevin, of the Collège de France, for his scientific work; the Hébert prize to Professor Edgar Haudié, of the Ecole Navale, for his *Cours d'Electricité Générale*; the grand prix des sciences mathématiques to Professor Paul Montel, of the University of Paris, for his work on sequences of analytic functions; the Vaillant prize to Professor Claude Guichard, of the University of Paris, for his work in geometry; a prize from the Hirn foundation to Professor Georges Giraud, of Clermont-Ferrand, for his scientific work; a prize from the Henri Becquerel foundation to Professor René Garnier, for his work in differential equations.

Professor G. Fubini, of the Turin Technical School, has presented a sum of money to the Unione Matematica Italiana to found a prize to be called the "Premio ing. Lazzaro Fubini"; the prize is to be awarded every two years, under conditions to be announced by the Unione, for some unpublished work by a young Italian mathematician.

It is announced that the second award of the Cesare Arzelà prize of the Bologna Academy of Sciences will be made in December, 1928, this prize is awarded to a doctor of an Italian University for work in the theory of functions of real variables completed within five years of his graduation.

Professor Emile Picard, permanent secretary of the Paris Academy of Sciences, has been elected a member of the Académie Française.

Professors L. Bianchi, of the University of Pisa, and E. Goursat, of the University of Paris, have been elected associates in the section of mathematical and physical sciences of the Royal Academy of Belgium.

Professors N. Bohr, of the University of Copenhagen, and M. von Laue, of the University of Berlin, have been elected foreign members of the Royal Academy dei Lincei of Rome.

Professor A. W. Bickerton, of London, has been elected an honorary member of the New Zealand Astronomical Society in recognition of his work on cosmic evolution.

Professor Harry Bateman, of the California Institute of Technology, has been elected a member of the American Philosophical Society.

At the University of Lille, Dr. H. Beghin has been promoted to the chair of rational mechanics.

Professor Elie Cartan, of the University of Paris, has been transferred from the professorship of rational mechanics to that of higher geometry.

Professor C. Riquier, of the University of Caen, has been designated honorary professor.

At the University of Florence, which has been established recently, Professor E. Ciani, of the University of Genoa, has been appointed professor of higher geometry.

At the University of Milan, which has been established recently, Professors L. Berzolari and G. Vivanti, of the University of Pavia, and G. A. Maggi, of the University of Pisa, have accepted the professorships of higher geometry, higher analysis, and mathematical physics, respectively.

At the University of Naples, Professor M. Picone, of the University of Pisa, has been appointed to the chair of calculus, as successor to Professor G. Torelli, who has retired from active teaching; Professor G. Sannia, of the University of Modena, has been transferred to the chair of descriptive geometry.

At the University of Padua, Professor G. Vitali, of the University of Modena, has been appointed professor of calculus as successor to Professor F. d'Arcais, who has retired from active teaching.

Professor A. Palatini, of the University of Parma, has been appointed professor of rational mechanics.

At the University of Pisa, Professor E. Daniele, of the University of Modena, and Professor O. Lazzarino, of the University of Catania, have been appointed to professorships of rational and celestial mechanics

The following appointments as privat-docents in Italian Universities are announced: University of Bologna, Dr. G. Belardinelli; University of Turin, Dr. U. Cassina; University of Naples, Dr. A. Finzi; University of Florence, Dr. G. Sansone.

The title of emeritus professor of mathematics in the University of London has been conferred on Dr. M. J. M. Hill, who held the chair of pure mathematics at University College from 1884 to 1923.

Professor E. T. Whittaker has been elected dean of the faculty of arts of the University of Edinburgh.

Professor L. P. Eisenhart has been elected dean of the faculty of Princeton University.

Dr. R. E. Gleason, of Princeton University, has been appointed professor of mathematics at Temple University, Philadelphia.

Dr. P. E. Heinke, of the United States Naval Academy, has accepted a position as mathematical analyst at Langley Field, Virginia.

At the University of Vermont, Mr. F. W. Householder has been promoted to an assistant professorship.

Dr. L. M. Kells and Mr. W. A. Conrad, of the United States Naval Academy, have been promoted to assistant professorships of mathematics.

Associate Professor Arthur Kiernan, of the United States Naval Academy, has accepted a position with Ginn and Company.

Professor Solomon Lefschetz, of the University of Kansas, has been appointed to an associate professorship of mathematics at Princeton University.

Mr. W. E. Loring, of the University of Maine, has been appointed assistant professor of mathematics at Colby College, to fill the vacancy caused by the illness of Professor B. E. Carter.

Professor R. G. D. Richardson, of Brown University, secretary of the American Mathematical Society, has been granted leave of absence for the second semester of 1924–25.

At the Agricultural and Mechanical College of Texas, Assistant Professors P. K. Smith and F. W. Sparks have resigned, and Dr. W. P. Udinski has been promoted to an assistant professorship.

The following appointments to instructorships are announced:
University of Vermont, Mr. H. A. Giddings;
Agricultural and Mechanical College of Texas, Messrs. Frank Ayres, A. A. Blumberg, and W. P. Stevens.

Professor E. A. Shaw, for twenty-five years professor of mathematics at Norwich University, has retired from active teaching.

Professor Hugo von Seeliger, of the University of Munich, died December 2, 1924, at the age of seventy-five.

Professor A. G. Hall, registrar of the University of Michigan, died January 11, 1925. Professor Hall had been a member of the American Mathematical Society since 1902.

Mr. D. H. MacPherson, instructor in mathematics at the Brooklyn Polytechnic Institute, died December 28, 1924, at the age of 22.

Mr. W. L. Putnam, attorney at law, of Boston, died July 26, 1924. Mr. Putnam had been a member of the American Mathematical Society since 1910, and had frequently been of great service to the Society in legal matters.

The death is announced of Dr. E. W. Stanton, formerly professor of mathematics at Iowa State College, at the age of seventy-four.

Dr. Brandreth Symonds, chief medical director of the Mutual Life Insurance Company, died August 10, 1924; Dr. Symonds was a member of the American Mathematical Society.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS.

AMODEO (F.). Vita matematica napoletana. Parte 2. Napoli, 1924. 384 pp.

APPELL (P.). Souvenirs d'un Alsacien. Paris, Payot, 1923.

CUNNINGHAM (A. J. C.). Binomial factorisations, giving congruence tables and factorisation tables. Volume 2, 4 + 79 + 215 pp. Volume 6 (supplement to volume 2), 3 + 103 pp. London, F. Hodgson, 1924.

FRÉCHET (M.). See ROSENTHAL (A.).

HILB (E.) und RIESZ (M.). Neuere Untersuchungen uber trigonometrische Reihen. (Sonderabdruck aus der Encyklopadie der mathematischen Wissenschaften.) Leipzig, Teubner, 1924. 40 pp.

JUNG (H. W. E.). Algebraische Flächen. Hannover, Helwingsche Verlagsbuchhandlung, 1924. 16 + 420 pp.

MONTEL (P.). See ROSENTHAL (A.).

PEARSON (E. S.). Tables of the logarithms of the complete Γ-function (for arguments 2 to 1200, i. e., beyond Legendre's range). (Tracts for Computers, No. 8.) Cambridge, University Press, 1922. 12 + 16 pp.

PICARD (E.). Pascal mathématicien. Paris, Gauthier-Villars, 1924. 23 pp.

REYMOND (A.). Histoire des sciences exactes et naturelles dans l'antiquité gréco-romaine. Paris, A. Blanchard, 1924. 8 + 238 pp.

RICKERT (H.). Das Eine, die Einheit, und die Eins. Bemerkungen zur Logik des Zahlbegriffs. 2te, umgearbeitete Auflage. Tubingen, J. C. B. Mohr (Paul Siebeck), 1924. 11 + 93 pp.

RIESZ (M.). See HILB (E.).

ROSENTHAL (A.). Neuere Untersuchungen über Funktionen reeller Veranderlichen nach den Referaten von L. Zoretti, P. Montel und M. Fréchet. (Sonderabdruck aus der Encyklopadie der mathematischen Wissenschaften.) Leipzig, Teubner, 1924. 351 pp.

DE SAINT-PAUL (H.). Tables de lignes trigonométriques naturelles des angles et des arcs variant de minute en minute depuis 0° jusqu'à 90°. Paris, Gauthiers-Villars, 1924. 32 pp.

SCHEFFERS (G.). Anwendung der Differential- und Integralrechnung auf Geometrie. 3te, verbesserte Auflage. 1ter und 2ter Band. Berlin, de Gruyter, 1922–23. 12 + 482 + 11 + 582 pp.

SCHUH (F.). Lessen over de hoogere algebra. Tweede deel. Groningen, Noordhoff, 1924. 358 pp.

ZORETTI (L.). See ROSENTHAL (A.).

PART II. APPLIED MATHEMATICS

Becqueler (J.). Cours de physique à l'usage des élèves de l'enseignement supérieur et des ingénieurs. Tome 1: Thermodynamique. Paris, Hermann, 1924. 8+431 pp.

Behm (H.). Kubiktabelle. Zur Bestimmung des Inhaltes von Rundholzern nach Kubikmetern und Hundertteilern des Kubikmeters, mit angehängten Reduktionstafeln. Berlin, Springer, 1924. 72 pp.

Berliner (A.) und Scheel (K.). Physikalisches Handwörterbuch. Berlin, Springer, 1924. 6+903 pp.

Beth (H. J. E.). Beknopt leerboek der cosmographie. Groningen, Noordhoff, 1923. 63 pp.

Bloch (E.). The kinetic theory of gases. Translated by P. A. Smith. London, Methuen, 1924. 14+178 pp.

Born (M.). Einstein's theory of relativity. Translated by H. L. Brose. London, Methuen, 1924. 11+293 pp.

Bouasse (H.). Capillarité. Phénomènes superficiels. Paris, Delagrave, 1924. 460 pp.

Boussinesq (J.). Théorie analytique de la chaleur. Epilogue (2e édition). Paris, Gauthier-Villars, 1923. 12 pp.

Bradley (H. C.). See Kenison (E.).

Brose (H. L.). See Born (M.).

Brunner (W.). Von Stern zu Stern. Eine Weltschau von verschiedenen Himmelskorpern aus. Zurich, Rascher, 1923.

Calvert (W. J. R.). Physics. Volume 1: Mechanics. heat and heat engines. London, John Murray, 1924. 10+260 pp.

Cantero-Villamel (—.). Aviation et probabilité. Les problèmes du vol sans moteurs. Paris, Gauthier-Villars, 1923. 144 pp.

Choisnard (P.). L'influence astrale et les probabilités. Paris, Alcan, 1924. 249 pp.

Christesco (S.). Les ondes fluorescents dans le substratum de la matière. Paris, Alcan, 1924. 104 pp.

Clementi (I.). I contattori elettrici. Teoria e pratica. Milano, Hoepli, 1924. 16+291 pp.

Eddington (A. S.). The mathematical theory of relativity. 2d edition. Cambridge, University Press, 1924. 9+270 pp.

——. Relativitatstheorie in mathematischer Behandlung. Autorisierte Übersetzung von A. Ostrowski und H. Schmidt. Mit Zusätzen und Erganzungen und mit einem Anhang von A. Einstein. (Die Grundlehren der mathematischen Wissenschaften, Band 18.) Berlin, Springer, 1924.

Einstein (A.). See Eddington (A. S.).

Forsyth (C. H.). Mathematical theory of life insurance. New York, Wiley, 1924. 8+74 pp. $1.25

FRAZIER (R. A.). See PANNELL (J. R.).

FRÉCHET (M.) et HALBWACHS (—.). Le calcul des probabilités à la portée de tous. Paris, Dunod, 1924. 11+297 pp.

GÉRARD (E.). Leçons d'électricité. 9e édition revue et mise à jour par E. Marec. Tome I et tome II. Paris, Gauthier-Villars, 1924. 495+588 pp.

GIBBS (R. W. M.). Engineering mathematics. Part 2. London, Blackie, 1924. 132 pp.

GIRAULT (F.). Le problème de la gravitation. Paris, Gauthier-Villars, 1924. 20 pp.

GRATZ (L.). Die Physik. 2te, verbesserte und vermehrte Auflage. (Die Naturwissenschaften und ihre Anwendungen, Band 1.) Berlin, de Gruyter, 1923. 12+582 pp.+12 Tafeln.

HALBWACHS (—.). See FRÉCHET (M.).

HALE (G. E.). The depths of the universe. New York, Scribner's, 1924. 15+98 pp.

HILL (J. G.). La transmission téléphonique. (Théorie et applications.) Traduit de l'anglais par G. Valensi. Paris, Gauthier-Villars, 1924. 516 pp.

HOOL (G. A.) and KINNE (W. S.), editors. Reinforced concrete and masonry structures, compiled by a staff of specialists. New York and London, McGraw-Hill, 1924. 19+786 pp.

JEFFERY (G. B.). Relativity for physics students. London, Methuen, 1924. 7+151 pp.

KANTHACK (F. E.). The principles of irrigation engineering with special reference to South Africa. London, Longmans, 1924. 299 pp.

KAYSER (H.) und KONEN (H.). Handbuch der Spectroscopie. Band VII, Teil 1. Leipzig, Hirzel, 1924. 10+498 pp.

KENISON (E.) and BRADLEY (H. C.). Descriptive geometry. Enlarged edition. New York, Macmillan, 1923. 19+407 pp.

KINNE (W. S.). See HOOL (G. A.).

KONEN (H.). See KAYSER (H.).

LAGRANGE (C.). Remarques sur la relativité. Paris, Gauthier-Villars, 1923. 52 pp.

LAMB (H.). Hydrodynamics. 5th edition. Cambridge, University Press, 1924. 16+687 pp.

LEVI-CIVITA (T.). Fragen der klassischen und relativistischen Mechanik. Vier Vorträge, gehalten in Spanien im Januar 1921. Autorisierte Übersetzung. Berlin, Springer, 1924. 6+110 pp.

LORENTZ (H. A.). Considerazioni elementari sul principio di relatività. Torino, Pietro Gobetti, 1923. 26 pp.

LUCAS (H. M.). See VASILIEV (A. V.).

MAREC (E.). See GÉRARD (E.).

MITTELSTEN SCHEID (F.). See WHITTAKER (E. T.).

OSTROWSKI (A.). See EDDINGTON (A. S.).

PANNELL (J. R.). The measurement of fluid velocity and pressure. Edited by R. A. Frazier. New York, Longmans, 1924. 12+132 pp. $3.50

PARENTY (H.). Les tourbillons de Descartes et la science moderne. Paris, Gauthier-Villars, 1923. 8+220 pp.

PICARD (E.). Les théories d'optique et l'oeuvre d'Hippolyte Fizeau. Paris, Gauthier-Villars, 1924. 64 pp.

PRESCOTT (J.). Applied elasticity. New York, Longmans, 1924.

ROTHE (R.). Elementarmathematik und Technik. Eine Sammlung elementarmathematischer Aufgaben mit Beziehungen zur Technik. (Mathematisch-physikalische Bibliothek, Band 54.) Leipzig, Teubner, 1924.

SANGER (C. P.). See VASILIEV (A. V.).

SAVINE (--). Réduction du système des trois équations différentielles d'équilibre d'un corps isotrope et élastique à trois équations avec une seule fonction inconnue dans chacune. Paris, Gauthier-Villars, 1923. 14 pp.

SCHEEL (K.). See BERLINER (A.).

SCHMIDT (H.). See EDDINGTON (A. S.).

SIEGBAHN (M.). Spektroskopie der Röntgenstrahlen. Berlin, Springer, 1924. 6+257 pp.

SMITH (P. A.). See BLOCH (E.).

STEPHAN (P.). Die technische Mechanik des Maschineningenieurs. Band 4: Die Elastizität gerader Stäbe. Berlin, Springer, 1922. 249 pp.

VALENSI (G.). See HILL (J. G.).

VARIN D'AINVILLE (J.). L'origine de l'atome et ses conséquences. Paris, Gauthier-Villars, 1924. 215 pp.

VASILIEV (A. V.). Space, time, matter. An historical introduction to the general theory of relativity. Translated from the Russian by H. M. Lucas and C. P. Sanger. London, Chatto and Windus, 1924. 256 pp.

WHETHAM (W. C. D.). The recent development of physical science. 5th edition. London, Murray, 1924. 16+313 pp.

WHITTAKER (E. T.). Analytische Dynamik der Punkte und starren Körper. Nach der zweiten Auflage übersetzt von F. und K. Mittelsten Scheid. (Die Grundlagen der mathematischen Wissenschaften, Band 17.) Berlin, Springer, 1924.

LIBRARY.

THE FRANK NELSON COLE PRIZE IN ALGEBRA

The first award of the Cole prize will be made at the end of 1927 for a notable contribution to the theory of linear algebras by a resident of the United States or Canada.

1. The contribution may be either an unpublished manuscript or a paper first published during 1925, 1926, or 1927.

2. Two copies of the manuscript or printed paper should be in the hands of the Secretary of the Society, 501 West 116th Street, New York City, on or before October 31, 1927.

3. The first award will be two hundred dollars ($200.00).

4. The prize will be withheld in the absence of a sufficiently meritorious memoir.

Without excluding others, the following topics are suggested as suitable ones:

a. Determination of all division algebras of rank 5. (For those of rank 4, see F. Cecioni, PALERMO RENDICONTI, vol. 47 [1923], pages 209–254.)

b. Direct proof, without the theory of the rank equation, that an algebra whose units form a group is semi-simple.

c. Classification of nilpotent algebras.

d. Every algebra A is the sum of a semi-simple sub-algebra S and the maximal nilpotent invariant sub-algebra N. Investigate the multiplicative relations between units of S and units of N.

e. Develop a rational theory of algebras A over any field by starting with the rank function of the general element of A expressed in terms of the rank functions of the simple components of S in problem d.

f. Extend the theory of arithmetics of algebras. In particular, invent a theory of ideals for division algebras.

The literature is accessible in G. Scorza's *Corpi Numerici e Algebre,* Messina, 1921, and Dickson's *Algebras and their Arithmetics,* University of Chicago Press, 1923.

H. S. WHITE,
For the Committee of the Council.

THE PAN-AMERICAN SCIENTIFIC CONGRESS

The third Pan-American Scientific Congress was held in Lima, Peru, from December 20, 1924 to January 6, 1925. It was divided into nine section: 1. Anthropology and History; 2. Physical and Mathematical Sciences (including Geology and Geography); 3. Mining and Metallurgy; 4. Engineering; 5. Medicine and Sanitation; 6. Biology and Agriculture; 7. Law; 8. Economics and Sociology; 9. Education.

At the impressive opening ceremony, the Congress was formally welcomed by President Leguia of the Peruvian Republic and Dr. Salamón, Minister of Foreign Affairs. All the republics in the western hemisphere except Chili were represented. There were 158 members from Peru and 151 from foreign countries, including 30 from the United States, 25 from Argentina, 15 from Cuba, 9 from Venezuela, 8 from Colombia, 7 from Uruguay, 6 from Brazil, 5 from Ecuador, 5 from Mexico, etc. It was decided to hold the next Pan-American Scientific Congress in San José, Costa Rica, in 1929.

The subject of Pure Mathematics was assigned to Subsection 1 under Section 2. This subsection held one meeting for the reading of papers on December 28, 1924. About fifteen members were present, and the following three papers were read.

1. *Conjugate ordinates and their geometric applications,* by Dr. Florencio D. Jaime, President of the recently organized Argentinian Mathematical Society, and delegate from the Ministry of Public Instruction of Argentina.

2. *Elementary types of order,* by Dr. Edward V. Huntington, vicepresident of the American Mathematical Society, and delegate from the American Mathematical Society, the Mathematical Association of America, and the American Academy of Arts and Sciences.

3. *On the descriptive geometry of the sphere,* by Dr. Alejandro Guevara, Honorary Professor of the School of Engineers in Lima.

E. V. HUNTINGTON.

THE FEBRUARY MEETING IN NEW YORK

The two hundred fortieth regular meeting of the Society was held at Columbia University, on Saturday, February 28, 1925, extending through the usual morning and afternoon sessions. The attendance included the following sixty-one members of the Society:

Alexander, R. L. Anderson, C. R. Ballantine, J. P. Ballantine, Birkhoff, Blake, Bowden, R. W. Burgess, G. A. Campbell, Alonzo Church, Cole, Dadourian, Eisenhart, Eshleman, Fite, Frink, Fry, Gafafer, Gehman, Gronwall, C. C. Grove, Guggenbuhl, Hausle, Hille, Himwich, Louis Ingold, Joffe, Kasner, O. D. Kellogg, Kline, Lamson, Langer, Langman, Lefschetz, Harry Levy, McGiffert, McMackin, MacColl, MacNeish, Meder, Mirick, Molina, C. L. E. Moore, Mullins, Olson, Pell, Pfeiffer, R. G. Putnam, Rainich, Reddick, Ritt, Schelkunoff, Seely, Siceloff, J. H. Taylor, H. D. Thompson, Veblen, H. E. Webb, Wedderburn, Whittemore, Wiener.

There was no meeting of the Council or of the Trustees.

At the beginning of the afternoon session, a paper was read by Professor J. W. Alexander, at the request of the Program Committee, on *Problems in the topological theory of manifolds.*

Professor Béla de Kerékjártó, of the University of Szeged, Hungary, spoke briefly of the mathematical journal recently founded at that university, the Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae, Sectio Mathematicarum; the articles in this journal are in French or German. He suggested that members of the Society might call the attention of their university libraries to this publication.

President Birkhoff presided at both sessions.

Titles and abstracts of the papers read at this meeting follow below. Professor Alexander's second paper and the papers of Bennett, Dodd, Franklin, Garabedian, James, Jeffery, Libman, Osgood, Reynolds, Thomas, and Weisner were read by title; Professor Morley's paper was communicated by Mr. Rainich. Professor de Kerékjártó was introduced by President Birkhoff.

1. Professor C. L. E. Moore: *Minimal varieties of two or three dimensions whose element of arc is a perfect square.*

After determining the euclidean space of lowest dimension in which minimal varieties of two or three dimensions whose element of arc is a perfect square can lie, the author discusses the intrinsic geometry of these varieties.

2. Professor L. P. Eisenhart: *Fields of parallel vectors in a Riemannian geometry.*

In accordance with the definition introduced by Levi-Civita, parallelism of vectors depends upon the metric of the space and upon the curve along which a vector is displaced so as to remain parallel to its initial position. In a general Riemannian space there does not exist a field of vectors which are parallel to one another regardless of the curves of displacement. However, there are certain Riemannian spaces in which there are one, or more, fields of vectors possessing this property. In this paper canonical forms of the metric of these spaces are determined.

3. Mr. G. Y. Rainich: *On the Riemann tensor.*

In a flat space, to every contour corresponds its *tensor area*, an alternating tensor of rank two. If a curved space $\sum$ is immersed in our flat space, we consider for every contour on $\sum$ its *contour variation of normals* (normal flat spaces). For infinitesimal contours in the vicinity of a given point, the contour variation of normals is a linear function of the tensor area, and this function is the *Riemann tensor* at the point considered. By integrating (in a sense previously introduced) the Riemann tensor over a surface, we obtain again the finite variation of normals corresponding to the contour limiting this surface. If the curved space is four-dimensional, the Riemann tensor is the sum of two parts, of which the first assigns to absolutely perpendicular planes equal curvatures and the second opposite curvatures. The first part involves 11 and the second 9 constants. The second part has the same coefficients as the "cosmological energy tensor"; its vanishing is equivalent to the cosmological equations. In physical space-time, the second part depends only on 5 constants and can be expressed quadratically through the electromagnetic tensor.

4. Mr. G. Y. Rainich: *Integrals in curved space.*

It has been previously shown how integrals can be formed when a tensor field has superfluous indices; this required the consideration of the containing flat space. However, when the tensor field depends on the superfluous indices in the same way as on the others, it is possible to form integral expressions without leaving the curved space, by presenting the tensor as the sum of products of tensors of which integrals can be taken and by forming the sum of the products of these integrals. This is first applied to the case of the indicator of a surface; it is shown that only for a minimal surface do the expressions considered vanish over a reducible contour. For an irreducible contour on a one-sided surface they give rise to residues; the residues are the same for all simple irreducible contours of a given surface. In the case of the physical space-time, the second part of the Riemann tensor when treated in a similar manner gives an expression which vanishes on a reducible closed surface, and for a surface surrounding a singularity it gives the square of the electric charge.

5. Professor Frank Morley: *Comitants of a curve under inversion.*

This paper deals with the comitants under inversion of a plane curve. The equation of such a curve, expressed in circular coordinates, is obtained by bordering a self-conjugate or hermitian matrix. The determinant of this matrix is an invariant. If with the curve we combine the special curve which represents a repeated point, the determinant of the pencil gives the reciprocal curve of the given one. The pencil formed by a curve and its reciprocal gives a series of invariants. The results are included in the memoir of Kasner (TRANSACTIONS OF THIS SOCIETY, vol. 1) but it may be of interest to consider this important question from all points of view.

6. Professor Edward Kasner: *Null geometry.*

A null manifold (all distances vanishing) of k dimensions cannot exist in a euclidean space of less than $2k$ dimensions. In space of $2k$ dimensions, null k-flats exist, and in higher spaces curved null spaces exist.

7. Professor Louis Ingold: *Extensions of the equations of Gauss and Codazzi.*

The paper considers Riemann spaces R_n determined by a general fundamental vector $F(u^1, u^2, \cdots, u^n)$. The first partial derivatives $\partial f/\partial u^i = f_i$ are regarded as independent tangent vectors to R_n. The first fundamental quantities E_{ij} are the scalar products $f_i f_j$. Second fundamental quantities L_{ijrs} are defined as the scalar products $N_{ij} N_{rs}$ of vectors normal to R_n, where N_{ij} is linear in the corresponding f_{ij} and the tangent vectors f_k. Third fundamental quantities are defined in a similar way from normals N_{ijk} linear in f_{ijk} and the normals and tangents defined above. These are orthogonal to the N_{rs}. In case the vector f satisfies a completely integrable system of $n-1$ linear differential equations of the second order, all of the normals N_{ij} are expressible in terms of a single normal N; and the first and second fundamental quantities satisfy the usual equations of Gauss and Codazzi. If it is assumed that this is not true, but that the normals N_{ijk} are expressible in terms of a single normal vector, then analogous necessary relations connecting the first, second, and third fundamental quantities are obtained. Further extensions are obvious.

8. Dr. Harry Levy: *Tensors determined by a hypersurface in a Riemann space.*

Consider a hypersurface, V_n, of n dimensions, embedded in a general Riemann space of $n+1$ dimensions. If we let V_n be represented by the equation $u^0 = 0$ and if we take for the parameter u^0 the arc of the geodesics normal to V_n, the linear element of the space becomes $ds^2 = (du^0)^2 + C_{\alpha\beta} du^\alpha du^\beta$ $(\alpha, \beta = 1, 2, \cdots, n)$. We find that the coefficients in the expansion of $C_{\alpha\beta}$ in a power series in u^0 are a sequence of tensors intimately connected with the geometry of the hypersurface. In the first part of this paper we deduce a number of identities which hold in general tensor analysis. In the second part we consider these tensors and obtain general properties of Riemann space.

9. Dr. Harry Levy: *Symmetric tensors of the second order whose covariant derivatives vanish.*

We give a complete geometric characterization of tensors of the kind mentioned in the title, showing that necessary and sufficient conditions that the covariant derivative of b_{ij} vanish are that b_{ij} be a first integral of the differential equations of the geodesics of the space, and that the space whose fundamental form is $b_{ij}\,dx^i\,dx^j$ admit of geodesic representation upon the given space. We prove further that, in a space of constant curvature, a tensor whose covariant derivatives are zero is a constant multiple of the fundamental tensor.

10. Professor H. L. Olson: *Congruences with constant absolute invariants.*

This paper is based on the researches of Wilczynski, chiefly on his memoir *Sur la théorie générale des congruences,* published by the Royal Academy of Belgium. Regarding the special type of congruences considered, it is shown that both sheets of the focal surface have constant absolute invariants, and that the cuspidal edges of the developables are two families of projectively equivalent anharmonic curves. The Laplace transforms of any such congruence have constant absolute invariants. Necessary and sufficient conditions that a congruence be projective to its 1st or (—1)st Laplace transform are that its absolute invariants be constants and that $\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}$ and $\overset{(y)}{\mathfrak{C}}{}'' - \overset{(z)}{\mathfrak{C}}{}''$, defined by Wilczynski, vanish identically. Any congruence of the latter type is contained in a tetrahedral complex or one of the limiting forms of the tetrahedral complex.

11. Professor W. F. Osgood: *On normal forms of differential equations.*

Linear differential equations of the second order (and their Schwarzian resolvents) are considered on an algebraic configuration which is first taken in the form of Noether's normal curve. The latter is now projected on a pencil of hyperplanes, and the algebraic configuration is treated (i) in the binary domain thus arising; and (ii) in the domain corresponding to a uniformization by means of automorphic functions with limiting circle. The invariant, to the existence of which Klein was led in the case of the group of the binary linear transformations of Case (i), is obtained explicitly and independently in terms of the uniformizing variables of Case (ii), and its properties are read off by means of the automorphic functions.

12. Professor A. A. Bennett: *Two new arctangent relations for* π.

In this paper the author gives two new formulas for use with Gregory's series in computing π. The first is not unsuited for pencil and paper computation, and gives much more rapidly convergent series than does Machin's well known relation. The second is intended for use with a computing machine. The first may be written in the form $\pi/4 = 12 \operatorname{arcctn} 18 + 3 \operatorname{arcctn} 70 + 5 \operatorname{arcctn} 99 + 8 \operatorname{arcctn} 307$. The second has more arguments, but has 200 for its smallest argument.

13. Professor A. A. Bennett: *Diophantine arccotangent relations.*

In this paper the author shows that a necessary and sufficient condition that the equation $\operatorname{arcctn} x_1 + \operatorname{arcctn} x_2 = \operatorname{arcctn} y_1 + \operatorname{arcctn} y_2$ be satisfied by given integers is that certain corresponding expressions u_1, u_2, v_1, v_2 satisfy the congruences $u_1v_1 \equiv u_1v_2 \equiv u_2v_1 \equiv u_2v_2 \pmod{x_1+x_2}$. Several infinite systems of solutions are obtained, and a complete table of all solutions, up to $x_1+x_2 = 25$, is given. The simpler case of the three term arccotangent relation is disposed of. The problem in both cases reduces to a utilization of a table of numerical factors of x^2+1, and the theory of the construction and checking of such a table is developed.

14. Professor E. L. Dodd: *The fitting of curves by the use of moments and conjugate moments.*

If f is the frequency of x, the conjugate rth moment is defined as $\sum f \cdot x^r \cdot \operatorname{sign} x$, with $r = 0, 1, 2, \cdots$. The author shows that conjugate moments are especially useful in curve-fitting, since double the usual number of parameters may be introduced. Suppose, for example, that the frequency function is $y = e^{-x^2/2}(A + Bx + Cx^2 + \cdots + Kx^{2s+1})$, where the arithmetic mean has been made the origin and the standard deviation the unit. Using moments up to the sth with their conjugates, the $s+1$ alternate parameters $A, C, E, \cdots$ may be obtained by solving $s+1$ linear equations, and the remaining parameters likewise. Thus we may fit a six-parameter curve to a distribution without going higher than second moments (it seems especially advisable to avoid high moments when extreme variates are irregular) while even twenty-two-parameter curves become tractable with the Pearson tables.* For comparison, the distribution of 1130 pensioners exhibited by Arne Fisher† was fitted by a six-parameter curve of the above form (with corrections for grouping) and 13 positive and 13 negative aberrations were obtained, the same absolute total (26) as was obtained by Fisher by the Charlier method involving moments up to the fourth.

15. Dr. E. E. Libman: *Linear complex of conics.*

A conic in space involves in its specification eight arbitrary constants. When these are expressed as integral linear non-homogeneous functions of three independent parameters, the system so obtained is called a linear complex. The linear complex of conics so defined has the following properties: (I) There is one and only one conic upon each plane in space. (II) The centers of the degenerate conics of the complex lie upon a quadric surface and their planes envelope another quadric surface; these are called the singular quadrics of the complex. (III) The conics through a point are in one to one correspondence with the points of a plane quartic curve associated with the point. The plane of the quartic, and the point are called polar and pole with respect to the complex. (IV) The points that lie upon their polars are on one of the singular quadrics, while the planes that contain their poles envelope the other.

The machinery used in the investigation is quaternions with the usual Hamiltonian symbols.

16. Professor C. N. Reynolds: *On the map-coloring problem, with particular reference to connected sets of pentagons.*

In this paper the author considers the problem of developing, by analytic methods, the implications of such geometric reductions of the map-coloring problem as have been published; i. e., those due to Kempe, Birkhoff, and Franklin, but not those of Errera. His method is that of applying linear difference equations to the study of the topological properties of irreducible connected configurations of pentagons. It is then shown that the reductions considered imply the possibility of coloring, in four colors, the map of any spherical surface which is divided into not more than twenty-seven regions. Two maps of twenty-eight regions each which can be colored in four colors but which are irreducible, in the sense defined above, are then exhibited as a check on the closeness of fit between the analytic methods presented and the geometric problem considered.

---

* *Tables for Statisticians and Biometricians*, Tables II and IX.

† *Frequency Curves*, 1922, p. 48.

17. Professor C. A. Garabedian: *Solution of the problem of the thick rectangular plate, clamped or supported at its edges and under uniform or central load.*

The author has already given a solution of the problem of the thick rectangular plate supported at its edges and under uniform pressure.* The method employed may be successfully applied to all four problems mentioned in the title above, and it is noteworthy that the resulting formulas of displacement are in each case simply expressible in terms of the solution (and its derivatives) of the corresponding thin plate problem. Since the four thin plate solutions at issue are given in complete form in Hencky's thesis,† rigorous and usable formulas for the thick plate are now available to the engineer. The solutions of the present note are analogous to the author's solutions of the same problems in circular plates,‡ and there are certain interesting remarks applicable to both rectangular and circular cases. Finally the method is capable of including plates of variable thickness. An abstract of results will be found in the COMPTES RENDUS of January 26, 1925; details will be given in an extended paper now in preparation.

18. Professor Glenn James: *A complete solution of the cubic equation.*

This paper determines an interval in which one and only one of the roots of a cubic equation lies, and then developes an interpolation theorem concerning a root in a given interval. Upon the basis of this theorem, a root is secured as a function of the coefficients which can be evaluated for numerical equations by a certain recurrence formula. The other roots are obtained in general form from the reduced equation.

---

* COMPTES RENDUS, vol. 178 (1924), p. 261.

† *Der Spannungszustand in rechteckigen Platten*, Darmstadt, 1913 (published by Oldenbourg, Munich).

‡ TRANSACTIONS OF THIS SOCIETY, vol. 25 (1923), pp. 379–388.

19. Professor R. L. Jeffery: *Functions of two variables for which the double integral does not exist.*

It is pointed out by E. W. Hobson* that even though a function of two variables is continuous in each variable separately, the double integral in the sense of Riemann need not exist. This is obviously true if the function is unbounded, but there seems to be nowhere in the literature a bounded function which illustrates Hobson's observation. In this note, a function $f(x, y)$ is constructed on the unit square which is bounded and continuous in each variable separately, but which is discontinuous in the two variables with a saltus equal to unity at every point of a two-dimensional set whose plane content is $\frac{1}{4}$. Hence, for this function, the Riemann double integral fails to exist.

20. Dr. Louis Weisner: *On the number of elements in a group which have a power in a given conjugate set.*

This paper will appear in full in an early number of this BULLETIN.

21. Dr. J. M. Thomas: *The number of even and odd absolute permutations of n letters.*

This paper appears in full in the present number of this BULLETIN.

22. Dr. J. M. Thomas: *Note on the projective geometry of paths.*

The projective curvature tensor discovered by Weyl (GOTTINGER NACHRICHTEN, 1921, p. 99) is derived in a manner analogous to the derivation of the ordinary curvature tensor, and a new projective tensor of rank 13 is obtained.

23. Professor J. W. Alexander: *Problems in the topological theory of manifolds.*

This paper will appear in an early issue of this BULLETIN.

24. Professor J.W. Alexander: *Combinatorial analysis situs.*

The author determines a necessary and sufficient condition for the homeomorphism of two complexes which is expressed in terms of cells and their incidence relations rather than in terms of points and limit points. He is then able to develop the theory of the connectivity numbers and intersection invariants without further appeal to continuity considerations.

---

* *Theory of Functions of a Real Variable,* 2d edition, vol. 1, § 365.

25. Professor O. D. Kellogg: *Note on the convergence of real power series representing harmonic functions.*

If a series of spherical harmonics in x, y, and z converges within the sphere $x^2+y^2+z^2<R^2$, then the power series in x, y, and z which represents the same function converges within the octahedron $|x|+|y|+|z|<R$. A corresponding result holds in the case of n variables. For $n=2$, the complete two-dimensional region of convergence is thus characterized, as has been shown by Bôcher. For $n>2$, however, the complete n-dimensional region is not, in general, given by the theorem indicated above.

26. Professor J. F. Ritt: *Transcendental transcendency of certain functions of Poincaré.*

Poincaré, in 1890, established the existence of a large class of meromorphic functions $y(x)$ for which an $m\neq 1$ exists such that $y(mx)$ is a rational function of $y(x)$. The functions e^x, cos x, and $\wp(x)$ belong to this class. In the present paper, it is proved that with the exception of the three functions just named and certain others closely related to them, the functions of Poincaré are transcendentally transcendental, that is, they do not satisfy algebraic differential equations.

27. Professor J. R. Kline: *Concerning the sum of a countable infinity of mutually exclusive continua.*

Sierpinski has proved that no bounded closed connected set can be the sum of a countable infinity of mutually exclusive closed connected sets. In her dissertation, Miss Anna Mullikin gave an example of a connected set which is the sum of a countable infinity of mutually exclusive closed connected sets. In the present paper it is proved that if S is a point set which is the sum of a countable infinity S_1, S_2, $\cdots$ of closed connected sets having the property that given any ε, greater than zero, there are at most a finite number of the sets S_1, S_2, $\cdots$ of diameter greater than ε, then S is not connected.

28. Dr. H. M. Gehman: *On extending a continuous* (1,1) *correspondence of two plane continuous curves to a correspondence of their planes.*

It is proved that if any two plane continuous curves are in continuous (1,1) correspondence in such a way that sides of arcs are preserved under the correspondence, then a continuous (1,1) correspondence of their planes can be defined in such a way that the correspondence of the two continuous curves is preserved. This generalizes a theorem previously announced for the case of continuous curves that contain no simple closed curves.

29. Professor S. Lefschetz: *On the problem of inversion of abelian integrals.*

This paper gives a proof of the existence and uniqueness of the solution of Jacobi's inversion problem by means of the implicit function theorem.

30. Dr. Philip Franklin: *Osculating curves and surfaces.*

In this paper several conditions are given under which a curve or surface, obtained as the limit of a sequence of curves or surfaces having a fixed number of points in common with a given curve or surface, osculates this given one. As a typical theorem we have the following: If in some neighborhood of a point P on a curve its defining function possesses a continuous fourth derivative, and if at this point there is a unique osculating conic ($y''' \neq 0$) a sequence of conics having five points in common with the given curve approaches the osculating conic at P when the five points close down on P.

31. Professor R. E. Langer: *On the momental constants of a summable function.*

This paper deals with the momental constants defined by Haskins for any summable function. A set of necessary conditions on the constants of any enumerably infinite set if they are to be the momental constants of a summable function with finite measurable bounds is deduced. These conditions are then shown to be sufficient as well. The determination of the function having a given set of momental constants is made to depend upon the solution of an associated "problem of moments".

32. Professor Norbert Wiener: *The solution of a difference equation by trigonometric integrals.*

In every case in which Norlund has solved the equation $f(x+1)-f(x) = \varphi(x)$ in real variables, his principal solution may be resolved into the sum of two functions $f_1(x)$ and $f_2(x)$, such that $f_1(x) = -\sum_{s=0}^{\infty}\varphi_1(x+s)$ and $f_2(x) = \int_a^x \varphi_2(z)\,dz + \sum_{\nu=1}^{\infty}(B_\nu/\nu!)\,\varphi_2^{(\nu-1)}(x)$, where $\varphi(x) = \varphi_1(x)+\varphi_2(x)$.

33. Dr. T. H. Gronwall: *On Gibbs' phenomenon.*

This paper investigates the condition for the occurrence of a Gibbs' phenomenon in the Cesàro sums of order k $(0 \leqq k \leqq 1)$ of a Fourier series.

34. Professor Einar Hille: *Some remarks on Dirichlet's series.*

This paper is concerned with certain relations between Dirichlet's, factorial, and binomial series. The starting point is a paper by Wigert (ARKIV FÖR MATEMATIK, ASTRONOMI, OCH FYSIK, vol. 7 (1911), No. 26) in which a binomial series is expressed as the sum of an entire function and a factorial series. The corresponding relation for Dirichlet's series can be obtained from Cahen's expression for such a series by means of a Laplace integral. The desired relation is obtained by applying Pincherle's results on *fonctions déterminates* to this integral. It is observed that Cahen's integral can be used for the purpose of summing a certain class of everywhere divergent Dirichlet's series. Some theorems on the analytic nature of the functions represented by certain Dirichlet's series are given as an application. These results were first derived by Hardy using Borel's method of summation. Various generalizations are considered.

35. Professor Béla de Kerékjártó: *Remarks on convex regions.*

The author discusses the theorems of E. Helly concerning common points of convex regions and of simply connected plane regions.

W. B. FITE,
Acting Secretary.

THE NUMBER OF EVEN AND ODD ABSOLUTE PERMUTATIONS OF n LETTERS*

BY J. M. THOMAS†

A permutation of n letters is called absolute if it leaves no letter fixed. A formula for the total number of such permutations is well known, but no separate formulas for the numbers of even and odd absolute permutations seem to be given in the literature. It is the purpose of this note to show how such formulas can be derived.

Consider the determinant of order n

$$D = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 0 \end{vmatrix},$$

all of whose elements are equal to 1, except those on the main diagonal, which equal 0. By writing it in the form

$$\begin{vmatrix} 1-1, & 1+0, & 1+0, & \cdots, & 1+0 \\ 1+0, & 1-1, & 1+0, & \cdots, & 1+0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1+0, & 1+0, & 1+0, & \cdots, & 1-1 \end{vmatrix},$$

we see that it is expressible as the sum of 2^n determinants which fall into three classes. Those determinants having more than one column of 1's are zero. Those having exactly one column of 1's are n in number, and their common value is $(-1)^{n-1}$. Finally there is just one determinant containing no column of 1's, and its value is $(-1)^n$. Hence the value of D is $(-1)^{n-1}(n-1)$.

On the other hand if we consider the expanded form of D, say

$$D = \sum \pm a_{1i}a_{2j} \cdots a_{nk},$$

---

* Presented to the Society, February 28, 1925.

† National Research Fellow in Mathematics.

we see that any term which does not correspond to an absolute permutation of the n indices $i, j, \cdots, k$ contains an element from the main diagonal and is therefore zero. The value of a term corresponding to an absolute permutation is $+1$ or -1 according as the permutation is even or odd. The value of D is therefore the difference between the number of even absolute permutations, N_e, and the number of odd absolute permutations, N_o, that is

$$(1) \qquad N_e - N_o = (-1)^{n-1}(n-1).$$

Moreover we have*

$$(2) \qquad N_e + N_o = n!\sum_{r=2}^{n}\frac{(-1)^r}{r!}.$$

From (1) and (2) N_e and N_o can be calculated.

PRINCETON UNIVERSITY

THE ABSOLUTE VALUE OF THE PRODUCT OF TWO MATRICES†

BY J. H. M. WEDDERBURN

1. *Introduction.* If $a = (a_{pq})$ is a matrix of order n whose elements are ordinary complex numbers, the absolute value of a is defined as $\sqrt{\sum a_{pq}\bar{a}_{pq}}$, where $\bar{a} = (\bar{a}_{pq})$ is the matrix whose coefficients are the conjugates of the corresponding coefficients in a; we shall denote it here by $\lfloor a \rfloor$, a special symbol being convenient since the absolute value of a scalar matrix λ is not $|\lambda| \equiv \operatorname{mod} \lambda$ but $n^{1/2}|\lambda|$. This definition has been freely used by writers on differential equations; but, in spite of this, its properties with regard to multiplication have seemingly escaped notice, or are at least not well known.

---

[*] Cf. Seelhoff, ARCHIV DER MATHEMATIK UND PHYSIK, (2), vol. 1, p. 100.

[†] Presented to the Society, May 2, 1925.

The principal properties of $\lfloor a \rceil$ are as follows:

$$
\begin{aligned}
&(1) && \lfloor a+b \rceil \leqq \lfloor a \rceil + \lfloor b \rceil, \\
&(2) && \lfloor \lambda \rceil = n^{1/2}|\lambda|, \\
&(3) && \lfloor \lambda a \rceil = |\lambda|\lfloor a \rceil, \\
&(4) && \lfloor ab \rceil \leqq \lfloor a \rceil \lfloor b \rceil.
\end{aligned}
\quad \left.\begin{matrix} \\ \\ \end{matrix}\right\} (\lambda \text{ scalar})
$$

Relation (4), which is the only one which is not obvious, follows immediately from the identity

$$
\lfloor a \rceil^2 \lfloor b \rceil^2 = \sum_{pr} a_{pr}\bar{a}_{pr} \sum_{qs} b_{sq}\bar{b}_{sq}
$$

$$
= \sum_{pq} \left\{ \frac{1}{2} \sum_{rs} (a_{pr}\bar{b}_{sq} - a_{ps}\bar{b}_{rq})(\bar{a}_{pr}b_{sq} - \bar{a}_{ps}b_{rq}) + \sum_{r} a_{pr}b_{rq} \sum_{s} \bar{a}_{ps}\bar{b}_{sq} \right\},
$$

whence, seeing that the terms of the first summation within the braces are real and positive, we have.

$$
\lfloor a \rceil^2 \lfloor b \rceil^2 \geqq \sum_{pq} \sum_{r} a_{pr}b_{rq} \sum_{s} \bar{a}_{ps}\bar{b}_{sq} = \lfloor ab \rceil^2.
$$

The equality sign holds in (4) only when a_{rs} and b_{rs} have the form $a_{rs} = \lambda_r\mu_s$, $b_{rs} = \bar{\mu}_r\nu_s$.

Similar matrices do not in general have the same absolute value; $\lfloor bab^{-1} \rceil$ and $\lfloor a \rceil$ are the same however if $b = uc$, where $ca = ac$ and u is a unitary matrix.*

2. *Negative Powers of a Matrix.* As pointed out by Schur, $\lfloor a \rceil^2$ may be defined as the trace of $\bar{a}'a$, which is a positive Hermitian matrix and therefore has positive real roots, say $g_1, g_2, \cdots, g_n$. If

$$
s_r = \sum g_1^r,\ p_r = \sum g_1 g_2 \cdots g_r,\ s \equiv s_1 = \lfloor a \rceil^2,\ p \equiv p_n = |\det a|^2,
$$

---

* Cf. Schur, *Über die characteristischen Wurzeln einer linearen Substitution*, MATHEMATISCHE ANNALEN, vol. 66 (1909), pp. 488–510, pp. 490, 492. Since replacing a matrix by a similar one corresponds to a change of coordinates when the matrix is regarded as a linear transformation, it follows from Schur's work that, when the elementary divisors of a are simple, $\lfloor a \rceil$ has its minimum value when a is represented as a diagonal matrix, that is, in its normal form; and it seems probable that the normal form also gives the minimum value even if the elementary divisors are not simple.

then it follows readily from the inequality connecting the arithmetic and geometric means that*

$$r^r p_r \leqq \sum (g_1 + g_2 + \cdots + g_r)^r \leqq \binom{n-1}{r-1} s^r,$$

$$p_r \geqq \binom{n}{r} p^{\binom{n-1}{r-1}/\binom{n}{r}} = \binom{n}{r} p^{r/n},$$

whence

$$\binom{n}{r} p^{r/n} \leqq p_r \leqq r^{-r} \binom{n-1}{r-1} s^r. \tag{5}$$

If $C_r(a)$ is the rth compound of a,† $p_r = \lfloor C_r(a) \rceil^2$; hence, if we put α for the determinant of a, (5) becomes

$$\binom{n}{r} |\alpha|^{2r/n} \leqq \lfloor C_r(a) \rceil^2 \leqq r^{-r} \binom{n-1}{r-1} \lfloor a \rceil^{2r}. \tag{6}$$

Now $a^{-1} = C'_{n-1}(a)/\alpha$, provided $\alpha \neq 0$; hence

$$n^{1/2} |\alpha|^{-1/n} \leqq \lfloor a^{-1} \rceil \leqq (n-1)^{-\frac{(n-4)}{2}} \lfloor a \rceil^{n-1} / |\alpha|. \tag{7}$$

This inequality enables us to deal with expressions containing negative powers of a matrix. The following inequality is, however, sometimes more convenient. Since $aa^{-1} = 1$, it follows from (4) that

$$\lfloor a^{-1} \rceil \leqq \frac{n^{1/2}}{\lfloor a \rceil}, \tag{8}$$

and, since‡ $a = e^{\log a}$, $a^{-1} = e^{-\log a}$, both $\lfloor a \rceil$ and $\lfloor a^{-1} \rceil$ are less than or equal to $e^{\lfloor \log a \rceil}$.

---

* For $r = n$ this inequality, which is an extension of Hadamard's expression for the maximum of the absolute value of a determinant, is given (loc. cit., p. 496) by Schur, who also derives a number of other interesting inequalities.

† It must be remembered here that the order of $C_r(a)$ is $\binom{n}{r}$, so that, if for example $a = 1$, then $\lfloor C_r(a) \rceil^2 = \binom{n}{r}$ and not n. References to the literature of compound matrices will be found in Pascal, Repertorium der hoheren Mathematik, vol. 1, pp. 138–146.

‡ An elementary discussion of the logarithm of a matrix is given by the author, *The automorphic transformation of a bilinear form*, Annals of Mathematics, (2), vol. 23 (1921), pp. 122–125.

3. *Infinite Series and Products of Matrices.* The use of (4) makes the discussion of infinite series and products of matrices somewhat more compact than is the case with the usual methods.[*] We shall give here only a few elementary illustrations.

If a_r $(r = 0, 1, 2, \cdots)$ is an infinite sequence of matrices and x a variable matrix, and we set $\lfloor a_r \rceil = \alpha_r$, $\lfloor x \rceil = \xi$, then $\sum a_r x^r$ converges absolutely if $\sum \lfloor a_r x^r \rceil$ converges, and hence also when $\sum \alpha_r \xi^r$ is convergent. In particular, $e^x = \sum_0^\infty x^r/r!$ converges for every matrix x and $\lfloor e^x \rceil \leqq e^{\lfloor x \rceil}$. Similarly $\log(1+x) = \sum_1^\infty (-1)^{r-1} x^r/r$ converges for $\lfloor x \rceil < 1$. Here, however, the condition is much too strong; for, if $x = 1/3$, say, then $\lfloor x \rceil = n^{1/2}/3$, which is greater than 1 if $n > 9$, while the series converges absolutely in the ordinary sense. A better form of the condition in this and similar cases is $\lfloor xc \rceil / \lfloor c \rceil < 1$, where c is some non-singular matrix. Any criterion of this sort has distinct limitations; for it is not difficult to show that all that is necessary for the convergence of $\sum a_r x^r$ is that the series converge when x is replaced by the ordinary absolute value of the root of x of greatest absolute value.

4. *Examples.* The following examples[†] illustrate the application to infinite products. Let c be an arbitrary matrix, and set

$$
\begin{aligned}
P_m &= (1+a_1)(1+a_2)\cdots(1+a_m),\\
Q_m &= (1+\lfloor a_1 \rceil)(1+\lfloor a_2 \rceil)\cdots(1+\lfloor a_m \rceil);
\end{aligned}
\tag{9}
$$

then

$$
\lfloor P_m - 1 \rceil \leqq Q_m - 1 < e^{\Sigma \lfloor a_r \rceil} - 1, \tag{10}
$$

---

[*] For applications to differential equations, for example, reference may be made to Schlesinger, *Vorlesungen über lineare Differentialgleichungen*, Leipzig, 1908, pp. 86–90, 41–42.

[†] The inequalities given here are those required in proving the existence of a solution of a system of linear differential equations by the method of successive approximations; cf. Schlesinger, loc. cit., pp. 3–16.

$$\lfloor P_m c \rceil \leqq e^{\sum \lfloor a_r \rceil} \lfloor c \rceil, \tag{11}$$

$$\lfloor P_m - P_k \rceil \leqq Q_m - Q_k \leqq \sum_k^m \lfloor a_r \rceil e^{\sum_1^m \lfloor a_r \rceil}. \tag{12}$$

It is sufficient here to give the proofs of (10) and (11). Expanding (9), we have

$$P_m = 1 + \sum_p a_p + \sum_{p<q} a_p a_q + \sum_{p<q<r} a_p a_q a_r + \cdots,$$

therefore

$$\lfloor P_m - 1 \rceil \leqq \sum_p \lfloor a_p \rceil + \sum_{p<q} \lfloor a_p \rceil \lfloor a_q \rceil + \cdots = Q_m - 1,$$

$$\lfloor P_m c \rceil \leqq \Big(1 + \sum_p \lfloor a_p \rceil + \sum_{p<q} \lfloor a_p \rceil \lfloor a_q \rceil + \cdots\Big) \lfloor c \rceil = Q_m \lfloor c \rceil.$$

As another example we may take the proof of the existence of a solution of the differential equation

$$\frac{dx}{dt} = ax, \tag{13}$$

where $a = a_0 + a_1 t + a_2 t^2 + \cdots$ is a matrix which is analytic in the scalar variable t at $t = 0$. Let C be the circle of convergence of the series for a, that is, if $\lfloor a_r \rceil = \alpha_r$; then $\alpha(t) \equiv \sum \alpha_r t^r$ converges in C; and set $x = \sum x_r t^r$ in (13). Comparing like powers of t we get

$$s x_s = a_0 x_{s-1} + a_1 x_{s-2} + \cdots + a_{s-1} x_0,$$

so that $s \lfloor x_s \rceil \leqq \alpha_0 \lfloor x_{s-1} \rceil + \alpha_1 \lfloor x_{s-2} \rceil + \cdots + \alpha^{s-1} \lfloor x_0 \rceil$. Let $\xi_0 = \lfloor x_0 \rceil$, $\xi(t) = \xi_0 e^{\int_0^t \alpha(t)\,dt} = \sum \xi_r t^r$, then $d\xi/dt = \alpha(t)\xi$, so that $s\xi_s = \alpha_0 \xi_{s-1} + \alpha_1 \xi_{s-2} + \cdots + \alpha_{s-1}\xi_0$. It follows that $\lfloor x_s \rceil \leqq \xi_s$; for if this is true for $s = 1, 2, \cdots, i-1$, then

$$i \lfloor x_i \rceil \leqq \sum_j \alpha_j \lfloor x_{i-j-1} \rceil \leqq \sum_j \alpha_j \xi_{i-j-1} = i\xi_i.$$

The series for x therefore also converges in C and $x = x_0$ when $t = 0$.

PRINCETON UNIVERSITY

ON THE NUMBER OF REPRESENTATIONS OF AN INTEGER AS A SUM OR DIFFERENCE OF TWO CUBES*

BY E T BELL

1. *Introduction and Summary.* Let $C(n)$ denote the number of integer solutions $(x, y) = (\xi, \eta)$, $\xi > 0$, $\eta > 0$, of $x^3 + y^3 = n$, the pair (ξ, η), (η, ξ) being considered as a single solution, and $D(n)$ the number of integer solutions (x, y), $x > 0$, $y > 0$ of $x^3 - y^3 = n$, $n > 0$. If the pair (ξ, η), (η, ξ) in $C(n)$ be counted as two solutions, the total number is evidently $2C(n)$ or $2C(n) - 1$ according as n is not or is the double of an integer cube > 0. No determination of $C(n)$, $D(n)$ seems to have been made. It will be of interest therefore to record forms of these functions *depending only upon the real divisors of n,* in analogy to the classical results for $x^2 \pm y^2 = n$. These forms also indicate fairly expeditious means for finding all the resolutions of n into a sum or difference of two cubes.

We denote by $\psi(z)$ the well known function whose value is 1 or 0 according as z is or is not an integer square $\geqq 0$. In the sequel only integer arguments z occur. For $S(n) \equiv C(n)$ or $D(n)$ we find the following:

$$(1) \qquad n \equiv 0 \bmod 3, \qquad n \not\equiv 0 \bmod 9, \qquad S(n) = 0.$$

$$(2) \qquad n \equiv 0 \bmod 9, \qquad S(n) = \sum \psi(4t - 3d^2),$$

the $\sum$ extending to all pairs (t, d) of conjugate divisors of $n/9$ such that

$$\frac{1}{3}\sqrt[3]{n} < d \leqq \frac{1}{3}\sqrt[3]{4n}, \text{ or } d < \frac{1}{3}\sqrt[3]{n},$$

according as $S = C$ or D.

$$(3) \qquad n \equiv \pm 1 \bmod 3, \qquad S(n) = \sum \psi\left(\frac{4t - 3d^2}{3}\right),$$

---

* Presented to the Society, San Francisco Section, June 19, 1925.

the $\sum$ extending to all pairs (t, d) of conjugate divisors of n such that $d \equiv \pm 1 \bmod 3$, and

$$\sqrt[3]{n} < d \leqq \sqrt[3]{4n}, \text{ or } d < \sqrt[3]{n},$$

according as $S = C, D$, the upper or the lower sign $\pm$ being taken throughout.

Corresponding to the exact formulas of Gauss and Eisenstein in the case of two squares for the number of lattice points in the circle of radius $\sqrt{n}$ with center at the origin, we have here

$$(4) \qquad \sum_{r=1}^{n} C(r) = \frac{1}{2}\left\{[\sqrt[3]{n/2}] + \sum_{r=1}^{N}[\sqrt[3]{n-r^3}]\right\},$$

where $N = [\sqrt[3]{n}]$, and $[z] =$ the greatest integer $\leqq z$. This and the formulas for the circle are obvious cases of the following. Let $M(n)$ denote the total number of pairs of integers (x, y), $x > 0$, $y > 0$, satisfying $n = ax^r + by^s$, where a, b, r, s are integers > 0. Then

$$(5) \qquad \sum_{\alpha=1}^{n} M(\alpha) = \sum_{\alpha=1}^{N}[\sqrt[r]{(n-b\alpha^s)/a}], \; N \equiv [\sqrt[s]{n/b}].$$

2. *Proofs of* (1)—(5). Putting

$$n = dt = (x+y)(x^2-xy+y^2) = x^3+y^3,$$

we take $x+y = d, \qquad x^2-xy+y^2 = t$, and hence

$$3x^2-3dx+(d^2-t) = 0.$$

The necessary and sufficient condition that this equation have an integer root is $4t-d^2 = 3w^2$, w an integer. Since $4t-d^2$ and d are of like parity, so also are d, w. Provided $d > w$, the values of (x, y) are then $\frac{1}{2}(d \pm w)$ in either order, and hence from the definition of $C(n)$ we may take $w \geqq 0$, $x = \frac{1}{2}(d+w)$, $y = \frac{1}{2}(d-w)$. From $w^2 \geqq 0$, $d > w$, we have at once the first inequality stated in (3).

For a particular d there is one and only one t, and hence not more than one w; so that each d gives not more than one (x, y), and the (x, y)'s for two different d's are distinct.

The total number of pairs (x, y) is therefore equal to the number of integer squares $(4t - d^2)/3$. Of the 9 possible cases only the following give $4t - d^2 \equiv 0 \bmod 3$,

$$(n, d, t) \equiv (0, 0, 0), \quad (1, 1, 1), \quad (2, 2, 1) \bmod 3.$$

In the case (0, 0, 0) we set $(d, t) = (3d_1, 3t_1)$, and hence $n = 9n_1$ $(n_1 = d_1 t_1)$. Reduction of the conditions upon d, and final replacement of (n_1, d_1) by (n, d) with the proper qualifications gives $C(n)$ as stated in (2); the $C(n)$ of (1), (3) are equivalent to what precedes.

For $D(n)$ we proceed similarly from

$$n = dt = (x - y)(x^2 + xy + y^2) = x^3 - y^3,$$

setting $x - y = d$, and hence $3x^2 - 3dx + (d^2 - t) = 0$. The condition for an integer root is as before, and it is seen immediately that we can take $x = \frac{1}{2}(w + d)$, $y = \frac{1}{2}(w - d)$, $w > 0$, with the condition $w - d > 0$, which includes necessarily $w > 0$. Hence the sole condition is $d < \sqrt[3]{n}$.

The formula (5) is either evident or it is the immediate consequence of equating coefficients of q^n in the expansions of

$$\frac{1}{1-q}\, x\{f(q^a)\ g(q^b)\}, \qquad \left\{\frac{f(q^a)}{1-q}\right\} x g(q^b),$$

where

$$f(q) = \sum_0^\infty q^{n^r}, \qquad g(q) = \sum_0^\infty q^{n^s}, \qquad |q| < 1.$$

3. *Decomposition into Cubes.* The least favorable situation is that in which n is so large that its resolution into factors is impracticable. It will be sufficient to indicate the process for $C(n)$ when $n \equiv 0 \bmod 9$. We write down the $[\sqrt[3]{4n}/3] - [\sqrt[3]{n}/3]$ integers satisfying the first inequality in (2), test which of these divide $n/9$, obtaining thus the pairs (d_i, t_i) $(i = 1, 2, \cdots)$ of conjugate divisors. If of these $i = a, b, \cdots$ give all pairs such that $4t_j - 3d_j^2$ $(j = a, b, \cdots)$ is a square, all the solutions of $x^3 + y^3 = n$ with $x > 0$, $y > 0$ are

$$x=\frac{1}{2}\left\{3d_j+\sqrt{4t_j-3d_j^2}\right\},\qquad y=\frac{1}{2}\left\{3d_j-\sqrt{4t_j-3d_j^2}\right\}.$$

Numerous devices for shortening the computations are suggested by numerical work, whether or not the prime factor resolution of n be feasible.

As an immediate consequence of (2) we note that 9 is the only prime multiple of 9 which is the sum of two cubes > 0; from (3) the only solution $x > 0$, $y > 0$ of $x^3+y^3=p^2$, p prime, is $(x, y, p) = (1, 2, 3)$, etc. It is not difficult to obtain from (1)—(3) the known types of impossible equations $x^3 \pm y^3 = n$, except when n is a cube, and some others that do not seem to have been stated.

THE UNIVERSITY OF WASHINGTON

CONTACT CURVES OF THE RATIONAL PLANE CUBIC*

BY L. W. GRIFFITHS

1. *Introduction.* Contact conics and hyperosculating curves of the rational cubic have been discussed by Winger.† Likewise some account has been given of curves of order n which cut the cubic, rational or elliptic, in $(3n-1)$ coincident points.‡ There remains the question of contact curves of order $n\,(n > 2)$ whose contacts are of lower orders. This paper considers that question for the rational cubic, with results which hold for $n \geqq 1$ and for contacts of any order.

If the cubic is taken in the canonical form

$$(1)\qquad x_1 = 3t^2,\quad x_2 = 3t,\quad x_3 = t^3+1,$$

a necessary and sufficient condition that a set of $3n$

---

* Presented to the Society, San Francisco Section, December 22, 1923.

† *Involutions on the rational cubic,* this BULLETIN, vol. 25 (1918), p. 27.

‡ Winger, *Some generalizations of the satellite theory,* this BULLETIN, vol. 26 (1919), p. 75.

points on the cubic be the intersection of a curve* of order n and the cubic is

$$s_{3n} = (-1)^n, \tag{2}$$

where s_{3n} is the product of the $3n$ parameters of the points. The intersecting curve will be a contact curve if the $3n$ points are not all distinct, i. e., if the exponent of at least one parameter in s_{3n} is > 1. For instance, hyperosculating curves correspond to the coincidence case $t_1 = t_2 = \cdots = t_{3n}$. It is from (2) that the results in this paper are derived.

2. *An Example.* As an example, consider a contact quartic and require that the 12 points common to it and the cubic coincide in two contacts, a 5-point contact at a point whose parameter is τ and a 7-point contact at a point whose parameter is t. Then, in virtue of (2), t and τ satisfy the equation

$$\tau^5 t^7 = (-1)^{12}. \tag{3}$$

Thus for a given τ there are in fact seven points t, which means that seven quartics can have 5-point contact at an arbitrary point of the cubic and 7-point contact elsewhere. However, it is easily seen from (3) that τ is merely one of a set of 5 points each of which determines this same set of 7 points t_i. That is, here are 35 related contact quartics, 5 with 7-point contact at each t_i and 7 with 5-point contact at each τ_i. It is easily verified also that the 12 parameters of the 12 points τ_i and t_i (in terms of τ) satisfy (2). That is, the 12 contacts of the 35 related contact quartics lie on a non-contact quartic. But τ was an arbitrary point of the cubic. Therefore all contact quartics of the type illustrated are related in sets of 35 each, and the 12 contacts for any one set lie on a quartic.

Similar relations hold for other types of contact quartics. For instance, if τ be supposed a 4-point contact and t an 8-point contact, it can be shown that all contact quartics of this type are related in sets of 32 each, etc. Indeed, similar relations hold if and only if the 12 points common to the quartic and the cubic coincide in two contacts.

---

* The curve will in fact be one of a pencil when $n > 2$. Hereafter the word curve is used with this understanding.

3. *Generalization.* These relations on contact quartics may be generalized immediately, for the method of proof is independent of the order n of the intersecting curve. Here it is convenient to give the name *complementary contacts* to a pair of points like τ and t which absorb all the intersections of the curve and the cubic; and to indicate that a definite number $r < 3n$ of single intersections coincide at τ by saying that τ is a P^r for the curve in question. Then two complementary contact curves are said to be of the *same type* if their values of n and r are respectively equal.

THEOREM I. *All complementary contact curves of the same type are related in sets of* $r\ (3n-r)$ *each, where* r *and* n *have the values for the type in question. The distinct contacts for any one such set of complementary contact curves are* $3n$ *in number,* r *of them being contacts* P^r *and the remaining* $(3n-r)$ *being contacts* P^{3n-r}. *Each point of the type* P^r *is a contact* P^r *for* $(3n-r)$ *of the curves of the set and for no other complementary contact curves of this type in this set or any other; similarly each point of the type* P^{3n-r} *is a contact* P^{3n-r} *for* r *of the curves of the set. That is, points of the cubic are related in non-overlapping sets of* $3n$ *each by complementary contact curves of a given type.*

THEOREM II. *The* $3n$ *contacts of a set of related complementary contact curves lie on a non-contact curve of order* n *when and only when* n *is even.*

4. *Several Sets of Contacts.* In contrast to this relation (Theorem II) among the $3n$ contacts of a single set of related complementary contact curves, there are relations between several such sets of contacts: for instance, in the satellite theory. In particular, the theorem that the tangentials of 3 points of a line are collinear relates 3 sets of contacts: each set consists of 3 points, a P^1 (the single

intersection of a tangent and the cubic) and two P^{2}'s (the contacts of tangents to the cubic from the P^{1}). It will be clear from Theorem 3 that the complete relation between these 3 sets of contacts is rather this theorem on tangentials *and* the less familiar theorem that the 6 contacts of tangents drawn from 3 collinear points lie by threes on 4 lines.

For the general case the several sets of contacts are determined as follows. First choose arbitrary integer values for n, $r < 3n$, k. Let τ_i $(i = 1, 2, 3, \cdots, 3k)$ be the $3k$ distinct points in which a non-contact curve of order k intersects the cubic. Then (Theorem I) determine the $3n$ distinct contacts of the set of related complementary contact curves of order n in which τ_1 is a P^r; similarly determine the set in which τ_2 is a P^r, etc. Since the τ_i are distinct, there are $3k$ distinct sets of complementary contact curves of the same type, each set of complementary contact curves having its set of $3n$ contacts. These are the contacts whose relations are required. Note that they are 3^2kn distinct points, of which $3kr$ are contacts of the type P^r and $3k(3n-r)$ are contacts of the type P^{3n-r}.

THEOREM III. *If, as above, any $3k$ contacts of the type P^r lie on a curve of order k, then the totality of contacts of the type P^r lie on precisely r^{3k-1} curves of order k; and, further, the totality of contacts of the type P^{3n-r} lie on precisely $(3n-r)^{3k-1}$ curves of order k.*

The proof consists in exhibiting two auxiliary sets of points, called the points T_i and T_i' $(i = 1, 2, \cdots, 3k)$, such that (1) a necessary condition that the points τ_i lie on a curve of order k is that the points T_i lie on a curve of order k; (2) a sufficient condition that another selection of points of the type P^r lie on a curve of order k is that the T_i lie on a curve of order k; (3) a sufficient condition that the points T_i' lie on a curve of order k is that the points T_i lie on a curve of order k; and (4) a sufficient condition that a selection of $3k$ points of the type P^{3n-r} lie on a curve of order k is that the points T_i' lie on a curve of order k. Define $T_i = -(-\tau_i)^r$. Then if S_{3k} represents the product of the parameters of the points T_i

$$(4) \qquad S_{3k} = (-1)^k(-1)^{kr}(\tau_1 \tau_2 \cdots \tau_{3k})^r.$$

And so $S_{3k} = (-1)^k$ if $(\tau_1 \tau_2 \cdots \tau_{3k}) = (-1)^k$. This proves (1). The proof of (2) is more difficult, and requires the specific parameters of the $3n$ contacts determined by τ_i. By a method similar to that used in discussing quartics they are found to be

$$(5) \; [\tau_i, \omega_r \tau_i, \omega_r^2 \tau_i, \cdots, \omega_r^{r-1}\tau_i, t_i, \omega_{3n-r} t_i, \omega_{3n-r}^2 t_i, \cdots, \omega_{3n-r}^{3n-r-1} t_i],$$

where ω_r is a primitive rth root of unity, ω_{3n-r} a primitive $(3n-r)$th root of unity, and $t_i = [(-1)^n/\tau_i^r]^{1/(3n-r)}$, also a primitive root. Now select $(3k-1)$ points, one from among $(\tau_1, \omega_r \tau_1, \cdots, \omega_r^{r-1}\tau_1)$, one from among $(\tau_2, \omega_r \tau_2, \cdots, \omega_r^{r-1}\tau_2)$, etc., omitting one from the set to which τ_j belongs. By these points a curve of order k is determined. It will be shown that the $3k$th point A in which this curve intersects the cubic is some one of the points $(\tau_j, \omega_r \tau_j, \cdots, \omega_r^{r-1}\tau_j)$. For, determine the points T for the points in which this curve intersects the cubic. They are $T_1, \cdots, T_{j-1}, T_a, T_{j+1}, \cdots, T_{3k}$. But by the hypothesis and proof of (1) the $3k$ points $T_1, \cdots, T_{j-1}, T_j, T_{j+1}, \cdots, T_{3k}$ lie on a curve of order k. Since $(3k-1)$ points on the cubic determine a curve of order k, T_a is identical with T_j. Now from the definition of T_j it is clear that each of the r points $(\tau_j, \omega_r \tau_j, \cdots, \omega_r^{r-1}\tau_j)$ determines this same T_j, and that no other points determine this T_j. Therefore A is some one of these points, say $\omega_r^m \tau_j$. But A and the $(3k-1)$ points selected lie on a curve of order k. Therefore (2) is proved. This method of proof of (2) leads to the enumeration r^{3k-1} stated in the theorem. For the choice from the points $(\tau_i, \omega_r \tau_i, \cdots, \omega_r^{r-1}\tau_i)$ can be made independently in r ways and independently of choosing from any of the other sets.

To prove (3) define $T_i' = -(-t_i)^{3n-r}$, where t_i has the value in (5). Then each of the $(3n-r)$ points $(t_i, \omega_{3n-r} t_i, \cdots, \omega_{3n-r}^{3n-r-1} t_i)$ gives the same point T_i', and they are the only ones giving this T_i'. Then if S_{3k}' represents the product of the parameters of the points T_i', it can be shown that $S_{3k}' = 1/S_{3k}$. Therefore (3) is proved. The proof of (4) is precisely the proof of (2), where t_i and T_i' replace τ_i and T_i respectively. The enumeration $(3n-r)^{3k-1}$ is also immediate. The proof of Theorem 3 is therefore complete.

It should be noted that some of the results of Theorem 3, but no enumerations, follow from the theory of residuation.

5. *Several Contacts of the Same Order.* Finally, consider contact curves whose intersections are not completely absorbed in two contacts. Then at least two contacts are arbitrary, and relations are correspondingly indeterminate. The interesting case of precisely three contacts is that in which the contacts are of the same order; then from (2)

$$(t_1t_2t_3)^n = (-1)^n. \tag{6}$$

Thus the necessary and sufficient condition that a triad of points be contacts of this sort is that it belong to one of the n involutions

$$s_3 = -\omega_n^i, \quad (i = 0, 1, \cdots, n-1), \tag{7}$$

where ω_n is a primitive nth root of unity. For $n = 2$ these contact curves are the familiar tritangent conics.

The interesting case of n contacts is also that in which the contacts are of the same order, when from (2)

$$(t_1t_2 \cdots t_n)^3 = (-1)^n, \tag{8}$$

$$s_n = (-1)^n\omega_3^i, \qquad (i = 0, 1, 2), \tag{9}$$

where ω_3 is a primitive cube root of unity.

6. *Analogous Theorems for the General Cubic.* For the general cubic, expressed rationally in terms of $\wp(u)$, analogous theorems hold. The equations in terms of the elliptic argument are not of the simple type (2) afforded by the rational cubic, but yield determinate enumerations.

THE UNIVERSITY OF WASHINGTON

NOTE ON THE PROJECTIVE GEOMETRY OF PATHS

BY T. Y. THOMAS*

1. *Projective Geometry of Paths.* It was first shown by Weyl† that the functions $\Gamma^i_{\alpha\beta}$ and the functions

$$(1) \qquad \varLambda^i_{\alpha\beta} = \Gamma^i_{\alpha\beta} + \delta^i_\alpha \psi_\beta + \delta^i_\beta \psi_\alpha,$$

where ψ_α is an arbitrary covariant vector, and

$$\delta^i_\alpha = 0, \quad \text{for } i \neq \alpha; \qquad = 1, \quad \text{for } i = \alpha,$$

define the same geometry of paths. This leads to the consideration of properties of the paths which are independent of the particular set of functions $\Gamma^i_{\alpha\beta}$ by means of which the paths are defined. Theorems expressing such properties constitute the projective geometry of paths. In the following note we give a few theorems belonging to the projective geometry of paths.

2. *Projective Tensors.* Theorems of the projective geometry of paths appear to have their statement in terms of what may be called *projective tensors,* i. e. tensors which are independent of the particular set of functions $\Gamma^i_{\alpha\beta}$ defining the paths. We shall show how a set of projective tensors may be derived by covariant differentiation from an n-uple of mutually independent vectors.

Let $h_{(\alpha)i}$ denote an n-uple of independent covariant vectors. Then the determinant

$$(2) \qquad h = |h_{(\alpha)i}|$$

does not vanish identically. We may therefore define an n-uple of contravariant vectors $h^{(\alpha)i}$ as the cofactors of the corresponding $h_{(\alpha)i}$ in the determinant h divided by h. Hence

---

* National Research Fellow in Mathematical Physics.

† H. Weyl, GOTTINGER NACHRICHTEN, 1921, p. 99. See also O. Veblen, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), p. 347; and O. Veblen and T. Y. Thomas, TRANSACTIONS OF THIS SOCIETY, vol. 25, p. 557.

$$(3) \qquad h_{(\alpha)i}\, h^{(\alpha)j} = \delta^j_i,$$

and

$$(4) \qquad h_{(i)\alpha}\, h^{(j)\alpha} = \delta^j_i,$$

where the left members represent a summation in α from $\alpha = 1$ to $\alpha = n$. It will be understood in the following that each index which appears twice in a term, once as a subscript and once as a superscript, is to be summed over the values 1 to n.

The change $[h_{(\alpha)i,j}]$ in the covariant derivative $h_{(\alpha)i,j}$ of the n-uple $h_{(\alpha)i}$, based on the functions $\Gamma^i_{\alpha\beta}$, when the Γ's are replaced by the above functions $\varDelta$, is

$$(5) \qquad [h_{(\alpha)i,j}] = -h_{(\alpha)i}\, \psi_j - h_{(\alpha)j}\, \psi_i.$$

Hence

$$(6) \qquad (n+1)\, \psi_i = -h^{(\alpha)\beta}\, [h_{(\alpha)\beta,\, i}].$$

The vector γ_i defined by

$$(7) \qquad (n+1)\, \gamma_i = -h^{(\alpha)\beta}\, h_{(\alpha)\beta,\, i}$$

will therefore change by ψ_i when $\Gamma^i_{\alpha\beta}$ is replaced by $\varDelta^i_{\alpha\beta}$, i.e.

$$(8) \qquad [\gamma_i] = \psi_i.$$

Hence

$$(9) \qquad h_{(\alpha)ij} = h_{(\alpha)i,j} + h_{(\alpha)i}\gamma_j + h_{(\alpha)j}\gamma_i$$

represents a set of n projective covariant tensors of the second order.

In a similar way we may treat the case of an n-uple of independent contravariant vectors $h^{(\alpha)i}$. This leads to a set of n projective mixed tensors of the second order given by

$$(10) \qquad h^{(\alpha)i}_j = h^{(\alpha)i}{}_{,j} - h^{(\alpha)i}\lambda_j - \delta^i_j h^{(\alpha)\beta}\lambda_\beta$$

where the vector λ_i is defined as

$$(11) \qquad (n+1)\lambda_i = h_{(\alpha)\beta}\, h^{(\alpha)\beta}{}_{,\, i},$$

in which the covariant n-uple $h_{(\alpha)i}$ is obtained from the contravariant n-uple $h^{(\alpha)i}$ by dividing the cofactors of the elements of the determinant

$$h = |h^{(\alpha)i}|$$

by the determinant h. Other projective tensors will appear in the following paragraphs.

3. *The n-uple of Parallel Vectors.*[*] If $h_{(\alpha)i}$ represents an n-uple of independent parallel covariant vectors for some set of functions $\Gamma^i_{\alpha\beta}$ defining the paths, then

$$h_{(\alpha)ij} = 0 \tag{12}$$

identically. The condition (12) is also sufficient for the n-uple of independent covariant vectors $h_{(\alpha)i}$ to be parallel. Hence we have the following theorem.s

THEOREM I. *A necessary and sufficient condition for the n-uple of independent covariant vectors* $h_{(\alpha)i}$ *to be parallel in the projective geometry of paths is that* (12) *be satisfied.*

Forming the equations

$$h^{(\alpha)i}_{\;\;j} = 0, \tag{13}$$

we may state the corresponding theorem for the case of a contravariant n-uple.

THEOREM II. *A necessary and sufficient condition for the n-uple of independent contravariant vectors* $h^{(\alpha)i}$ *to be parallel in the projective geometry of paths is that* (13) *be satisfied.*

4. *Reduction to the Euclidean Geometry.* The affine geometry of paths becomes a euclidean geometry if there exists an n-uple of independent parallel covariant vectors $h_{(\alpha)i}$. This is obviously a necessary condition and it is seen to be sufficient since the existence of an n-uple of parallel vectors $h_{(\alpha)i}$ leads to the equations

$$h_{(i)\sigma}B^{\sigma}_{\alpha\beta\gamma} = 0, \tag{14}$$

where $B^{\sigma}_{\alpha\beta\gamma}$ is the affine (ordinary) curvature tensor, and hence

---

[*] L. P. Eisenhart, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), p. 207. See also O. Veblen and T. Y. Thomas, loc. cit., p. 589.

$$B^i_{\alpha\beta\gamma} = 0, \tag{15}$$

owing to the independence of the vectors $h_{(\alpha)i}$. Hence we have the following theorem.

THEOREM III. *A necessary and sufficient condition for the projective geometry of paths to be a euclidean geometry is that the Γ's be such that there exists an n-uple of independent covariant vectors $h_{(\alpha)i}$ satisfying* (12).

A similar theorem may be stated for the n-uple of contravariant vectors.

5. *Reduction to the Riemann Geometry.* For the projective geometry of paths to be a Riemann geometry it is necessary and sufficient that the Γ's be such that there exists a symmetric tensor $g_{\alpha\beta}$ whose determinant

$$g = |g_{\alpha\beta}| \tag{16}$$

does not vanish identically, and a vector ψ_i such that

$$\tilde{g}_{\alpha\beta,\gamma} = 0, \tag{17}$$

where $\tilde{g}_{\alpha\beta,\gamma}$ is the covariant derivative of $g_{\alpha\beta}$ based on the functions $\mathcal{A}^i_{\alpha\beta}$ given by (1). Writing (17) in the form

$$g_{\alpha\beta,\gamma} - 2g_{\alpha\beta}\psi_\gamma - g_{\gamma\beta}\psi_\alpha - g_{\alpha\gamma}\psi_\beta = 0, \tag{18}$$

where $g_{\alpha\beta,\gamma}$ is the covariant derivative of $g_{\alpha\beta}$ based on the functions $\Gamma^i_{\alpha\beta}$, and multiplying by the contravariant tensor $g^{\alpha\beta}$, formed in the ordinary manner from the tensor $g_{\alpha\beta}$, we have

$$2(n+1)\psi_\gamma = g^{\alpha\beta} g_{\alpha\beta,\gamma}. \tag{19}$$

Substituting this value of ψ_γ in the left member of (18) and denoting the resulting expression by $g_{\alpha\beta\gamma}$, we find

$$g_{\alpha\beta\gamma} = g_{\alpha\beta,\gamma} - \frac{g^{\mu\nu}}{n+1}\left(g_{\alpha\beta} g_{\mu\nu,\gamma} + \frac{1}{2} g_{\gamma\beta} g_{\mu\nu,\alpha} + \frac{1}{2} g_{\alpha\gamma} g_{\mu\nu,\beta}\right). \tag{20}$$

Hence (18) takes the form

$$g_{\alpha\beta\gamma} = 0. \tag{21}$$

The tensor $g_{\alpha\beta\gamma}$ defined by (20) may be shown to be projective.

The equations (21) constitute a necessary condition for the projective geometry of paths to be a Riemann geometry based on the tensor $g_{\alpha\beta}$ as the fundamental metric tensor. This condition is easily shown to be sufficient. Hence we have the following theorem.

THEOREM IV. *A necessary and sufficient condition for the projective geometry of paths to be a Riemann geometry is that the Γ's be such that there exists a tensor $g_{\alpha\beta}$ which satisfies the equations* (21).

6. *Reduction to the Weyl Geometry.** Let us denote by $g_{\alpha\beta}$ a covariant symmetric tensor whose determinant g does not vanish identically, as in the preceding paragraph, and also by φ_α a covariant vector. Let us then form the equations

$$g_{\alpha\beta\gamma} = g_{\alpha\beta}\varphi_\gamma, \tag{22}$$

where $g_{\alpha\beta\gamma}$ is now defined by

$$\begin{aligned} g_{\alpha\beta\gamma} = g_{\alpha\beta,\gamma} - \frac{g^{\mu\nu}}{n+1}\left(g_{\alpha\beta}g_{\mu\nu,\gamma} + \frac{1}{2}g_{\gamma\beta}g_{\mu\nu,\alpha} + \frac{1}{2}g_{\alpha\gamma}g_{\mu\nu,\beta}\right) \\ + \frac{n}{n+1}\left(g_{\alpha\beta}\varphi_\gamma + \frac{1}{2}g_{\gamma\beta}\varphi_\alpha + \frac{1}{2}g_{\alpha\gamma}\varphi_\beta\right), \end{aligned} \tag{23}$$

in which $g_{\alpha\beta,\gamma}$ is the covariant derivative of $g_{\alpha\beta}$ based on the functions $\Gamma^i_{\alpha\beta}$. The tensor $g_{\alpha\beta\gamma}$ defined by (23) is a projective tensor. We may prove the following theorem.

THEOREM V. *A necessary and sufficient condition for the projective geometry of paths to be a Weyl geometry is that the Γ's be such that there exists a tensor $g_{\alpha\beta}$ and vector φ_α which satisfy the equations* (22).

THE UNIVERSITY OF CHICAGO

---

* By the Weyl geometry is meant the geometry used by Weyl as the basis of his combined theory of gravitation and electricity. See H. Weyl, *Raum, Zeit, Materie*, 4th ed., p. 113.

THE TENSOR CHARACTER OF THE GENERALIZED KRONECKER SYMBOL*

BY F. D. MURNAGHAN

1. *Introduction.* In a previous paper† we have considered the use of the generalized Kronecker symbol $\delta^{r_1 r_2 \cdots r_m}_{s_1 s_2 \cdots s_m}$ in presenting the theory of determinants and we now proceed to show that it is an arithmetic tensor of the type indicated by its subscripts and superscripts, i. e., it is covariant of rank m and contravariant of rank m. By the statement that a tensor is arithmetic, we mean that its presentation is independent of the particular coordinate system in use and that it has the same numerical values for its various components at all points of space.‡

The generalized Kronecker symbol may be defined by means of the equation

$$
\delta^{r_1 r_2 \cdots r_m}_{s_1 s_2 \cdots s_m} = \begin{vmatrix} \delta^{r_1}_{s_1} & \cdots & \delta^{r_1}_{s_m} \\ \delta^{r_2}_{s_1} & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \delta^{r_m}_{s_1} & \cdots & \delta^{r_m}_{s_m} \end{vmatrix} . \tag{1}
$$

Here the labels r and s run independently over a set of n numbers $1, 2, \cdots, n$ and $\delta^r_s = 1$ if $r = s$, and $= 0$ if $r \neq s$; δ^r_s is the ordinary Kronecker symbol and it is usually denoted by g^r_s in the theory of relativity. It is there derived as the scalar product of the metric tensor g_{rs} and its reciprocal g^{rs}, but this mode of presentation is somewhat

---

[*] Presented to the Society (under a different title), March 1, 1924.

[†] *The Generalized Kronecker symbol and its application to the theory of determinants,* AMERICAN MATHEMATICAL MONTHLY vol. 32 (1925), p. 233. This paper will be denoted by the symbol (*A*) in references below.

[‡] Cf. P. FRANKLIN, PHILOSOPHICAL MAGAZINE, (6), vol. 45 (1923), p. 998.

unfortunate since the tensor character of the symbol has nothing to do with the metric properties of the space.

The simplest procedure in proving the tensor character of the symbol $\delta^{r_1 r_2 \cdots r_m}_{s_1 s_2 \cdots s_m}$ is first to prove the theorem when $m = n$, the number of dimensions of the space. To do this we shall define a tensor of rank $2n$, contravariant of rank n and covariant of rank n, by the statement that its presentation in a particular coordinate system $(x^1, x^2, x^3, \cdots, x^n)$ is furnished by the values of the generalized Kronecker symbol $\delta^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$. Denoting the presentation of this tensor in any other coordinate system $(y^1, y^2, \cdots, y^n)$ by $\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$, we have to show that

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \delta^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$$

where the labels $(r_1, r_2, \cdots, r_n)$ and $(s_1, s_2, \cdots, s_n)$ may each be assigned independently any one of the values $(1, 2, \cdots, n)$. We have, from the definition of a tensor,

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_n} \frac{\partial y^{r_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{r_n}}{\partial x^{\alpha_n}} \frac{\partial x^{\beta_1}}{\partial y^{s_1}} \cdots \frac{\partial x^{\beta_n}}{\partial y^{s_n}}$$

where we have adopted the convention of the previous paper (A), according to which *Greek* labels occurring twice in any term indicate summations. Writing*

$$\delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_n} = \frac{1}{n!} \delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\lambda_1 \lambda_2 \cdots \lambda_n} \delta^{\lambda_1 \lambda_2 \cdots \lambda_n}_{\beta_1 \beta_2 \cdots \beta_n} \tag{2}$$

we have

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \frac{1}{n!} \left(\delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\lambda_1 \lambda_2 \cdots \lambda_n} \frac{\partial y^{r_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{r_n}}{\partial x^{\alpha_n}} \right) \left(\delta^{\lambda_1 \lambda_2 \cdots \lambda_n}_{\beta_1 \beta_2 \cdots \beta_n} \frac{\partial x^{\beta_1}}{\partial y^{s_1}} \cdots \frac{\partial x^{\beta_n}}{\partial y^{s_n}} \right).$$

Now the expression on the right hand side is a summation of products of two factors of the type

$$\delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{l_1 l_2 \cdots l_n} \frac{\partial y^{r_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{r_n}}{\partial x^{\alpha_n}} \quad \text{and} \quad \delta^{l_1 l_2 \cdots l_n}_{\beta_1 \beta_2 \cdots \beta_n} \frac{\partial x^{\beta_1}}{\partial y^{s_1}} \cdots \frac{\partial x^{\beta_n}}{\partial y^{s_n}}$$

---

* See paper (A), equation (2 · 3).

respectively. Each of these factors is a determinant of order n (see (3·4), paper (A)) and since an inversion of any two of the labels $(l_1, l_2, \cdots, l_n)$ merely changes the sign of both factors we may write

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$$
$$= \delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{1\,2\,\cdots\,n} \frac{\partial y^{r_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{r_n}}{\partial x^{\alpha_n}} \cdot \delta^{1\,2\,\cdots\,n}_{\beta_1 \beta_2 \cdots \beta_n} \frac{\partial x^{\beta_1}}{\partial y^{s_1}} \cdots \frac{\partial x^{\beta_n}}{\partial y^{s_n}}.$$

The product of the two determinants on the right is a determinant of which the element in the pth row and the qth column is

$$(3) \qquad \frac{\partial y^{r_p}}{\partial x^{\sigma}} \cdot \frac{\partial x^{\sigma}}{\partial y^{s_q}} \quad \text{or} \quad \delta^{r_p}_{s_q}.$$

Hence

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \delta^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$$

which proves the theorem stated.

The tensor character of the symbols for the cases $m < n$ follows at once by contraction of the tensor $\delta^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$. Thus it is immediately apparent that

$$(4) \qquad \delta^{r_1 r_2 \cdots r_{n-1}}_{s_1 s_2 \cdots s_{n-1}} = \delta^{r_1 r_2 \cdots r_{n-1} \alpha}_{s_1 s_2 \cdots s_{n-1} \alpha}$$

for in the summation on the right all the terms vanish unless $(r_1, r_2, \cdots, r_{n-1})$ are all different and $(s_1, s_2, \cdots, s_{n-1})$ all different and also α different from any of the $(r_1, r_2, \cdots, r_{n-1})$ and the $(s_1, s_2, \cdots, s_{n-1})$. The only term which has a value different from zero occurs, therefore, when $(r_1, r_2, \cdots, r_{n-1})$ and $(s_1, s_2, \cdots, s_{n-1})$ are arrangements of the same group of $n-1$ out of the n numbers $(1, 2, \cdots, n)$ and α is the remaining number. The equation (4) shows that $\delta^{r_1 r_2 \cdots r_{n-1}}_{s_1 s_2 \cdots s_{n-1}}$ is an arithmetic mixed tensor contravariant of rank $n-1$ and covariant of rank $n-1$. Proceeding similarly we arrive at the simple Kronecker symbol $\delta^{r}_{s} = \delta^{r \alpha_1 \alpha_2 \cdots \alpha_{n-1}}_{s \alpha_1 \alpha_2 \cdots \alpha_{n-1}}$ which is an arithmetic mixed tensor of the second rank.

It may not be superfluous to call attention again to the fact that these tensors are non-metric. The space for which they are defined is the general space of analysis situs, or topology, in which a point is merely a set of n ordered numbers. The usual presentation of, and notation for, the mixed tensor δ^r_s is therefore unfortunate. This starts with a symmetric covariant tensor g_{rs} of the second rank which furnishes the metric ground form $(ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ of a Riemann space. From this is derived a contravariant presentation g^{rs} of the metric tensor and the inner product $g^r_s = g^{r\alpha}g_{\alpha s}$ furnishes the mixed arithmetic tensor which we have denoted by δ^r_s.

2. *The Outer Multiplication of Tensors.* If we have two covariant tensors $a_{r_1r_2\cdots r_p}$ and $b_{s_1s_2\cdots s_q}$ of ranks p and q respectively we may derive from them, by means of the generalized Kronecker tensor, an alternating covariant tensor of rank $p+q$ as follows:

$$c_{r_1r_2\cdots r_ps_1\cdots s_q} = \delta^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}_{r_1\cdots r_ps_1\cdots s_q}a_{\alpha_1\cdots\alpha_p}b_{\beta_1\cdots\beta_q}; \tag{5}$$

$p+q$ must be $\leqq n$ in order that the alternating tensor thus arrived at may not vanish identically.

Similarly we derive from two contravariant tensors $a^{r_1\cdots r_p}$ and $b^{s_1\cdots s_q}$ of ranks p and q respectively an alternating contravariant tensor of rank $p+q$

$$c^{r_1\cdots r_ps_1\cdots s_q} = \delta^{r_1\cdots r_ps_1\cdots s_q}_{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}a^{\alpha_1\cdots\alpha_p}b^{\beta_1\cdots\beta_q}. \tag{6}$$

If the original tensors are alternating considerable simplification results. It is convenient to remove a numerical factor $p!\,q!$ and we may define the derived tensor $c_{r_1\cdots r_ps_1\cdots s_q}$ by the equations

$$\begin{aligned} c_{r_1\cdots r_ps_1\cdots s_q} &= \frac{1}{p!}\frac{1}{q!}\delta^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}_{r_1\cdots r_ps_1\cdots s_q}a_{\alpha_1\cdots\alpha_p}b_{\beta_1\cdots\beta_q}, \\ &= \sum_{l,m}\delta^{l_1\cdots l_pm_1\cdots m_q}_{r_1\cdots r_ps_1\cdots s_q}a_{l_1\cdots l_p}b_{m_1\cdots m_q}. \end{aligned} \tag{7}$$

In the last expression $(l_1, l_2, \cdots, l_p)$ is any group of p out of the $p+q$ numbers $(r_1, \cdots, r_p, s_1, \cdots, s_q)$ and $(m_1, \cdots, m_q)$

is the remaining group of q numbers. No group $(l_1, \cdots, l_p)$ is to be repeated in the summation. The tensor derived in this way may be called the outer product of the two alternating covariant tensors. Similarly the outer product of two alternating contravariant tensors is given by

$$
\begin{aligned}
(8) \qquad c^{r_1 \cdots r_p s_1 \cdots s_q} &= \frac{1}{p!}\frac{1}{q!}\,\delta^{r_1 \cdots r_p s_1 \cdots s_q}_{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q}\, a^{\alpha_1 \cdots \alpha_p}\, b^{\beta_1 \cdots \beta_q} \\
&= \sum_{l,\, m} \delta^{r_1 \cdots r_p s_1 \cdots s_q}_{l_1 \cdots l_p m_1 \cdots m_q}\, a^{l_1 \cdots l_p}\, b^{m_1 \cdots m_q}.
\end{aligned}
$$

The alternating tensor derived as in (5) from two non-alternating tensors is not essentially different from the outer product of two alternating tensors. For we may write it, on using the result*

$$
\delta^{l_1 \cdots l_p m_1 \cdots m_q}_{r_1 \cdots r_p s_1 \cdots s_q} = \frac{1}{p!}\frac{1}{q!}\,\delta^{l_1 \cdots l_p}_{\lambda_1 \cdots \lambda_p}\,\delta^{m_1 \cdots m_q}_{\mu_1 \cdots \mu_q}\,\delta^{\lambda_1 \cdots \lambda_p}_{r_1 \cdots r_p}\,\delta^{\mu_1 \cdots \mu_q}_{s_1 \cdots s_q},
$$

in the form

$$
\frac{1}{p!}\frac{1}{q!}\,\delta^{\lambda_1 \cdots \lambda_p \mu_1 \cdots \mu_q}_{r_1 \cdots r_p s_1 \cdots s_q}\left(\delta^{\alpha_1 \cdots \alpha_p}_{\lambda_1 \cdots \lambda_p}\, a_{\alpha_1 \cdots \alpha_p}\right)\left(\delta^{\beta_1 \cdots \beta_q}_{\mu_1 \cdots \mu_q}\, b_{\beta_1 \cdots \beta_q}\right)
$$

showing that $c_{r_1 \cdots r_p s_1 \cdots s_q}$, as defined in (5), is really the outer product of the two alternating tensors

$$
f_{t_1 \cdots t_p} = \delta^{\alpha_1 \cdots \alpha_p}_{t_1 \cdots t_p}\, a_{\alpha_1 \cdots \alpha_p}
$$

and

$$
g_{u_1 \cdots u_q} = \delta^{\beta_1 \cdots \beta_q}_{u_1 \cdots u_q}\, b_{\beta_1 \cdots \beta_q}.
$$

It is well known that alternating covariant tensors may be used as the coefficients of invariant integral forms. In fact, we may, from p arbitrary contravariant tensors $a^r, b^s, \cdots, g^t$, of rank one, form the alternating contravariant tensor of rank p

$$
e^{r_1 \cdots r_p} = \delta^{r_1 \cdots r_p}_{\alpha_1 \cdots \alpha_p}\, a^{\alpha_1} b^{\alpha_2} \cdots g^{\alpha_p}.
$$

When $(a^r, b^s, \cdots, g^t)$ are differential vectors tangent to the parametric lines of a spread of p dimensions so that

---

* See equation $(2\cdot4^{\text{bis}})$, paper (A).

$$a^r = \frac{\partial x^r}{\partial u_1} du_1,$$

etc., the scalar product $a_{\varrho_1 \cdots \varrho_p} e^{\varrho_1 \cdots \varrho_p}$ is called *an integral form of order* p (in this case $e^{r_1 \cdots r_p}$ is usually denoted by $d(x^{r_1} x^{r_2} \cdots x^{r_p})$). Its integral over the spread is called *the integral of the alternating covariant tensor* $a_{r_1 \cdots r_p}$ *over the spread of* p *dimensions.* In this connection the result (7), which enables us to derive from two integral forms of orders p and q respectively an integral form of order $p+q$, is known as the *law of the outer product of two integral forms.**

3. *Connection between the Kronecker Tensor and Reciprocation with respect to a Quadratic Differential Form.* If we have any symmetric covariant tensor of the second rank a_{rs} we may denote by a the value of the determinant which has a_{rs} as the element in its rth row and sth column. It follows at once that the product $\sqrt{a}\ d(x^1 x^2 \cdots x^n)$ is invariant, and so we may introduce an alternating covariant tensor $\varepsilon_{r_1 \cdots r_n}$ which is defined by the statement that its components have the value $\pm \sqrt{a}$ according as the arrangement $(r_1, r_2, \cdots, r_n)$ of the n numbers $(1, 2, \cdots, n)$ is of the even or odd class. For the integral form

$$\varepsilon_{\alpha_1 \cdots \alpha_n} d(x^{\alpha_1} x^{\alpha_2} \cdots x^{\alpha_n})$$

is invariant, its value being $n! \sqrt{a}\, d(x^1 x^2 \cdots x^n)$. From this alternating covariant tensor we derive an alternating contravariant tensor of rank n as follows. Solving the system of n^2 equations

$$a_{r\alpha} a^{\alpha s} = \delta_r^s$$

for the n^2 unknowns a^{rs}, we obtain a contravariant tensor of rank two which is said to be the reciprocal of a_{rs} with respect to the quadratic differential form $a_{\alpha\beta} dx^\alpha dx^\beta$. The tensor $\varepsilon^{r_1 \cdots r_n} = a^{r_1 \alpha_1} a^{r_2 \alpha_2} \cdots a^{r_n \alpha_n} \varepsilon_{\alpha_1 \cdots \alpha_n}$ is said to be the reciprocal of $\varepsilon_{r_1 \ldots r_n}$ with respect to the quadratic differential form. It is alternating and has the value $\pm 1/\sqrt{a}$ according as the arrangement $(r_1, \cdots, r_n)$ of the n numbers $(1, 2, \cdots, n)$ is of the even or odd class. Then the simple product $\varepsilon^{r_1 \cdots r_n}\varepsilon_{s_1 \cdots s_n}$ is the generalised Kronecker tensor

---

* Reference may be made to E. Cartan, *Leçons sur les Invariants Intégraux*, Paris, 1922. E. Cartan, ANNALES DE L'ECOLE NORMALE, 1899. H. Bateman, *Differential Equations*, Chapter 7. London, 1918. H. Bateman, PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 8 (1910).

$$\delta^{r_1 \cdots r_n}_{s_1 \cdots s_n}.$$

It may be remarked that the operation of finding tensors reciprocal to any quadratic differential form is a possible one so long as the differential form is non-singular $(a \neq 0)$. When the quadratic differential form is the metrical ground form $(ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ of a Riemann space we may say that the reciprocal covariant and contravariant tensors are but different representations of the same physical idea which is the tensor. This is by analogy with the case of rectangular cartesian coordinates in euclidean space where the distinction between covariance and contravariance breaks down and the components of two tensors which are reciprocal with respect to the metric ground form $(ds)^2 = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2$ coincide.

We have endeavored to show in the preceding paragraphs the utility of the alternating tensor $\delta^{i_1 \cdots i_m}_{s_1 \cdots s_m}$ in proving theorems of tensor algebra. It will be readily recognised that there is an intimate connection here with Grassmann's *Ausdehnungslehre,* and we believe, in fact, that a systematic exposition of this theory with the aid of the generalized Kronecker symbol would help to make it more widely understood. The use of the symbol in connection with the discussion of the orientation of cells in analysis situs[*] is also recommended.

JOHNS HOPKINS UNIVERSITY

---

[*] O. Veblen. *Cambridge Colloquium Lectures,* New York, 1922.

TWO GENERAL FUNCTIONAL EQUATIONS*

BY W. H WILSON

The object of this paper is to discuss the functional equations

(1) $f(x+y)+g(x-y)=h(x)k(y), \quad h(x)\neq 0, \quad k(x)\neq 0,$

and

(2) $F(x+y)G(x-y)=H(x)+K(y), \quad F(x)\neq 0, \quad G(x)\neq 0,$

in which x and y are independent variables and $f(x)$, $g(x)$, $h(x)$, $k(x)$, $F(x)$, $G(x)$, $H(x)$, $K(x)$ are functions to be determined. Special cases of equations (1) and (2) have been discussed in the literature. Some of the more familiar special cases are

$$h(x)\equiv k(x)\equiv f(x)\equiv \psi(x), \quad g(x)\equiv 0,$$
$$g(x)\equiv k(x)\equiv f(x), \quad h(x)\equiv 2f(x),$$
$$G(x)\equiv 1, \; H(x)\equiv K(x)\equiv F(x)\equiv \varphi(x),$$
$$G(x)\equiv F(x), \; H(x)\equiv F^2(x), \; K(x)\equiv F^2(x)-1,$$
$$G(x)\equiv F(x), \; H(x)\equiv F^2(x), \; K(x)\equiv -F^2(x).$$

In this paper no relationships are assumed between the functions in equation (1) or the functions in equation (2). Furthermore, no restrictions (such as continuity, differentiability, etc.) are imposed on the functions. The variables, x and y, are not assumed real nor must they necessarily be complex. The author shows that the functions of equations (1) and (2) are expressible in terms of the functions $\varphi(x)$ and $\psi(x)$ which satisfy the Cauchy equations

$$\varphi(x+y)=\varphi(x)+\varphi(y), \tag{3}$$
$$\psi(x+y)=\psi(x)\psi(y), \tag{4}$$

given above as special cases of (1) and (2). The results of the paper are of sufficient generality to permit immediate

---

[*] Presented to the Society, November 29, 1924.

application of information concerning solutions of (3) and (4) to equations (1) and (2).

If equation (1) *is satisfied, there are functions* $F(x)$, $G(x)$, $H(x)$, $K(x)$, *related to the functions* $f(x)$, $g(x)$, $h(x)$, $k(x)$, *which satisfy equation* (2).

To prove this replace x by $x+y$ and y by $x-y$ in equation (1). Then

$$h(x+y)k(x-y) = f(2x)+g(2y)$$

which is equation (2) if we define the functions of (2) by the equations

$$F(x) \equiv h(x), \quad G(x) \equiv k(x), \quad H(x) \equiv f(2x), \quad K(x) \equiv g(2x).$$

In like manner, *if equation* (2) *is satisfied, there are functions* $f(x)$, $g(x)$, $h(x)$, $k(x)$, *related to the functions* $F(x)$, $G(x)$, $H(x)$, $K(x)$, *which satisfy equation* (1). In view of these statements, it is sufficient if we discuss equation (1).

Since $k(x) \not\equiv 0$, there is some a such that $k(a) \neq 0$. We may replace y by $y+a$ in equation (1) with the result

$$(5) \qquad f(x+y+a)+g(x-y-a) = h(x)k(y+a).$$

If we write

$$\begin{aligned} k(a)\bar{f}(x) &= f(x+a), \\ k(a)\bar{g}(x) &= g(x-a), \\ k(a)\bar{k}(x) &= k(x+a), \end{aligned}$$

equation (5) becomes at once of form (1) with the added condition that $\bar{k}(0) = 1$. *We may therefore suppose, in what follows, that this transformation has been made and that* $k(0) = 1$.

In equation (1), let $y = x$. Then

$$(6) \qquad f(2x) = h(x)k(x) - g(0).$$

In like manner, if we let $y = -x$ in (1),

$$(7) \qquad g(2x) = h(x)k(-x) - f(0).$$

Equations (6) and (7) show that $f(x)$ and $g(x)$ are determined, except for constants, by $h(x)$ and $k(x)$. In (1), replace x by $x+y$ and y by $x-y$. It follows that

$$h(x+y)k(x-y)=f(2x)+g(2y) \tag{8}$$
$$=h(x)k(x)+h(y)k(-y)-[f(0)+g(0)].$$

The auxiliary equation

$$h(2x)=[k(x)+k(-x)]h(x)-[f(0)+g(0)] \tag{9}$$

may be obtained by letting $y=x$ in (8). Interchanging x and y in (8), we have

$$h(x+y)k(y-x)=h(y)k(y)+h(x)k(-x)-[f(0)+g(0)]. \tag{10}$$

If, now, we add equations (8) and (10) and use equation (9), we find

$$h(x+y)[k(x-y)+k(y-x)]=h(2x)+h(2y), \tag{11}$$

which is transformed into

$$h(x+y)+h(x-y)=2h(x)E(y), \tag{12}$$

where $E(y)$ is the even component of $k(y)$, by replacing $2x$ by $x+y$ and $2y$ by $x-y$. The relation of equation (12) to equations (3) and (4) has been discussed by the writer.*

In order to find the odd component $S(x)$ of $k(x)$, we may subtract equation (10) from equation (8), whence

$$h(x+y)S(x-y)=h(x)S(x)-h(y)S(y). \tag{13}$$

In equation (13), replace x by $-x$ and y by $-y$ and add the equation thus found to (13). If we denote the odd component of $h(x)$ by $S_h(x)$, we have

$$S_h(x+y)S(x-y)=S_h(x)S(x)-S_h(y)S(y). \tag{14}$$

Replacing y by $-y$ in (14) leaves the right-hand member of the equation unchanged. Therefore

$$S_h(x+y)S(x-y)\equiv S_h(x-y)S(x+y), \tag{15}$$

whence, if $S_h(x)\not\equiv 0$,

$$S(x)\equiv\lambda S_h(x), \tag{16}$$

where λ is a constant.

---

* This BULLETIN, vol. 27 (1920), p. 302, equation (7).

If $S_h(x) \equiv 0$, $h(x)$ is an even function. Replacing x by $-x$ in (1), we have

$$f(-x+y)+g(-x-y) = h(x)k(y) = f(x+y)+g(x-y),$$

from which

$$g(x) = f(-x)+\mu,$$

where μ is a constant. Equation (1) may now be written in the form

$$f(y+x)+f(y-x)+\mu = h(x)k(y). \tag{17}$$

Since $h(x) \not\equiv 0$, there is some value, say b, of x such that $h(b) \neq 0$. Replace x by b and y by $y+x$ in equation (17). The result is

$$h(b)k(y+x) = f(y+x+b)+f(y+x-b)+\mu. \tag{18}$$

If we replace x by $-x$ in (18), we have

$$h(b)k(y-x) = f(y-x+b)+f(y-x-b)+\mu \tag{19}$$

whence, from (18) and (19),

$$\begin{aligned} h(b)[k(y+x)+k(y-x)] &= [f(y+x+b) \\ &\quad +f(y-x-b)+\mu]+[f(y+x-b)+f(y-x+b)+\mu] \\ &= h(x+b)k(y)+h(x-b)k(y) \\ &= [h(x+b)+h(x-b)]k(y). \end{aligned} \tag{20}$$

We may now define a function $C(x)$ by the equation

$$2h(b)C(x) = h(x+b)+h(x-b).$$

It is easily seen that $C(x)$ is an even function. Equation (20) may now be written in the form

$$k(y+x)+k(y-x) = 2C(x)k(y). \tag{21}$$

If we interchange x and y in (21), we have

$$k(x+y)+k(x-y) = 2k(x)C(y), \tag{22}$$

which is of the same form as equation (12). Since we are concerned, however, with the odd component of $k(x)$, we may change x to $-x$ and y to $-y$ in (22), and subtract the equation thus found from (22); whence we shall have

$$S(x+y)+S(x-y) = 2S(x)C(y), \tag{23}$$

which is also of form (12), but which has the advantage of oddness (see reference under (12)).

The relationships of functions $f(x)$, $g(x)$, $h(x)$, $k(x)$ to the functions $\Phi(x)$ and $\psi(x)$ may now be summarized as follows (bearing in mind the transformation by which $k(0)=1$):

CASE A: $S_h(x) \not\equiv 0$. Either

$$E(x)=\frac{\psi(x)+\psi(-x)}{2}, \qquad S_h(x)=\frac{\psi(x)-\psi(-x)}{2\alpha},$$

where α is a constant different from zero; or else

$$E(x)\equiv 1, \qquad S_h(x)=\Phi(x).$$

The even component of $h(x)$ is $\eta E(x)$, where η is a constant. In addition, we have

$$S(x)=\lambda S_h(x),$$

so that

$$\begin{aligned} h(x) &= \eta E(x)+S_h(x), \qquad \bar{k}(x)=E(x)+S(x),\\ k(x) &= k(a)\bar{k}(x-a), \qquad k(a)\neq 0,\\ f(2x) &= h(x)k(x)-g(0), \qquad g(2x)=h(x)k(-x)-f(0). \end{aligned}$$

CASE B: $S_h(x)\equiv 0$. In this case we have

$$E(x)=\frac{\psi_1(x)+\psi_1(-x)}{2},$$

and either

$$S(x)=\frac{\psi_2(x)-C(x)}{\alpha},$$

where α is a constant different from zero; or else

$$S(x)=\Phi(x) \text{ and } C(x)\equiv 1,$$

where each of the functions $\psi_1(x)$ and $\psi_2(x)$ satisfies (4) and

$$2h(b)C(x)=h(x+b)+h(x-b), \ h(b)\neq 0.$$

In this case,

$$h(x)=\eta E(x),$$

where η is a constant; and

$$\begin{aligned} &\bar{k}(x)=E(x)+S(x), \qquad k(x)=\bar{k}(a)k(x-a), \qquad k(a)\neq 0,\\ &f(2x)=h(x)k(x)-g(0), \qquad g(2x)=h(x)k(-x)-f(0). \end{aligned}$$

THE UNIVERSITY OF IOWA

FUNCTIONAL INVARIANTS, WITH A CONTINUITY OF ORDER p, OF ONE-PARAMETER FREDHOLM AND VOLTERRA TRANSFORMATION GROUPS*

BY A. D. MICHAL

1. *Object of the Paper*. In the present paper sufficient conditions are given for the invariance of functionals $f[y(\tau_0'), y'(\tau_0'), y''(\tau_0'), \cdots, y^{(p)}(\tau_0')]$ with a Volterra variation under given arbitrary Fredholm groups

$$\delta y(x) = \left[\int_0^1 H(x,s)y(s)\,ds\right]\delta a \tag{1}$$

with a kernel $H(x,s)$ of the type†

$$H(x,s) = \eta(x)\psi(s). \tag{2}$$

These sufficient conditions are given in the form of func-

---

* Presented to the Society, October 7, 1924.

† In this BULLETIN, vol. 30 (July, 1924), pp. 338–344, the writer considered analytic functionals $f[y(\tau_0'), y'(\tau_0')]$ admitting Fredholm groups (1). In the process of finding sufficient conditions for invariance, a certain amount of specialization on the kernel $H(x,s)$ was necessary. The specialization however was erroneously made more than necessary due to an error in formula (16). The reader will verify the statement that the corrected formula (16) becomes

$$f_{\iota,\lambda}(t_1, \cdots, t_\iota; t_{\iota+1}, \cdots, t_{\iota+\lambda-1}, t)$$
$$= (-1)^\lambda \varphi(t)\varphi(t_{\iota+k-1})\varphi(t_{\iota+\lambda-2})\cdots\varphi(t_{\iota+1})f_{\iota+\lambda,0}(t_{\iota+1}, t_{\iota+2}, \cdots, t_{\iota+\lambda-1}, t, t_1, \cdots, t_\iota).$$

With the aid of this corrected formula, we can show readily that the kernel $H(x,s)$ will have to be of the form $\eta(x)\psi(s)$ and hence $\varphi(t) = \eta(t)\Big/\frac{d\eta(t)}{dt}$. Consequently the $f_{\iota,\lambda}$'s need not be symmetric in all their arguments. Theorem II of the cited paper becomes true when the above changes are introduced.

In the cited paper we also need to make the following correction: substitute "It is sufficient that" for the wording "We may now apply Lemma 2 of I.D.I.V.; doing so we find" immediately preceding equation (12).

tional equations with partial functional derivatives. Of special interest are the linear functionals in which case necessary as well as sufficient conditions for invariance are given. We further demonstrate a theorem which shows the unique rôle played by a linear functional of $y(\tau)$ and $y'(\tau)$ that admits a given arbitrary *Volterra* group of transformations

$$\delta y(x) = \left[\int_0^x H(x,s)\,y(s)\,ds\right]\delta a \tag{3}$$

with

$$H(s,s) \neq 0.$$

2. *A Sufficient Condition for Invariance.* We proceed to prove the following theorem.

THEOREM I. *A sufficient condition that a functional* $f[y(\tau_0'), y'(\tau_0'), \cdots, y^{(p)}(\tau_0')]$ *with a Volterra variation and continuous functional derivatives admit a given arbitrary Fredholm group of transformations* (1) *with a kernel* $H(x,s)$ *of type* (2) ($\eta(x)$ *being assumed* to be such that all the* $\eta^{(i)}(x)/\eta(x)$*'s are continuous in the interval* 0,1) *is that it satisfy the completely integrable functional equation with partial functional derivatives*

$$f_y(t) = -\frac{\eta'(t)}{\eta(t)}f_{y'}(t) - \frac{\eta''(t)}{\eta(t)}f_{y''}(t) - \cdots - \frac{\eta^{(p)}(t)}{\eta(t)}f_{y^{(p)}}(t), \tag{4}$$

where $f_{y^{(i)}}(t)$ *represents the partial functional derivative of* $f[y, y', y'', \cdots, y^{(p)}]$ *with respect to* $y^{(i)}(\tau)$ *taken at the point* t, *and where*

$$\eta^{(i)}(t) \equiv \frac{d^i\eta(t)}{dt^i}.$$

---

* To shorten the statements of the theorems that follow in this paper we shall always assume this restriction on the $\eta(x)$ found in the Fredholm kernel (2). Moreover it is scarcely necessary to state here that we assume that the derivatives of $\eta(x)$ up to and including the pth derivative exist or else our functional equations will have no meaning.

PROOF. A necessary and sufficient condition that $f[y, y', \cdots, y^{(p)}]$ admit a given arbitrary Fredholm group (1) with a kernel $\eta(x)\psi(s)$ is that

$$\delta f[y, y', \cdots, y^{(p)}] \equiv \delta a \int_0^1 \Big[f_y(t)\eta(t)\int_0^1 \psi(s)y(s)\,ds + f_{y'}(t)\eta'(t)\int_0^1 \psi(s)y(s)\,ds + \cdots + f_{y^{(p)}}(t)\eta^{(p)}(t)\int_0^1 \psi(s)y(s)\,ds\Big]dt \equiv 0.$$

Hence it is sufficient that

$$\eta(t)f_y(t) + \eta'(t)f_{y'}(t) + \eta''(t)f_{y''}(t) + \cdots + \eta^{(p)}(t)f_{y^{(p)}}(t) = 0,$$

which can be written in the form (4).

Paul Lévy* has treated extensively the integrability conditions and the Cauchy problem for functional equations with partial functional derivatives in the case of functionals of two independent functions. Cauchy problems and integrability conditions for equations of type (4) involving functionals of p functions can be considered by an extension of Lévy's work.

We now consider the functional equation

$$(5)\quad f_{y_p}(t) = g_1(t)f_{y_1}(t) + g_2(t)f_{y_2}(t) + \cdots + g_{p-1}(t)f_{y_{p-1}}(t).$$

Assume that each of the p functional arguments $y_1(\tau), y_2(\tau), \cdots, y_p(\tau)$ is expressed in terms of two parameters λ and μ. Let δ_λ be the variation operator when λ changes and let δ_μ be the variation operator when μ changes. By an obvious extension of Lévy's work, it can be shown that a necessary and sufficient condition that the functional equation (5) be integrable† is that

---

* Cf. his book *Leçons d'Analyse Fonctionnelle*, Part II.

† Cf. Lévy's book, § 78, Part I; note at the end of Chapter I, Part II, §§ 45, Part II. Lévy assumes the existence of a solution in getting his integrability conditions. In our case, however, it is possible to get explicit solutions by a direct computation. See § 4 of this paper.

$$(6) \qquad \delta_\mu \delta_\lambda f[y_1, y_2, \cdots, y_p] \equiv \delta_\lambda \delta_\mu f[y_1, y_2, \cdots, y_p].$$

A necessary and sufficient condition* that the integrability condition (6) hold is that E_{qk} be adjoint† to $E_{kq}(k, q = 1, 2, \cdots, p)$, where E_{qk} stands for the linear functional of δy_k found in the expression for $\delta f_{y_q}(t)$.

If for $y_p(\tau) \equiv y_{0p}(\tau), f[y_1, y_2, \cdots, y_p]$ is a given arbitrary functional of $y_1, y_2, \cdots, y_{p-1}$, then the $f_{y_i}(t)$'s $(i = 1, 2, \cdots, p-1)$ are known for $y_p(\tau) \equiv y_{0p}(\tau)$ and hence the linear functionals $E_{kq}(k, q = 1, 2, \cdots, p-1)$ are known and are furthermore respectively adjoint to the E_{qk}'s. To compute the E_{pi}'s and the E_{ip}'s $(i = 1, 2, \cdots, p)$, we take the variation of both sides of (5). Doing so, we get

$$(7) \quad \delta f_{y_p}(t) = g_1(t)\delta f_{y_1}(t) + g_2(t)\delta f_{y_2}(t) + \cdots + g_{p-1}(t)\delta f_{y_{p-1}}(t),$$

an identity in $\delta y_1, \delta y_2, \cdots, \delta y_p$.

Consequently on equating terms in δy_i, we get

$$(8) \quad \begin{aligned} E_{pi}[\delta y_i(\tau)/t] &= g_1(t)E_{1i}[\delta y_i(\tau)/t] + g_2(t)E_{2i}[\delta y_i(\tau)/t] + \\ &\cdots + g_{p-1}(t)E_{p-1,i}[\delta y_i(\tau)/t], \qquad (i = 1, 2, \cdots, p). \end{aligned}$$

From (8) we see that the E_{pi}'s $(i = 1, 2, \cdots, p-1)$ are given in terms of known linear functionals. Hence, imposing the condition that E_{pi} be adjoint to $E_{ip} (i = 1, 2, \cdots, p-1)$, we get from (8)

$$(9) \quad \begin{aligned} E_{ip}[\delta y_i(\tau)/t] &= E_{i1}[g_1 \delta y_i/t] + E_{i2}[g_2 \delta y_i/t] + \\ &\cdots + E_{i,p-1}[g_{p-1}\delta y_i/t], \quad (i = 1, 2, \cdots, p-1). \end{aligned}$$

Finally in order that the integrability conditions (6) be surely satisfied we must have $E_{pp}[\delta y_p(\tau)/t]$ self-adjoint. On substituting the expression for E_{ip}, as given in (9), in the expression for E_{pp} in (8), we can see with a little computation that E_{pp} is automatically self-adjoint. Hence the functional equation (5) is completely integrable. Since p can be any positive integer, it follows that (4) is also a completely integrable equation. This completes the proof of our theorem.

---

* This condition can readily be found by computation.

† For a definition of adjointness see Lévy's book, § 73, Part I.

The following corollaries are immediate.

COROLLARY 1. *There is one and only one solution* $f[y, y', y'', \cdots, y^{(p)}]$ *of* (4), *regarded momentarily as a functional of* $p+1$ *independent arguments, such that for a given initial functional value of the first argument*

$$y(\tau) \equiv y_0(\tau),$$

$f[y, y', y'', \cdots, y^{(p)}]$ *has arbitrary values as a functional of its last* p *arguments* $y'(\tau), y''(\tau), \cdots, y^{(p)}(\tau)$.

COROLLARY 2. *There always exist functionals* $f[y(\tau_0'), y'(\tau_0'), \cdots, y^{(p)}(\tau_0')]$, *with a Volterra variation, admitting a given arbitrary group of Fredholm transformations* (1) *with a kernel of type* (2).

3. *Linear Functionals Continuous of Order* p; *Necessary and Sufficient Conditions.** Of special interest are the linear functionals with a Volterra variation; i. e., functionals of the form

$$f_0 + \int_0^1 f_1(t) y(t)\,dt + \int_0^1 f_2(t) y'(t)\,dt + \cdots + \int_0^1 f_{p+1}(t) y^{(p)}(t)\,dt, \tag{10}$$

where p denotes any positive integer or zero; and all the $f_i(t)$'s are assumed continuous functions of t in the interval (0, 1). We shall demonstrate the following theorem.

THEOREM II. *A necessary and sufficient condition that a linear functional* (10) *of continuity order* p *admit a given arbitrary group of Fredholm transformations* (1) *with a kernel of type* (2)† *is that it satisfy the completely integrable functional equation*

$$f_y(t) = -\frac{\eta'(t)}{\eta(t)} f_{y'}(t) - \frac{\eta''(t)}{\eta(t)} f_{y''}(t) - \cdots - \frac{\eta^{(p)}(t)}{\eta(t)} f_{y^{(p)}}(t) + \frac{\omega(t)}{\eta(t)}, \tag{11}$$

---

* Cf. G. C. Evans, *The Cambridge Colloquium*, 1916, Lecture III, § 3.

† We further restrict ourselves in this theorem to kernels of type (2) with $\eta(x) \neq 0$ in the interval (0, 1).

where $\omega(t)$ *is a given arbitrary continuous function satisfying the equation*

$$\int_0^1 \omega(t)\,dt = 0. \tag{12}$$

PROOF. Evidently a necessary and sufficient condition for invariance is that

$$\left[\int_0^1 y(s)\psi(s)\,ds\right]\left[\int_0^1 \{f_y(t)\eta(t)+f_{y'}(t)\eta'(t)+ \cdots + f_{y^{(p)}}(t)\eta^{(p)}(t)\}\,dt\right] \equiv 0$$

in y. Hence it follows, since $\psi(s) \not\equiv 0$* and since all the functional derivatives of $f[y, y', y'', \cdots, y^{(p)}]$ are point functions, that

$$\int_0^1 \{f_y(t)\eta(t)+f_{y'}(t)\eta'(t)+\cdots+f_{y^{(p)}}(t)\eta^{(p)}(t)\}\,dt = 0,$$

a condition which evidently can be expressed in the form of the functional equation (11)†.

The following important theorem is immediate.

THEOREM III. *Given any continuous function* $\omega(t)$ *such that* (12) *holds, and given the initial conditions in the Cauchy problem for equation* (11), *then there is one and only one linear functional* (10) *of continuity order* p *admitting a given arbitrary Fredholm group of transformations* (1) *with a kernel of type* (2) *and taking on the given initial conditions.*

We remark here that since $\psi(s)$ does not enter into the final conditions for invariance for the linear as well as for the non-linear functional, there is an infinitude of Fredholm groups (1) with a kernel of type (2) leaving one functional, of the types considered, invariant.

---

* For if $\psi(s) \equiv 0$, then our transformation degenerates into the identical one and hence there is no problem.

† Equation (11) can readily be shown to be completely integrable by a slight modification of the reasoning involved in treating equation (4).

4. *Explicit Expressions for the Functional Invariants.* To get explicit expressions for the functional invariants, we can assume $f[y, y', y'', \cdots, y^{(p)}]$ to be expansible in a series of functionals. For instance we can assume $f[y, y', y'', \cdots, y^{(p)}]$ to be an analytic functional of its $p+1$ functional arguments $y(\tau), y'(\tau), y''(\tau), \cdots, y^{(p)}(\tau)$ and then by a reasoning similar to that found in one* of the writer's papers, recurrence formulas can be computed yielding the desired functional invariants in terms of the given initial conditions. For the sake of brevity, we here give only a simple illustration.

Consider the problem of finding the functional $f[y(\tau_0'), y'(\tau_0'), y''(\tau_0')]$ of the form

$$
\begin{aligned}
&\int_0^1 f_{100}(t)y(t)\,dt + \int_0^1 f_{010}(t)y'(t)\,dt + \int_0^1 f_{001}(t)y''(t)\,dt \\
&\qquad + \frac{1}{2!}\Bigg[\int_0^1\int_0^1 f_{200}(t_1, t_2)y(t_1)y(t_2)\,dt_1\,dt_2 \\
&\qquad + \int_0^1\int_0^1 f_{020}(t_1, t_2)y'(t_1)y'(t_2)\,dt_1\,dt_2 \\
(13)\qquad &\qquad + \int_0^1\int_0^1 f_{002}(t_1, t_2)y''(t_1)y''(t_2)\,dt_1\,dt_2 \\
&\qquad + 2\int_0^1\int_0^1 f_{110}(t_1, t_2)y(t_1)y'(t_2)\,dt_1\,dt_2 \\
&\qquad + 2\int_0^1\int_0^1 f_{101}(t_1, t_2)y(t_1)y''(t_2)\,dt_1\,dt_2 \\
&\qquad + 2\int_0^1\int_0^1 f_{011}(t_1, t_2)y'(t_1)y''(t_2)\,dt_1\,dt_2\Bigg]
\end{aligned}
$$

admitting a given arbitrary Fredholm group of transformations (1) with a kernel of type (2).

We can, without any loss of generality, assume $f_{200}(t_1, t_2)$, $f_{020}(t_1, t_2)$ and $f_{002}(t_1, t_2)$ to be symmetric in t_1 and t_2. We further assume that all the f_{ikl}'s are continuous functions of all their arguments.

---

* *Integro-differential expressions invariant under Volterra's group of transformations,* ANNALS OF MATHEMATICS, vol. 26, p. 181 (March, 1925).

From our previous theory we see that for invariance it is sufficient that $f[y(\tau_0'), y'(\tau_0'), y''(\tau_0')]$ of type (13) satisfy the completely integrable functional equation

$$f_y(t) = -\frac{\eta'(t)}{\eta(t)} f_{y'}(t) - \frac{\eta''(t)}{\eta(t)} f_{y''}(t). \tag{14}$$

Assume the initial conditions for this equation to be given for

$$y(\tau) \equiv 0$$

in the form

$$\begin{aligned} f[0, y'(\tau_0'), y''(\tau_0')] \equiv & \int_0^1 f_{010}(t) y'(t)\,dt + \int_0^1 f_{001}(t) y''(t)\,dt \\ & + \frac{1}{2!}\Big[\int_0^1\int_0^1 f_{020}(t_1, t_2) y'(t_1) y'(t_2)\,dt_1\,dt_2 \\ & + \int_0^1\int_0^1 f_{002}(t_1, t_2) y''(t_1) y''(t_2)\,dt_1\,dt_2 \\ & + 2\int_0^1\int_0^1 f_{011}(t_1, t_2) y'(t_1) y''(t_2)\,dt_1\,dt_2\Big]. \end{aligned} \tag{15}$$

Calculating the partial functional derivatives of $f[y(\tau_0'), y'(\tau_0'), y''(\tau_0')]$ and substituting them in the functional equation (14) we get, on equating coefficients of similar terms in y, y', and y'', the unique determination of the four unknown f_{ikl}'s in terms of the five initially known f_{ikl}'s. Thus we get after computation

$$\begin{aligned} f_{100}(t) &= -\frac{\eta'(t)}{\eta(t)} f_{010}(t) - \frac{\eta''(t)}{\eta(t)} f_{001}(t), \\ f_{110}(t_1, t_2) &= -\frac{\eta'(t_1)}{\eta(t_1)} f_{020}(t_1, t_2) - \frac{\eta''(t_1)}{\eta(t_1)} f_{011}(t_2, t_1), \\ f_{101}(t_1, t_2) &= -\frac{\eta'(t_1)}{\eta(t_1)} f_{011}(t_1, t_2) - \frac{\eta''(t_1)}{\eta(t_1)} f_{002}(t_1, t_2), \\ f_{200}(t_1, t_2) &= \frac{\eta'(t_1)}{\eta(t_1)} \frac{\eta'(t_2)}{\eta(t_2)} f_{020}(t_2, t_1) \\ & \quad + \frac{\eta'(t_1)}{\eta(t_1)} \frac{\eta''(t_2)}{\eta(t_2)} f_{011}(t_1, t_2) + \frac{\eta'(t_2)}{\eta(t_2)} \frac{\eta''(t_1)}{\eta(t_1)} f_{011}(t_2, t_1) \\ & \quad + \frac{\eta''(t_1)}{\eta(t_1)} \frac{\eta''(t_2)}{\eta(t_2)} f_{002}(t_2, t_1). \end{aligned}$$

5. *Functional Invariants of Volterra Groups.* It is of interest at this point to give certain results as regards linear functionals of continuity order p of type (10) which admit a given arbitrary one-parameter Volterra group of infinitesimal transformations.

The author has treated extensively in another paper* the problem of finding functionals $f[y(r_0'), y'(r_0')]$ which admit a given arbitrary Volterra group of transformations defined by (3).

In particular it was shown that a necessary and sufficient condition that a linear functional $f[y(r_0'), y'(r_0')]$ of the type

$$(16) \qquad f_0 + \int_0^1 f_1(t)y(t)\,dt + \int_0^1 f_2(t)y'(t)\,dt$$

($f_1(t)$ and $f_2(t)$ assumed continuous in $(0, 1)$) admit a given arbitrary Volterra group of transformations (3) with $H(s, s) \neq 0$ is that it satisfy the completely integrable functional equation

$$(17) \qquad f_{y'}(s) = \int_0^1 L(s, r) f_y(r),$$

where

$$\begin{aligned} L(s, r) &= \int_s^r k(s, r') \frac{H(r, r')}{H(r', r')}\,dr' - \frac{H(r, s)}{H(s, s)}, & r \geqq s, \\ &= 0, & r < s, \end{aligned}$$

and where $k(s, r')$ is the reciprocal kernel of

$$-\frac{1}{H(s, s)} \frac{\partial H(r', s)}{\partial r'}.$$

It was also shown that there are no linear functionals

$$f_0 + \int_0^1 f(t) y(t)\,dt$$

($f(t)$ assumed continuous in $(0, 1)$) other than constants, which admit a given arbitrary Volterra group of transformations (3) with $H(s, s) \neq 0$.

With these facts in mind we proceed to prove the following theorem.

---

* *Integro-differential expressions,* loc. cit.

THEOREM IV. *The linear functionals* (16) *are the only linear functionals* (10) *other than constants that can admit a given arbitrary Volterra group of transformations** (3) *with* $H(s, s) \neq 0$.

The proof of this theorem can be made to depend on the following lemma.

LEMMA. *Let* $F_0(x), F_1(x), \cdots, F_k(x)$ *be continuous functions in the interval* $(0, 1)$. *If*

$$(18) \quad \int_0^1 [F_0(x)y(x) + F_1(x)y'(x) + \cdots + F_k(x)y^{(k)}(x)]dx = 0$$

for all possible forms of the function $y(x)$, *having a continuous kth derivative* $y^{(k)}(x)$ *in* $(0, 1)$, *then it is necessary*† *that* $F_k'(x)$ *exist and be continuous in* $(0, 1)$ *and that* $F_k(0) = F_k(1) = 0$.

PROOF. Assume

$$(19) \quad \begin{aligned} & y(0) = y(1) = y'(0) = y'(1) = \\ & \qquad \cdots = y^{(k-1)}(0) = y^{(k-1)}(1) = 0. \end{aligned}$$

Then making use of a well known formula‡ and by an evident use of Dirichlet's formula, we get

$$(20) \quad \begin{aligned} \int_0^1 y^{(k)}(t)\Bigg[F_k(t) + \int_t^1 \Big\{ & F_{k-1}(x) + F_{k-2}(x)(x-t) + \\ & \cdots + F_1(x)\frac{(x-t)^{k-2}}{(k-2)!} + F_0(x)\frac{(x-t)^{k-1}}{(k-1)!}\Big\} dx\Bigg] dt = 0 \end{aligned}$$

for all y of our hypotheses satisfying (19). It follows§ therefore that

---

* We note, without any further statement, that whenever we shall need the pth extended Volterra group of transformations, we shall assume that all the derivatives of $H(x, s)$ and $H(s, s)$ that are found in this extended transformation exist and are continuous.

† The reader can, without difficulty, push the argument until necessary and sufficient conditions on all the $F_i(t)$'s are found by successive applications of our lemma and by successive valid integrations by parts; but the lemma as stated is sufficient for our purpose.

‡ Cf. Hadamard, *Calcul des Variations*, vol. I, p. 201.

§ Cf. Hadamard, loc. cit.

$$\begin{aligned} F_k(t) = -\int_t^1 \Big\{ & F_{k-1}(x) + F_{k-2}(x)\,(x-t) + \\ (21)\qquad & \cdots + F_1(x)\frac{(x-t)^{k-2}}{(k-2)!} + F_0(x)\frac{(x-t)^{k-1}}{(k-1)!}\Big\}\,dx \\ & + l_0 + l_1(1-t) + l_2(1-t)^2 + \cdots + l_{k-1}(1-t)^{k-1}, \end{aligned}$$

where the l_i's are arbitrary constants. The derivative of the right hand side of (21) exists and is continuous in $(0,1)$ and hence the derivative $F_k'(t)$ of $F_k(t)$ exists and is continuous in $(0,1)$. Dropping now the restrictions (19) and calculating $F_k'(t)$ from (21) we proceed as follows. We substitute the expression for $F_k(t)$ in (18) and perform the now valid integration by parts. Carrying out $k-1$ valid successive integrations by parts starting with the term involving $y^{(k-1)}(t)$ and ending with the resultant term involving $y'(t)$, we get, on putting

$$(22)\qquad \begin{aligned} y(0) = y(1) = y'(0) = y'(1) = & \\ \cdots = y^{(k-2)}(0) = y^{(k-2)}(1) = 0&, \end{aligned}$$

$$(23)\qquad [F_k(t)y^{(k-1)}(t)]_0^1 = 0$$

for all y's satisfying (22). In order that (23) may hold it is necessary and sufficient that $F_k(0) = F_k(1) = 0$. This completes the proof of our lemma.

A necessary and sufficient condition that a linear functional (10) admit a given arbitrary Volterra group of transformations (3) is that

$$\begin{aligned} \int_0^1 [& y(t)h_1(t) + y'(t)h_2(t) + \\ & \cdots + y^{(p-2)}(t)h_{p-1}(t) + y^{(p-1)}(t)H(t,t)f_{p+1}(t)]\,dt \equiv 0 \end{aligned}$$

in y, where $h_i(t)$ $(i = 1, 2, \cdots, p-1)$ are easily calculated expressions whose explicit form is not necessary to the argument. On applying the lemma just proved we conclude that it is necessary that the derivative of $H(t,t)f_{p+1}(t)$ exist and be continuous and that

$$H(1,1)f_{p+1}(1) = H(0,0)f_{p+1}(0) = 0.$$

Since by hypothesis $H(t,t) \neq 0$, we conclude that the derivative of $f_{p+1}(t)$ exists and is continuous and that $f_{p+1}(1) = f_{p+1}(0) = 0$. Hence it is necessary that the linear functional of continuity order p reduce, by an integration by parts, to a linear functional of continuity order $p-1$. The lemma applied again to this new functional reduces it to a linear functional of continuity order $p-2$. Applying the lemma successively in such a manner, we finally get, as a necessary condition for invariance, the result that the original linear functional of continuity order p has to reduce at least to a linear functional of continuity order one. This result coupled with the existence theorems cited in the beginning of § 5, establishes our theorem.

THE RICE INSTITUTE

ON THE DISTRIBUTION OF QUADRATIC AND HIGHER RESIDUES*

BY H. S VANDIVER

1. *Introduction.* In the present paper theorems will be obtained regarding the distribution of quadratic and higher residues. Special cases of the theorems yield results concerning the class number of quadratic forms of determinant $(-d)$ where $d \equiv 3 \pmod 4$.

2. *Conjugate Sets of Residues.* In a previous article† the writer considered the notion of *conjugate set* in a *finite algebra*. Applied to the finite algebra represented by the residue classes of a rational integer as modulus, we may define a *conjugate set of residues* of the modulus m to be a set

$$a_1, a_2, \cdots, a_t \tag{1}$$

---

* Presented to the Society, October 28, 1916.

† ANNALS OF MATHEMATICS, (2), vol. 18 (1917), p. 106.

which has the property that

$$na_1,\ na_2,\ \cdots,\ na_i$$

may be expressed, modulo m, as a permutation of the original set, n being some rational integer $\not\equiv 1 \pmod{m}$. In this case n is called a *multiplier* of the set. Let r, s, and $(r+s)$ all be multipliers of the set (1), modulo $m=p$, a prime, and $0<a_c<p$, $(c=1, 2, \cdots, i)$. Consider the expression

$$\sum_h \frac{|rh|+|sh|-|(r+s)h|}{p} h^{kp},$$

where h ranges over the elements of (1), and $|x|$ is the least positive residue of x, modulo p. Then

$$rh = |rh| + pt,$$

where t is an integer, and

$$(2) \qquad \begin{cases} (rh)^{pk} \equiv |rh|^{pk} & \pmod{p^2}, \\ h^{pk} \equiv r^{p(p-1)-pk}|rh|^{pk} & \pmod{p^2}. \end{cases}$$

Similarly

$$(3) \qquad h^{pk} \equiv s^{p(p-1)-pk}|sh|^{pk} \pmod{p^2},$$

$$(4) \qquad h^{pk} \equiv (r+s)^{p(p-1)-pk}|(r+s)h|^{pk} \pmod{p^2}.$$

Since r is a multiplier of (1), we have $\sum_h |rh| = \sum_{c=1}^{i} a_c$; and similarly, for $\sum_h |sh|$ and $\sum_h |(r+s)h|$. Hence

$$\sum_h (|rh|h^{pk} + |sh|h^{pk} - |(r+s)h|^{pk}) \equiv (r^e + s^e - (r+s)^e) \sum_{c=1}^{i} a_c^{pk+1} \pmod{p^2},$$

or

$$\sum_h \frac{(|rh|+|sh|-|(r+s)h|)h^{pk}}{p} \equiv \frac{(r^e+s^e-(r+s)^e)\sum_{c=1}^{i} a_c^{pk+1}}{p} \pmod{p},$$

where $e=p(p-1)-pk$. The coefficient of h^{pk} on the left is 0 or 1 according as $|rh|+|sh|$ is less than or greater than p. Hence we have proved the following theorem.

THEOREM I. *If $a_1, a_2, \cdots, a_i$ is a conjugate set modulo p, and $|x|$ is the least positive residue of x modulo p, then*

$$(5) \qquad \sum_{h_1} h_1^k \equiv \frac{(r^e + s^e - (r+s)^e)\sum_{c=1}^{i} a_c^{pk+1}}{p} \pmod{p},$$

where h_1 ranges over the a's which satisfy $|rh_1| + |sh_1| > p$ and $e = p(p-1) - pk$.

3. *The Number of j–ic Residues.* In Theorem I, let $a_1, a_2, \cdots, a_i$ be the i roots of $x^i \equiv 1 \pmod{p}$, $p = ij + 1$; and suppose that r, s, and $(r+s)$ are each congruent, mod p, to an integer included among the a's. Set $k = i$. Then (2) gives

$$H \equiv \frac{\sum_{c=1}^{i} a^c}{p} \pmod{p},$$

where H is the number of j–ic residues included in the integers $h < p$ such that $|rh| + |sh| > p$. Each side of this congruence is a positive integer less than p; consequently

$$H = \frac{\sum_{c=1}^{i} a_c}{p};$$

hence we have the following theorem.

THEOREM II. *If $p = ij + 1$ and $a_1, a_2, \cdots, a_i$, $(0 < a_c < p$, $c = 1, 2, \cdots, i)$, are the incongruent roots of $x^i \equiv 1 \pmod{p}$, r, s, and $(r+s)$ being j–ic residues of p, then*

$$H = \frac{\sum_{c=1}^{i} a_c}{p},$$

where H is the number of j–ic residues included in the integers $h < p$ which satisfy $|rh| + |sh| > p$.

This result was previously known for the case $r = 1$, $s = 1$, $j = 2$.

4. *Generalization for Quadratic Residues.* For quadratic residues we shall obtain a more general theorem. Let the set (1) range over the distinct positive quadratic residues of p which are less than p, and suppose now that r and s are not necessarily quadratic residues of p. Then if $(r/p)=1$,

$$\sum_h |rh| = \sum a.$$

If however $(r/p) = -1$, then

$$\sum_h |rh| = \sum b = p(p-1)/2 - \sum a,$$

where b ranges over the distinct quadratic non-residues $0<b<p$. Each of these relations is included in the statement

$$\sum_h |rh| = \left(1-\left(\frac{r}{p}\right)\right)p\mu + \left(\frac{r}{p}\right)\sum a,$$

where $\mu = (p-1)/4$, (not necessarily an integer), and r is any integer prime to p. If we take s and $(s+r)$ in lieu of this relation we get two other equations, which, with the original, give

$$\sum_h \frac{|rh|+|sh|-|(r+s)h|}{p} = (1-R)\mu + \frac{R\sum a}{p},$$

$$R = \left(\frac{r}{p}\right) + \left(\frac{s}{p}\right) - \left(\frac{r+s}{p}\right).$$

A term in the expression on the left of (6) is one or zero according as $|rh|+|sh|$ greater than or less than zero. We may then state the following theorem.

THEOREM III. *If p is an odd prime and K is the number of quadratic residues h included in the set* 1, 2, $\cdots$, $p-1$, *and satisfying* $|rh|+|sh|>p$, *then*

$$K = (1-R)\mu + R\frac{\sum a}{p}$$

where $\mu = (p-1)/4$, *and*

$$R = \left(\frac{r}{p}\right) + \left(\frac{s}{p}\right) - \left(\frac{r+s}{p}\right),$$

r, s, and $(r+s)$ being prime to p, with the a's ranging over the positive quadratic residues of p which are less than p.

For the case $s=1$, this theorem was previously stated in another form and without proof by the writer*. For $s=r=1$, the relation reduces to an equality which is well known.

We shall now show that the number of distinct integers h in the set $1, 2, \cdots, p-1$, which satisfy $|rh|+|sh|>p$ are $(p-1)/2$ in number. If $|rh|+|sh|<p$ then

$$rh=i_1p+\varepsilon_1,\ |sh|=i_2p+\varepsilon_2,$$

ε_1 and ε_2 being each less than p. Also

$$(p-h)r=pr-rh=p(r-i_1)-\varepsilon_1,$$

and

$$|(p-h)r|=p-\varepsilon_1.$$

Similarly,

$$|(p-h)s|=p-\varepsilon_2;$$

hence

$$|(p-h)r|+|(p-h)s|=2p-\varepsilon_1-\varepsilon_2.$$

Also, by definition, $|rh|=\varepsilon_1$, $|sh|=\varepsilon_2$ and $\varepsilon_1+\varepsilon_2<p$, therefore $2p-\varepsilon_1-\varepsilon_2>p$. Hence if h satisfies $|rh|+|sh|>p$, then $p-h$ will not. This shows that there are $(p-1)/2$ distinct values h.

5. *Conclusion.* From Theorem III we see that K is immediately determined if $\sum a/p$ is known, since, for $p=4n+3$,

$$2\left(1-\left(\frac{2}{p}\right)\frac{1}{2}\right)\frac{\sum b-\sum a}{p}=h(-p)$$

is the expression for the number of classes of properly primitive quadratic forms of determinant $(-p)$. Since also $\sum b+\sum a=(p^2-p)/2$, it is possible to give a relation between h and K. In particular we obtain a variety of expressions for h.

THE UNIVERSITY OF TEXAS

---

* This BULLETIN, vol. 23 (1916), p. 113.

COOLIDGE ON COMPLEX GEOMETRY

The Geometry of the Complex Domain. By Julian Lowell Coolidge. New York, Oxford University Press, American Branch, 1924. 242 pp.

The various geometries of a complex space are, with certain definite exceptions, essentially the same as the corresponding geometries of a real space, provided that one agrees, tacitly or wittingly, to admit to consideration only those configurations in complex space which are the direct generalizations of configurations in real space in that they arise from these by the replacement of real by complex numbers. This agreement restricts, for example, the discussion of manifolds of complex points to those manifolds depending on a fixed number of complex parameters. But a system of points depending on n complex parameters is but a very special case of a system depending on the equivalent number, $2n$, of real parameters, and its properties are in no way indicative of those of the general $2n$-parameter system. In other words, when the scope of complex geometry is widened to cover all possibilities, it takes on an entirely different aspect from that of real geometry.

Though there are recent books on complex geometry which pursue the subject along the narrow path outlined by the geometry of reals, the one now under review is the first to follow consistently the wider and, for the progress of mathematics, the more important, point of view. It aims to bring the reader abreast of the times in the advances, from this point of view, of the last half-century. The pioneers in these researches have been Segre and Study, and to them the book is informally dedicated. Among their disciples are to be mentioned Autonne, Benedetti, Coolidge, Fubini, Loewy, Sforza, and J. W. Young.

Closely connected with, if not strictly speaking a part of, complex geometry is the classical problem of representing complex points by real elements. An historical and critical account of the many attempts, ancient and modern, successful and unsuccessful, to solve this problem forms the second integral part of the book. The writers here are legion; we content ourselves with the names of the more important: Wallis, Wessel, Argand, Gauss, Poncelet, von Staudt, Laguerre, Marie, Klein, Segre, and Study.

The material has been well digested and excellently ordered. The complex line (Chapters I, II), the complex plane (Chapters III–VI), and complex three-dimensional space (Chapter VII) are taken up in turn. In each case, the real representations of complex points are first considered and then later put to work in throwing light on the complex geometry itself. The author writes in his characteristic, red-blooded style; his historical accounts and criticisms are, in particular, entertaining and stimulating. He has at various stages, notably in connection with the differential geometry of manifolds of points, contributed original results.

The developments of the wider aspects of complex geometry pertain largely to manifolds of points depending on a fixed number of real parameters. The study of algebraic manifolds and their projective properties has received the lion's share of attention and forms a dominant feature of the present book. The general analytic manifolds and their differential properties have been discussed only recently, by the author himself.

If $x_1, \cdots, x_n, x_{n+1}$ are complex variables and $\bar{x}_1, \cdots, \bar{x}_n, \bar{x}_{n+1}$ are their conjugates, a polynomial which is homogeneous and of degree m in each set of variables and has the property that it is a constant multiple of the conjugate-complex polynomial, is called a hyperalgebraic form. Using symbolic notation, we can write: $(ax)^m(\bar{a}\bar{x})^m$, where $(ax) = a_1x_1 + \cdots + a_nx_n + a_{n+1}x_{n+1}$. When the order, $2m$, of the form is equal to two, it reduces to the better known Hermitian form, $(ax)(\bar{a}\bar{x})$.

If we think of $x_1, \cdots, x_n, x_{n+1}$ as homogeneous coordinates of a point in a complex space of n dimensions, and set our hyperalgebraic form equal to zero, the locus of the equation obtained, if it has any locus other than singular points, is an algebraic manifold of the type just mentioned. The number of real parameters on which it depends is $2n-1$, one less than the total number of real degrees of freedom of a point in the space in question. Manifolds coming under this head are the predominant algebraic manifolds in their respective spaces; among them, the simplest and most interesting are those represented by Hermitian forms.

With this orientation concerning the general scope of the book, we can proceed with greater appreciation to a detailed discussion of its contents.

The first chapter deals with the attempts to represent by real elements the complex points of a line. Outstanding is, of course, the classical representation of analysis, first discovered by Wessel and later rediscovered by Gauss and Argand. It is interesting to note that it was Gauss who first introduced the representation by means of points; Wessel and Argand contented themselves with vectors.

The projective geometry of the complex line is the image of the real geometry of inversion in the Gauss plane, and we can best outline the content of the second chapter from this point of view. To a circle or a straight line in the Gauss plane corresponds on the complex line a system of points depending on one real parameter, known as a chain. A chain is characterized, analytically, by the fact that it can be represented by a Hermitian form, and, geometrically, by the properties that the cross ratios of any four of its points are real and that it contains the point making with any three of its points any given real cross ratio. The direct circular transformations of the Gauss plane correspond to the collineations of the line, whereas the indirect circular transformations, when interpreted on the line, are called anti-collineations. Collineations preserve a given cross ratio, anti-collineations change it to the conjugate-complex value; both carry a chain into a chain, and, taken together, they are the only continuous one-to-one transformations of the complex line which preserve harmonic separation.

Among the many further developments of this chapter we mention only the study of algebraic threads. A thread is a system of points depending on one real parameter. An algebraic thread can be represented by a hyperalgebraic form. It corresponds in the Gauss plane to an algebraic curve, of order equal to that of the form. The same order is given to the thread and a geometrical significance is found for it. Class is also defined for the thread, by the application of a transformation on the line corresponding to the polar reciprocation in the circle $x^2+y^2+1=0$ in the plane.

In the projective geometry of the complex plane (Chapter V), we have, besides the chain of points, its dual, the chain of lines; the two are projectively equivalent, respectively, to the range of real points and the pencil of real lines. The projective equivalents of the real points (lines) of the plane play also fundamental roles; these are manifolds of points (lines) depending on two real parameters, known as chain congruences of points (lines). As in the previous case, we have here both collineations and anti-collineations.

Just as ordinary polar systems lead to conics, the anti-polarities yield the so-called hyperconics. These are represented by Hermitian forms. They consist, then, not of ∞^2 but of ∞^3 points. On the other hand, there are no curves which are entirely contained in one of them. A straight line meets one either in a chain of points, a single point, or nowhere; the case of intersection in a single point occurs only when the line contains its pole in the given anti-polarity, and so such a line is called a tangent. Discussion of further developments, of the theory of linear systems of hyperconics, and of the work on hypercurves, three-parameter systems of points represented by hyperalgebraic forms, we must forego.

The geometry of the subgroup of the group of collineations which possesses the Hermitian form $(x\bar{x})$ as a relative invariant is known as a Hermitian metric of elliptic type. This metric and the parabolic metric resulting from it in the usual way are developed in detail. In the elliptic case, to which we confine ourselves, the governing group depends on three complex and three real parameters. Distance and angle are always real. A geodesic thread turns out to be a particular type of chain, characterized by the fact that its points are, in pairs, conjugate with respect to the fundamental form. The Hermitian trigonometry of a triangle is found to be identical with that of ordinary elliptic trigonometry only in case the altitudes are concurrent. In the discussion of the metric properties of hyperconics, it is discovered that the general hyperconic has a whole chain of foci and a chain of corresponding directrices, and that the ratio of the sines of the distances of a point on the hyperconic to a focus and to the corresponding directrix is not merely constant, but the same for all foci!

A representation of the complex points of the plane which is peculiarly applicable to the projective and metric theories just reviewed, is that of Segre. Thereby, each point is represented by a real point of a four-dimensional variety of the sixth order in a space of eight dimensions. Through each point of this S_4^6 pass two conjugate imaginary planes of the S_4^6. The collineations and anti-collineations of the plane are represented in S_8 by real collineations, which, in the one case, permute the planes of each of these systems and, in the other, interchange the two systems. A straight line is represented by a quadric surface, a chain by a conic, a chain congruence by a surface of Veronese (Cayley), and a hyperconic by the intersection of the S_4^6 by a hyperplane. The Hermitian metrics appear on S_4^6 as classical non-euclidean metrics!

Chapters III and IV are devoted to a review of the various methods of representing the complex points on a curve and in a plane, respectively. Aside from the representation of Segre, just described, and that of Klein and Study, to which, for lack of space, we can give but passing mention, the most important of these methods are those of von Staudt, Marie, and Laguerre. According to von Staudt, an imaginary point is represented by the elliptic involution on the real line through the imaginary point which has as its double points the imaginary point and its conjugate, or, more accurately, by this involution provided with a sense. The Laguerre representation consists of the two real points, taken in a specified order, in which the isotropic lines through the given imaginary point intersect those through the conjugate-imaginary point. The Marie representation is also an ordered pair of points, lying on the real line through the given imaginary point. These three representations of an imaginary point, though apparently essentially different, are closely related. The points of the Laguerre representation, when rotated through a right angle about the mid-point of their segment, become those of the Marie representation; these are, in turn, a pair of points in the von Staudt involution, in fact, that pair whose distance apart is least.

The Laguerre and Marie representations of a curve have been exhaustively treated by Study. They both consist, in general, of real transformations of the plane, inversely conformal transformations in the one case, and special directly equi-areal transformations in the other. These results are developed in Chapter VI, on the differential geometry of the plane. The remainder of the chapter consists of the author's recently published work on the differential geometry of two- and three-parameter manifolds of points.

The geometry of three-dimensional complex space is considered in Chapter VII. The projective theory follows, generally speaking, the lines of that in the plane. The representation of Marie still consists of an ordered pair of points, when applied to a curve, it yields two (translation) surfaces corresponding by an elliptic parallel map which is equi-areal and of invariant -1.* The Laguerre representation is not so simple, consisting now of an oriented circle; the results thus far obtained concerning it are interesting and there is still room for further labors.

An interesting feature of this chapter is the geometry of the minimal plane (Beck, C. L. E. Moore). The chapter closes with some previously unpublished work of the author dealing with analytic manifolds depending on n real parameters, $n = 2, 3, 4, 5$, and establishing in particular the conditions that such a manifold contain one or more curves or surfaces.

A book on complex geometry can hardly be considered complete unless it establishes a logical foundation for the subject. The simplest foundation is an analytical foundation, based on the complex number system, and such a foundation is inherent in the opening chapters of the book. In the last chapter, the author carries through a more adventurous method, by means of postulates. A system of postulates for the real projective domain, that of his book on Non-Euclidean Geometry, is first developed and projective coordinates for real elements introduced. Imaginary elements are then defined by means of the elliptic involutions of von Staudt and finally acquire coordinates through the medium of von Staudt throws.

The reviewer has but one serious quarrel with the author. To be sure, he would have been happier if the author had followed Study in choosing the word membrane instead of the over-worked word congruence to designate a system of points depending on two real parameters. But this is a matter of minor importance. What caused much greater mental anguish was the author's indiscriminate use of the words complex and imaginary. He seems to have made no conscious attempt to distinguish between the two and repeatedly uses complex as the opposite of real. In a book on complex geometry this is especially to be regretted.

---

* Cf. Graustein, *Real representations of complex curves,* ANNALS OF MATHEMATICS, (2), vol. 26 (1924), p. 143. The previous description by the reviewer, reproduced by Coolidge, has the disadvantage that it employs imaginary curves (the generators) on the representing surfaces.

The book is marred by many, very many, typographical errors. Those in the text itself are not seriously disturbing; in fact, some of them, as, for example, "any value linearly descendent on two given values" on p. 108, and "two intersectional minimal lines" on p. 190, are decidedly the opposite. But those in the formulas, though they may be only irritating to the mature reader, are likely to prove a stumbling block for the budding mathematician.

Aside from these matters, the book is full of solid, stimulating mathematics. Moreover, it is particularly welcome in that it brings up to date a field of geometry which is comparatively new and by no means exhausted. May it prove the inspiration and basis of departure for fresh endeavors.

W. C. GRAUSTEIN

GAUSS AND NON-EUCLIDEAN GEOMETRY

A question of historical accuracy is raised by Professor Emch in his review of my *Projective Geometry*[*] where he says:

"It is proper to point out an error which is common in histories of mathematics and which is contained in the following statement on page 420: 'Little progress was made until about a hundred years later when Gauss (1777–1855), his friends and pupils became deeply interested in the subject.' (1) Now the fact is that Gauss's deeper interest in the subject was subsequently aroused by the brilliant discoveries of Lobatchevsky and Bolyai. (2) As a matter of fact, Gauss, in the beginning, hoped to be able to prove what is known as the euclidean parallel axiom and (3) assumed a rather skeptical attitude towards the new discoveries. (4) Subsequent deeper meditations, however, led Gauss to his own establishment or verification and acceptance of the new theory."[†]

An examination of the sources on which my statement is based will, I believe, substantiate the sentence quoted as well as the context from which it is taken. Among these sources are the letters of the two Bolyais and Gauss himself.* Of the four points in Professor Emch's own statement, (2) is correct but irrelevant. For it applies equally to all the pioneers in the subject, including Lobachevsky and young Bolyai too † who may be said to have inherited the problem of parallels from his father, Wolfgang Bolyai.

---

[*] This BULLETIN, vol. 30 (1924), p. 81.

[†] The numbering is mine.

In Gauss's works (loc. cit.) are found some twenty letters dealing with the foundations of geometry and the theory of parallels, the first written in 1799. From these we learn that he had attacked the problem as early as 1792 (letter to Schumacher, 1846) while he first learned of the work of John Bolyai when he received from the father a copy of the famous *Appendix* in 1832 (letter to Gerling).‡ Prior to this time he expresses his doubts concerning the validity of Euclid (letter to W. Bolyai, 1799) and states his conviction that "the necessity of our geometry cannot be proved" and hence "we must place geometry somewhat in the category with mechanics" (letter to Olbers, 1817). He obtained many of the fundamental properties of the hyperbolic geometry, e. g. the impossibility of similar figures, the existence of an absolute *a priori* measure of length (letter to Gerling, 1816), the connection between the defect of the angle sum and the area of a triangle, the formula for the area of the maximum triangle (letter to Gerling, 1819) and the formula for the length of a circle (letter to Schumacher, 1831). He even originated the name *non-euclidean geometry*, declared it to be self-consistent throughout and that it reduced to the euclidean when the space constant is infinite (letter to Taurinus, 1824). Various reasons are assigned for not publishing his results, such as the lack of time for refining them, the indifference and incompetence of the mathematical public and the anticipated "clamor of the Boeotians".

This seems to me ample justification of my statement that Gauss was deeply interested in the problem of parallels and had made substantial progress toward its solution prior to his knowledge of the work of J. Bolyai and Lobachevsky. It would be interesting, if space permitted, to listen at length to Gauss's own testimony but a few extracts from his letters will have to suffice:

---

* Stackel and Engel, *Gauss, die beiden Bolyai und die nichteuklidische Geometrie,* MATHEMATISCHE ANNALEN, vol 49 (1897), p. 149; Gauss, WERKE, vol. 8, pp. 159–239. Many of these are available in English, e. g. in Carslaw, *Non-Euclidean Geometry,* pp. 19–37.

† Three proofs of the parallel postulate by Lobachevsky are contained in notes of his lectures (1815–1817) and another apparently in a paper, now lost, but read at Kasan in 1826, entitled *Exposition succincte des principes de la géometrie, avec une démonstration rigoreuse du théorème des parallèles.*—Carslaw, loc. cit. John Bolyai was seeking a proof of the postulate at about 1820 according to a statement in his autobiography, quoted by Stackel and Engel, loc. cit., p. 155.

‡ He appears not to have heard of the work of Lobachevsky until some years later (letter to Encke, 1841) and notes by Stackel.

1. To Schumacher, May 17, 1831. "Some of my own meditations which in part are now about 40 years old—whereof however I have never recorded anything and therefore have been obliged to think out many of them three or four times anew—I have begun in the past few weeks to write down. For I did not wish it to perish with me."

2. To Gerling, February 14, 1832. ... "I have this day received from Hungary a little treatise on the Non-Euclidean geometry wherein I find again all my own ideas and results, developed with great elegance, howbeit for one to whom the subject is strange, in a form somewhat difficult to follow because of the concentration. .. I regard this young geometer von Bolyai as a genius of the first order."

3. To Wolfgang Bolyai, March 6, 1832 (in response to a letter accompanying the *Appendix* by John Bolyai). .. "If I should begin by saying that I am unable to praise it, you would doubtless be astonished for a moment. But I cannot do otherwise—to praise it would be to praise myself. For the whole content of the work, the method which your son has employed and the results to which he is led, agree almost completely with my own meditations pursued in part now for 30-35 years." . .*

4. To Schumacher, November 28, 1846. ... "I have recently had occasion to examine again the little work by Lobatschewsky (*Geometric Researches on the Theory of Parallels,* Berlin, 1840). It contains the elements of that geometry which must exist and with strict consistency may exist if the euclidean geometry is not the true one. A certain Schweikart called such a geometry *astral geometry*, Lobatschewsky *imaginary geometry*. You know that for 54 years (since 1792) I have held the same conviction (with a certain later extension which I shall not mention here). I have not found therefore in Lobatchevsky's work material new to me but the development is made on other lines than I have followed and indeed by Lobatchevsky in amasterly manner with true geometric genius." ...

R. M. Winger

---

* John Bolyai was only 29 years old when this was written.

JULIA ON ESSENTIAL SINGULARITIES

Leçons sur les Fonctions Uniformes à Point Singulier Essentiel Isolé By Gaston Julia. Redigées par P. Flamant. (Borel Monograph.) Paris, Gauthier-Villars, 1924. vii + 149 pp.

This monograph deals with Picard's famous theorem of 1879 to the effect that an analytic function, in the neighborhood of an isolated essential singularity, assumes every value, with two possible exceptions. an infinite number of times. It presents this theorem, and perfections of it due to Landau, Schottky, Caratheodory, Lindelof, Iversen, and to Julia himself.

A first chapter develops as much of the theory of modular functions as is needed for the proof of Picard's theorem. The theorem that any simply connected region can be mapped conformally upon a circle is here used, and familiarity with it by the reader is assumed. Of course, an elementary proof of Picard's theorem, by the methods of Borel, Landau and Schottky, would not require as much preliminary work, but when one considers the intuitive qualities of Picard's own proof, one can understand Julia's preference. Besides, as Caratheodory has shown, (this is dealt with in the second chapter), the modular function does not play an artificial rôle, for it leads to the determination of an important least upper bound, in the expression for which the modular function actually appears.

The second chapter opens with the proof of Picard's theorem for the special case of an integral function. Then comes the following generalization, due to Landau:

There exists a function $\varphi(a_0, a_1) > 0$, *such that, if the series*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots, \tag{1}$$

with $a_1 \neq 0$, *converges for* $|x| < \varphi(a_0, a_1)$, *the series assumes at least one of the values* 0 *or* 1 *for* $|x| < \varphi(a_0, a_1)$.

Caratheodory's determination of the best $\varphi(a_0, a_1)$ is then given. As stated above, the modular function appears in this expression. The next theorem is that of Schottky:

There exists a function $M(a_0, \theta)$, *such that, if* (1) *converges for* $|x| \leq R$ *and does not assume either of the values* 0 *or* 1 *for* $|x| \leq R$, *we have, for any positive* θ *less than unity, and for* $|x| < \theta R$, *the inequality* $|f(x)| < M(a_0, \theta)$.

The chapter closes with the proof of Picard's theorem for any isolated essential singularity.

The third chapter, one of the most interesting in the book, gives an account of Montel's *normal families* of functions, which Julia used in his own researches. A set of functions, meromorphic in an area, is said to be *normal* in the area if from every sequence of functions of the family a sub-sequence can be extracted which converges uniformly in every area interior to the given one. Here the notion of uniform convergence has to be loosened somewhat, because of the poles. Also, the infinite constant is an allowable limit for the sub-sequence.

Montel's chief theorem states that *a sufficient condition for a family to be normal in an area is that three distinct values, a, b and c exist, none of which is assumed in the area by any function of the family.*

Montel's work on normal families has been singularly fruitful in many connections. One might mention conformal mapping, iteration and Picard's theorem. Still it seems to the present reviewer that Montel's principal theorem is little more than a combined statement of Schottky's theorem and of well known results on equally continuous families of functions.

An interesting detail of the third chapter is the crediting of M. B. Porter with the independent discovery of the theorem on sequences of analytic functions usually known as Vitali's theorem.

Chapter IV deals with researches of the Scandinavian school of analysts on sectors with vertex at an essential singularity. We pass over these results.

We come now to Julia's own work, which occupies the last three chapters. Julia's first problem amounts to this: *If* $f(z)$ *is a meromorphic function, and if* $|\sigma| > 1$, *what can be said as to the values assumed by the sequence of functions* $f(\sigma^n z)$, $n = 1, 2, \cdots$, *in any given area?* Those familiar with the recent work on the iteration of rational functions will see a relation between this problem and the theory of Poincaré's functions with rational multiplication theorems, which functions were introduced into iteration theory by S. Lattès and by the present reviewer. Julia's work deals principally with the set of points at which the family $f(\sigma^n z)$ is not normal. The results are numerous and interesting. In the final chapter, a set of multipliers not of the form σ^n is used. As a capital yield of Julia's methods, we cite the following extension of Picard's theorem for integral functions: *Given an integral function* $f(z)$, *and any point of the plane, there exists a ray joining the point to infinity such that, in any sector containing the ray in its interior,* $f(z)$ *assumes every finite value, with one possible exception, an infinite number of times.*

Julia's monograph is probably one of the finest in the Borel collection. The student who reads it together with Landau's *Neuere Ergebnisse der Funktionentheorie* will acquire dominion over a region of analysis whose exploitation is not yet complete.

J. F. Ritt

THE N. R. C. HANDBOOK ON STATISTICS

Handbook of Mathematical Statistics. By H. L. Rietz, editor in chief, and eight other members of the Committee on the Mathematical Analysis of Statistics of the Division of Physical Sciences of the National Research Council. Boston, Houghton Mifflin Company, 1924. viii + 221 pp.

The general purposes of this book may be inferred from statements in its preface and from its form. It deals "with the mathematical analysis of data", not with its collection or interpretation. "Special emphasis is laid on the limitations surrounding the proper applications of the various methods." It is not a treatise or a text-book. It is to be used as a reference book, probably also as a background for a course of lectures. At the same time, it provides sufficient explanation of the fundamental ideas, and especially sufficient illustration of the methods, and sufficient bibliography, to enable the reader to begin his acquaintance with statistical theory here, provided he has already a knowledge of collegiate mathematics. This last proviso is, in the mind of the reviewer, a distinct advantage that this book has over some others; it is possible to make the subject so much clearer with the use of mathematical language than without it. Apparently, also, having been written by a committee of nine authors, to each of whom a piece was allotted, the book is intended to be representative of expert knowledge in the several portions of its field.

This was an obvious and decided advantage of multiple authorship, and one cannot but recognize the appropriateness of choice of author in each case. There were also some obvious disadvantages, especially as the several authors were scattered about the country from the Atlantic seaboard to the Pacific. If the book is to serve well as a basis for lectures, it is important that there should be consistency both of notation and of language throughout. This is the more desirable because at best the notation of mathematical statistics is complicated. It was a difficult thing for a group of nine to achieve, and in fact it was not very well achieved. One is led to feel that, from a practical point of view, it would have been better to have had a smaller number, or at least a different division of labor among the nine. The chapters on multiple and simple correlation ought not to be widely different from each other in style, method of approach, notation and language. There would seem to be no good reason why the subject of frequency curves should be introduced by one writer and carried on by another, or why Bernoulli, Poisson, and Lexis distributions should be handled by a different author from the one who deals with probability. As a result there are such inconsistenties as the following. In Chapter I, page 15, moments are defined as in mechanics, not as usually in statistics. This definition is in agreement, though the notation is slightly in disagreement, with the usage by the same author in Chapter IV, page 69. In Chapter VII, however, the statistical definition is used, but the notation of Chapter I. That is, there are two different definitions with the same notation, and two different notations with the same definition. There is also the so-called "Bernoulli Theorem" in two different notations, pages 16 and 72, so that the misprint in the first statement is not immediately obvious by comparison with the second. Further, there is a confusion of statements so that the reader cannot be sure that the expressions really include areas on both sides of the mode until he has examined the illustrative example on page 73.

Another considerable difficulty with which the committee had to deal was that of securing uniform adherence to their excellent ideal of stating and emphasizing the limitations of methods. It is more important that this be done meticulously in a handbook than either in an elementary text-book or in a treatise. for, were the proofs of the theorems given, the careful reader could discover these limitations for himself. Since the proofs are not usually given the reader has a right to demand extraordinary care in the statements. It is a joy to find the many places where special care is manifest, e. g., in the statements regarding the χ test on page 80, and that generally, throughout the book, there is the spirit of mathematical rigor. There are some slips. In presenting the formulas for probable errors, for example, no author states that the parameters appearing on the right are, strictly, the parameters of the universe from which the sample is drawn, in some cases (p. 32; p. 77, probable errors of r, η, $r\ \sigma y/\sigma x$, and Y) the condition that the universe from which the sample is drawn be normal is omitted, and there is little disposition to recognize that, especially for the higher moments, the expressions are derivable only on the assumption that the sample is very large. Indeed, neglect of this very important condition permits the author of Chapter VII to recommend a substitute for the χ test of goodness of fit which cannot be accepted unless this difficulty has been rigorously dealt with. Not only do the probable errors of his high α_s's mount up, as he points out, with increasing index, but, n being fixed, his formulas for them become increasingly uncertain. This is due to the fact that, strictly, there should be more terms in his formula for σ_{μ_s}, which involve higher powers of s. If n is fairly large, these terms are negligible when s is small, because they involve higher powers of $1/n$ than the terms retained, but when s is large they are very troublesome. There is the more need of insisting on the use of large samples because one of the common errors of practical workers in statistics is the error of using in a more or less wooden way book formulas with inadequate data A more grievous instance of the Handbook's failure to warn its readers against the use of small samples occurs in the chapter on multiple correlation, cited below.

There are a considerable number of places in the book where it seems to the reviewer that the statements should be modified or supplemented, and some of them will be noted as the several chapters are considered in turn. It is hoped that the reader will recognize, however, that the adverse criticisms are of details, and that, for the most part, they are offered with constructive purpose, rather than as indicative of general features. In fact, at first reading these details were seldom noticed. The general effect of the book was stimulating, and it was welcomed enthusiastically. Adverse criticism may probably be summarized in the statement that the book seems to have been written a bit too hastily. This book is such a useful contribution, and will occupy for some time to come such an important position in statistical literature, that it was well worth that extra effort which would have made that position permanent. It is much more than an elementary text, yet it just falls short of being a digest of a treatise, which one feels it might have been. The material is carefully selected and apportioned. The chapters differ markedly, but almost everywhere there is clearness, thoroughness, and simplicity of treatment, combined with mathematical exactness of statement which it is not usual to find in books on statistics. Numerical examples are used to illustrate almost every important idea.

The first chapter on "Mathematical Memoranda" comprises a collection of many mathematical formulas useful in statistics, and a very brief account of probability. The latter is rather too brief; at least it should include the point binomial theorem, and the probability theorem leading to the hypergeometric in finite form (Elderton, W. P., *Frequency Curves and Correlation*, p. 37). There are one or two mistakes noted later under "Errata." It should also be stated that the C's, at the bottom of page 16, are mutually exclusive. There should be a reference, at least in the bibliography, for the important table of Burgess, mentioned on page 14, viz., TRANSACTIONS OF THE ROYAL SOCIETY OF EDINBURGH, vol. 39 (1897), pp. 257–321. A valuable reference on quadrature formulas is Irwin, J. O., *Tracts for Computers*, No. 10 (1923). Irwin gives the constants of the Euler-Maclaurin theorem in two useful forms. On page 12, line 4, the words "absolute error" are used in a different sense from that in which they were defined on page 2.

Chapter II is an almost perfect introduction to the subject of frequency distributions. There is an obvious slip on page 32. The author should insert after several formulas, as he does on page 77, the words, "normal distribution". This restriction has an importance here, for on the next page he says "preference should be given to the average which has the smallest probable error." Glancing back at page 32, the reader would infer that the arithmetic mean is therefore always to be preferred, but this it not a desirable inference; for on page 155 a case is cited where the distribution is not normal and the median appears to have the smaller probable error.

The third chapter deals with interpolation, summation, and graduation, and is clearly the work of one who has had much experience in this field. Considering the small space this chapter occupies, it is remarkably readable and comprehensive. The notation ${}_xC_k = x_k$ is confusing, since by u_k the author means the value of u at $x = k$. All his formulas are either successive averages or else based on interpolation by means of finite differences. As there are some functions to which the latter method does not well apply, a word of caution about the desirability of rapid convergence is needed. There is no mention of methods of interpolation on a surface (cf. Pearson, K., *Tracts for Computers*, No. 3, 1920). The author's formula for subdivision of an interval (p. 45) can be improved in form, provided the table on which it is to be used does not already contain its successive differences, and provided the computer is working with a machine which will subtract as well as it will add (Pearson, K.; *Tracts for Computers*, No. 2, 1920). The author should give the dates in the references in his foot-notes. For example, at the bottom of page 45, it is misleading to cite the reference "Explanation of a New Formula for Interpolation" without the date, which is 1880!

Chapter IV on curve fitting is in Huntington's best style, simple, clear, and adequate. There follow two short chapters by Rietz, a thoroughly good account both of random sampling and of the distributions of Bernoulli, Poisson, and Lexis. The theorem attributed to Bernoulli on page 72 is not truly his. If it were, the normal law, which is implicit in it, and which the Handbook (following a bad custom) calls Gauss's, on page 11, would be Bernoulli's too. The *Ars Conjectandi* antedated Gauss by more than a century. This theorem is proved by Laplace, but subject to limitations which may make the second term worse than useless if c is not small. E. g., for the point binomial $(1/3 + 2/3)^{450}$, the approximation is better without the second term at $c = 3\sigma$. Somewhere in this book, probably in the chapter on sampling, there should be an explanation of the (different) formulas for probable errors commonly accepted by engineers, physicists, and astronomers (e. g., Chauvenet, *Practical Astronomy*, vol. 2, pp. 494, 506). The ordinary "theory of errors" is a special case of the theory of statistics, and is too frequently divorced from it. The desirability of severer warnings against the use of probable error formulas for small samples has already been mentioned. It is also desirable that there should be some warning against the usual *interpretation* of such formulas.

What is meant by the usual interpretation is indicated in the next chapter, on "frequency curves." At the top of page 100 there is a table giving the probability that a variable will differ in absolute value from its mean value by less than k times its probable error, provided the variable be normally distributed. Now, when the value of the probable error of a frequency constant has been determined, it is not surely correct to estimate by means of this table the probability that this constant will differ in absolute value from its mean value by less than k times its probable error. It is necessary to know first that this constant is normally distributed, and for this to be true it is not always sufficient to know that the distribution to which the constant belongs is normal. One may, of course, use the table to get an approximate estimate if one knows that the distribution of the constant is approximately normal, but this is frequently not true for small samples, as has been shown in the case of the coefficient of correlation by R. A. Fisher, (BIOMETRIKA, vol. 10 (1915), pp. 507–521). The author uses this table in connection with his high moment test of goodness of fit, to which an objection was made in an earlier paragraph. Waiving the question, there raised, as to whether, in a concrete case, he is finding the values of his probable errors correctly, it is now fair to raise the additional question: can he show that the distribution of his high α_s's is approximately normal? It is the opinion of the reviewer that the author's idea here, though interesting and original, is not as simple a matter as it appears, and that there was not space available in a handbook for the arguments which would be necessary to justify it. The author's account of Pearson's curves is an admirable simplification of the usual methods. His argument with regard to Sheppard's corrections (p. 94) is not very clear. Certainly the function $f(x)$ used in his analysis on the preceding page is *a* graduating function, and so the conclusion that Sheppard's corrections do not demand the vanishing of the "chosen' graduating function does not seem to follow necessarily from the analysis. The author admittedly has a point here, but it is a delicate one, and requires more careful handling. There should also be a warning against the use of these corrections in "abrupt" cases, where they sometimes would make the approximation worse.

Some mention has already been made of the two chapters following on simple and on multiple correlation. The first is a brilliantly clear chapter, perhaps the most striking in the book. The second suffers from an attempt to compress too much into its short space. If the space was necessarily restricted, it would seem to have been better to have attempted first to show what a solid of frequency is (which is not described at all), perhaps by the analogy of distribution of mass in a physical solid, to show what are meant, geometrically, by total, partial, and multiple regression, and by the other constants to be used, and then merely to recite the formulas with numerical illustration. There appears, however, no good reason for such a strict compression of a subject which is well known to be difficult to present. The author does not always define his terms before using them. The term, "total correlation coefficient" is used on page 140, and first defined on page 141; "partial alienation coefficients" are defined in terms of "partial correlation coefficients" on page 141, but these in turn are first defined on page 142. The reader must be careful to note that the word "cell" is used in this chapter in a different sense from that in which it is used in the preceding chapter, and that the words "assigned values" and "measures" are used in the sense of observed values or measurements. The example chosen to illustrate the methods of computation contains only 136 data. This would not be objectionable if it were carefully pointed out that it is for purposes of numerical illustration only, but the serious discussion of the meaning of the results gives the impression that it is to be taken as an adequate and approved piece of statistical investigation in the field of education. This is extremely unfortunate, even though accidental, since it was hoped that the Handbook would be of special value in improving the quality of current statistical practice. Roughly speaking, it is as futile to attempt to find the parameters of a frequency solid of three dimensions from 136 data as it would be to seek the parameters of a frequency curve from $\sqrt[3]{136} = 5.14$ data. The situation is made worse in this case by the further fact that one of the variables does not represent a measurable character; there is only a division into three categories.

The last three chapters of the book are written by economic statisticians, on correlation in time series, periodogram analysis, and index numbers. They clearly set forth and illustrate methods that have been used by them, and will be a welcome addition to the earlier mathematical theory, especially in a country where the theory of statistics has been so largely identified with business.

Errata (partly supplied by the Editor). Page 16, line 12: *For* at least *read* some. Page 16, line 14: *For* at least one 2-spot with two dice *read* a 1-spot or a 2-spot with one die. Page 16, line 7 from bottom: *For* $\sqrt{2\pi npq}$ *read* $\sqrt{2npq}$. Page 54, line 5: *For* coefficients *read* coefficient. Page 56, line 5 from bottom: The minus sign (—) in $\frac{1}{2m}(u_0 - u_\omega)$ *should be changed to* $+$. Page 70, line 7 from bottom: *For* m^2 *read* m_1^2. Page 73, line 15: *For* 688 *read* 668. Page 74, line 6 of Ex. 2: *After* number *insert* of women's numbers; *to avoid ambiguity from the use of number in two senses.* Page 78, line 4 of second paragraph: *For* fit *read* fitted. Page 92: The integrals are definite and should have upper limits. Page 97, line 15: *For* $\frac{\ }{N}y$ *read* $\frac{\sigma}{N}y$. Page 103, line 7:

For 80 *read* 81. Page 105, line 3 from bottom. Omit x and the two commas following. Page 107, line 4: *For* V *read* VI *and for* VI *read* VII. Page 108, line 1: *For* VII *read* VIII. Page 121, line 4 from bottom· *For* $\frac{y'_1}{\sigma_y}$ *read* $\frac{y_1}{\sigma_y}$; *i. e., delete the prime.* Pape 126, formula (9): *For* ^2_x *read* σ_x^2. Page 142: The last line should begin with $\beta_{12\cdot34}$. Page 149, line 2· *For* (13) *read* (19). Page 187, formula (8): Insert Σ before parenthesis in the denominator.

B. H. CAMP

FOUR BOOKS ON PROBABILITIES

Éléments de la Théorie des Probabilités. Third edition. By Emile Borel. Paris, J. Hermann, 1924. vii + 266 pp.

Probabilités, Erreurs By Emile Borel and Robert Deltheil. Paris, Librairie Armand Colin, 1923. vi + 197 pp.

Wahrscheinlichkeitsrechnung. By Otto Knopf. Sammlung Goschen. Berlin, Walter de Gruyter and Co., 1923. Two volumes, 112 and 112 pp.

Grundlagen der Wahrscheinlichkeitsrechnung und der Theorie der Beobachtungsfehler. By F. M. Urban. Leipzig, B. G. Teubner, 1923. vi + 274 pp.

The present time is a time of decided activity in the publication of books on the theory of probability. While two or three works of considerable originality and merit have appeared, the majority are merely text-book rearrangements of the traditional course in probability adapted to various classes of readers.

The first of the books in this review is the third edition of Borel's *Théorie des Probabilités.* This well known work has been before the public since 1909 The chief change in this third edition is the addition of four notes at the end of the book on applications to radioactivity, on a problem leading to a Stieltjes integral, on games of chance in which the ability of the players is taken into consideration, and on what Borel calls a differential method in statistics.

The book by Borel and Deltheil is number 34 of the series of little books known as the "Collection Armand Colin". The purpose of the series is to present to the educated person readable monographs on special topics in philosophy, history, science, mathematics, literature and other branches of learning. As might be expected, the book covers much the same ground as the more extended work of Borel.

It emphasizes the applications to statistics, biology and physics. The last chapter is on the method of least squares and the theory of errors from the Gaussian standpoint. A number of interesting exercises and problems, including some of the well known problems in the theory of probability, are given at the ends of the chapters. The work is a very readable little book for the non-specialist, although the reader must have some knowledge of calculus.

The third work is one of the series of books well known as the SAMMLUNG GÖSCHEN. Two volumes of the series are used. This work covers the traditional topics in the orthodox formal manner of many English and German text books. It seems to replace the one volume work on probability by Hack in the SAMMLUNG GOSCHEN.

The work by Urban is an attempt to present to students of the sciences the fundamentals of the theory of probability. The book has rather a curious history. It was originally written in English while the author was connected with the faculty of the University of Pennsylvania, and the intention was to publish in the United States a rather complete treatise on the theory of probability. The war intervened and the author returned to Germany and published in German this abridged work emphasizing the underlying principles and the theory of errors. This was a distinct loss to the English reading student. It still seems easier to find a publisher in Germany than in the United States. The reviewer feels that the more complete book would have been much more acceptable than the part finally published. The book is not like a formal text book on the theory of probability. It is the aim of the author to make clear the fundamental notions underlying his subject, and in doing so he uses the "wordy" method, using the term in a good sense. He is not always successful in this clearing process, especially when one considers the layreader, but he does thoroughly discuss the debatable topics in probability from all points of view. For example, he devotes many pages to the word *Wahrscheinlichkeit*, and reviews the definitions, views, objections and remarks of various writers on this fundamental word. It is an interesting and valuable discussion but many objections could be raised to the author's insistence upon his Mengenlehre definition of probability, especially when the non-technical student is considered. The chapter on the theory of errors is unique. It is not the type of chapter which tells the student how to handle errors of observation but one which discusses the hypotheses and notions at the foundation of the theory in an informal, open-minded manner.

A. R. CRATHORNE

SHORTER NOTICES

Geschichte der Elementarmathematik. By Johannes Tropfke. Berlin and Leipzig, Walter de Gruyter. Bd. V: *Ebene Trigonometrie. Sphaerik und Sphaerische Trigonometrie.* i+185 pp. 1923. Bd. VI: *Analysis. Analytische Geometrie.* i+169 pp. 1924.

The fifth and sixth volumes of the revision of Tropfke's history of elementary mathematics maintain the high standard set by the first four volumes*. Here is the same wealth of well-arranged material, the same concise and yet vivid style, the same care in evaluating all the contributions of previous workers in the field. The revision includes large amplifications, as the topics contained in these two volumes of 354 pages were covered in 251 pages of the same size in the first edition. The number of references to the literature in the form of footnotes has been increased from 971 to 1922.

The topics are: Vol. 5, plane trigonometry, pp. 3–98; spherical geometry and trigonometry, pp. 101–185; Vol. 6, series, pp. 3–55; compound interest, pp. 56–62; permutations and probability, pp. 63–74; continued fractions, pp. 74–84; maxima and minima (in elementary geometry), pp. 84–91; analytic geometry, pp. 92–169.

The discussion of analytic geometry is a notable example of the improvement introduced in the new edition. The algebraic-geometric problems of al-Khowarizmi and Abu Kamil are given in some detail; the descriptions of Descartes's *Géométrie* and of Fermat's *Isagoge* have been made somewhat fuller and clearer; and the influence of Descartes's work upon his contemporaries and successors has been traced in a more satisfactory manner.

A detail that is not without general interest is in reference to "Heron's formula" for the area of a plane triangle in terms of its sides: $F=\sqrt{s(s-a)(s-b)(s-c)}$. In the revised edition, Tropfke accepts the statement of an Arab writer of the 11th century† that this formula is not original with Heron, but is due to Archimedes. Heath in his *History of Greek Mathematics*‡ also accepts this statement. In spite, however, of these excellent precedents, and of the fact that there is obviously nothing inherently improbable in the ascription to Archimedes, the reviewer would prefer to await more conclusive evidence before rejecting the tradition which is supported by so many ancient writers, because of this one contrary testimony.

---

* Reviewed in this BULLETIN, Vol. 29 (1923), pp. 476–477.

† BIBLIOTHECA MATHEMATICA, (3), vol. 11 (1910–11), p. 39.

‡ Vol. 2, p. 322.

There are a few misprints; in volume 5, p. 61, lines 15 and 16, sin 2α and cos 2α schould be sin α and cos α respectively; p. 74, line 4 from end, Gegenseiten should read Gegenwinkel; and p. 85, line 16, 1675 should be 1765.

The seventh and final volume is to contain Stereometry and a complete index. The appearance of this index will be impatiently awaited, as it will increase the value of the set many fold. Even as it is, the six volumes are indispensable for the teacher or student of the history of elementary mathematics.

R. B. McClenon

Principles of Geoemtry. By H. F. Baker. Vol. II: *Plane Geometry, Circles, Non-Euclidean Geometry.* Cambridge University Press, 1922. xv + 243 pp.

This book, in continuation of the first volume, aims to present the main theorems of plane geometry and to develop logically the results of the principles explained in the first volume. In both purposes the author has succeeded admirably.

The preliminary chapter of the present text reviews in a brief manner enough of the matter of the first volume to enable a reader to use the present volume without reference to the first one, provided he has an elementary knowledge of projective geometry. In both volumes the treatment is first synthetic. The fundamental notation is that of projective, or (as the author calls them) related ranges. The notions of distance and congruence are not assumed. These notions and coordinate systems are developed later with a study of the logical principles underlying them.

In chapter one of the present volume, the general properties of conics are deduced from their definition as the locus of the intersection of the corresponding rays of two projective pencils and a wealth of theorems are presented.

In chapter two the relation of geometric figures to two given points of reference are studied. Let us assume any two points I and J as the absolute. Then if a line AB meets the line IJ in a point K, the point C which is the harmonic conjugate of K with respect to A and B is the midpoint of AB. Two lines which meet on IJ are parallel, and two lines are perpendicular if they meet IJ in two points which are harmonic conjugates with respect to I and J. A circle is a conic through I and J. From these definitions the usual properties of circles are deduced and a discussion of coaxial circles, inversion on a circle and the like are given. Similarly, projective definitions of foci of a conic, of a rectangular hyperbola, of a parabola and the like may be given, and the so-called metrical properties of conics obtained.

In chapter three, under the heading of the equation of a line and of a conic, a coordinate system is for the first time introduced. From the author's viewpoint the use of symbols is unnecessary but is convenient as rendering a demonstration easier to follow. If A, B, C are symbols attached to any three points of a plane, not lying in the same straight line, then any other point P in the plane has a symbol of the form $xA+yB+zC$ where x, y, z are algebraic quantities. This has been shown in volume one as a consequence of postulates of incidence and the theorem of Pappus. Then (x, y, z) or (mx, my, mz) are the coordinates of P. With the coordinates thus introduced the usual analytic geometry is readily developed.

In chapter four there is discussion of the use of imaginary elements in geometry and the logical questions involved.

Chapter five treats of projective measurement of distance and angle with reference to a fundamental conic, and discusses somewhat fully the types of non-euclidean geometry which result.

F. S. WOODS

Mathematical Theory of Life Insurance. By C. H. Forsyth. New York, John Wiley and Sons, 1924. 6 + 74 pp.

Books dealing with the mathematics of finance and life insurance may be divided roughly into two classes: (1) those that present thousands of formulas; (2) those that attempt to select a comparatively few as fundamental. While certain students may learn best from the first type of book, the second type is more attractive and more inspiring. Forsyth's text is of the second type; and in the reviewer's opinion, the selection of topics and formulas to be stressed is decidedly felicitous. The first half of the book presents with great simplicity and direct nessthe rudiments of probability, the mortality table, and the premiums for annuities and insurances. As an analogue for ${}_nE_x$, Forsyth introduces the symbol nIx as the single premium for insurance covering the nth year only; and thus $Ax = \sum nIx$, just as $ax = \sum nEx$. Chapter IV explains rather briefly but comprehensively the valuation of policies in accordance with the laws of different states, and this chapter is decidedly important. The instructor will have to supplement this to some extent, and may prefer to give the prospective method before the retrospective method. In using the "shuttle" ${}_nE_x$, it should be noted that the "x" refers to the earlier age, not to the age from which the change is made. Chapter V explains in a simple manner the Makeham formula and its use in joint insurance, leaving for the two-page appendix the derivation of the Makeham formula. The American Experience Table is given, with columns for l_x, d_x, q_x, p_x, e_x, D_x, N_x, M_x, the three latter on the basis of $3\frac{1}{2}$ %. Numerous exercises appear throughout the text.

E. L. DODD

Lehrbuch der Algebra, verfasst mit Benutzung von Heinrich Webers gleichnamigem Buche. Erster Band: *Allgemeine Theorie der algebraischen Gleichungen.* By Robert Fricke. Braunschweig, F. Vieweg und Sohn, 1924. viii + 468 pp.

When the last edition of the *Lehrbuch der Algebra* by the late H. Weber became out of print, the publishers invited Professor Fricke to write a treatise on algebra in three volumes to replace that by Weber. The subject matter of the present volume 1 is essentially the same as Weber's volume 1 and the earlier chapters on abstract groups in Weber's volume 2. Fricke states that his second and third volumes will differ more essentially from those by Weber. Volume 2 will treat those algebraic equations which are not solvable by radicals and possess groups representable by binary and ternary substitutions; in particular, Klein's theory of the icosahedron and equations of the fifth degree will be presented from the geometrical standpoint of Klein (in contrast to Weber's algebraic treatment). Volume 3 will develop the theory of algebraic numbers and the class equation of complex multiplication of elliptic functions.

Professor Fricke is an experienced writer of unusual clearness. In addition to his recent book on elliptic functions, he was joint author with Klein of extended treatises on elliptic modulur functions and automorphic functions. These earlier books show that the author has long been familiar with the field covered by the new algebra. The latter will be somewhat simpler to read than Weber's algebra. General determinants are given a better notation, and their rank is employed systematically, thus avoiding the circumlocution of Weber. Certain proofs are replaced by simpler or more natural ones; for example, the product of two determinants, and the determination of the number of positive signs in the canonical form $\sum \pm x_i^2$ of a quadratic form.

There are brief treatments of the following topics omitted by Weber: k-fold bordered determinants, linear dependence, rank of symmetric matrices, besides the additions on pages 6–7, 10–13, 67.

Certain topics in Weber are omitted by Fricke: elimination of two unknowns from three equations; domains defined by an imaginary cube root or fourth root of unity; most of Weber's chapter 11 on continued fractions (perhaps those domains and the theory of the equivalence of quadratic numbers will be treated in volume 3).

Although the general theory of transformations of Tschirnhaus is developed at length, there is no mention of the chief application to the reduction of the general equation of the fifth degree to a normal form involving a single parameter, whether the Bring-Jerrard form $y^5 + y + c = 0$ or the still more important Brioschi form. Doubtless these normal forms will be given in volume 2. But it is far preferable that the reader should meet them as the culmination of the elaborate theory of transformations of Tschirnhaus (Weber I, 2d ed., p. 205, p. 263)

Since the present volume covers nearly all the ground of Weber's volume 1 of 703 pages, as well as the first two chapters (of 68 pages) on abstract groups in Weber's volume 2, it may seem strange that Fricke's present book contains only 461 pages, besides the index. But Fricke's pages are larger than Weber's and the type is materially smaller.

The owner of a copy of Weber's algebra who uses it chiefly as a reference book will not find it essential to buy Fricke's book. But a person who expects to do considerable reading in one of these books will find it to his advantage to own the simpler book by Fricke. The advantage in favor of Fricke's volumes 2 and 3 will doubtless be relatively greater than in the case of volume 1.

It is a pity we do not have in English a treatise on algebra as good as either Weber's or Fricke's.

L. E. Dickson

Leçons de Géométrie Élémentaire. By Édouard A. Fouet. Paris, Vuibert, 1924. xvi + 348 pp.

This is an elementary geometry for pupils who are familiar with theorems *les plus simples* on the straight line and circle. It consist of eight books: I and II on the straight line and circle, III on similar figures, IV on plane areas, V on the plane and line, VI on polyhedra, VII on the sphere, cylinder, and cone, VIII on the conic sections, together with a preliminary book (to be read last, or at least later), a résumé of definitions, transformations, loci, and methods for solution of problems. Intermingled in each book are paragraphs of three distinct natures, (1) an ordinary course in geometry, (2) a course in problems, (3) an excellent set of pedagogical and historical notes.

The ordinary course (1) needs only the obvious commentary that like most similar Continental works it is incomparably better than American geometries in the development of broad underlying principles. In appropriateness of application and in development of power the problem course (2) compares favorably with even such a specialized book as Petersen's *Methods and Theories for the Solution of Problems of Geometrical Construction*, while in wealth of advanced material it is only slightly inferior to Hadamard's *Leçons de Géométrie Élémentaire.* A superior book; but demanding superior teaching, for the course (1) must be accompanied by not too difficult problems (of which there is perhaps a scarcity), and those demanding more maturity must be left until later. But still these last are there, forming a part of the text, not grouped at the end of the chapter or book. This is perhaps the outstanding feature of this text.

B. H. Brown

An Introduction to the Mathematical Analysis of Statistics. By C. H. Forsyth. New York, John Wiley and Sons, 1924. 8 + 241 pp.

The preface advises us: "This book is offered as a textbook in mathematics and not as a reference book for the statistician". A knowledge of calculus is assumed, but the text lies strictly within the grasp of an undergraduate. Before taking up the usual topics of a book on statistics, a substantial mathematical background is established — as may be seen from the chapter headings: Errors and numerical computation; Finite differences; Interpolation; Gamma and beta functions; Probability; Averages and aids in their computation; Moments; The normal curve; The binomial $(p+q)^n$. Statistical series; Correlation theory. Few texts on statistics have been written in English which have such an admirable selection of topics for a junior course in mathematics. Without going into intricate details, the book is decidedly comprehensive, and introduces the student to a multitude of problems in statistical research. It is written in an attractive style — quite informal — and on this account an instructor will need to supplement some statements, by using more technical phraseology. For example, the student should be put on notice that "compensating errors" (p. 3) do not compensate, and that in tangential interpolation (p. 42) the continuity of the curve is not involved. Moreover, for (27) on page 57 $a > 0$; the illustration on page 81 is not felicitous; normality (p. 93) involves a great deal more than symmetry; the "set of observations" at the foot of page 94 is a set of deviations; the bisection of area (p. 119) is effected by the ordinate through the *median*; the argument (p. 139) for the differential equation of the normal curve is hardly convincing. Besides the useful 6-place tables of logarithms and antilogarithms at the back of the book, there are some half-dozen other tables that might well be placed at the back.

E. L. DODD

Methodik des mathematischen Unterrichts. Dritter Teil: *Didaktik der angewandten Mathematik.* By W. Lietzmann. Leipzig, Quelle und Mayer, 1923. xi + 234 pp.

This is the third and last volume of a work of which the second volume has already been reviewed in this BULLETIN, (1923, p. 479). The general characteristics of the present volume on "Applied Mathematics" are similar to those of its predecessor. Accordingly, it will suffice to mention here the various fields whose mathematical aspects are discussed in so far as they fall within the scope of secondary school mathematics, namely: linear drawing, manual training, geodesy and astronomy, commerce and politics, mechanics, physics, philosophy.

J. W. A. YOUNG

NOTES

At the Summer Meeting of the Society at Cornell University, on September 8–12, the Tenth Colloquium of the Society will consist of two courses of lectures: *The new differential geometry*, by Professor L. P. Eisenhart; and *The theory of approximation*, by Professor Dunham Jackson.

The Western Christmas Meeting of the Society and the meeting of the Southwestern Section of the Society will be held at Kansas City, in conjunction with the meeting of the American Association for the Advancement of Science. At the time of these meetings, Professor James Pierpont will deliver the annual Gibbs lecture, on the topic *Some modern views of space.* It is planned to hold the ordinary meeting on Tuesday, December 29, to have a joint session with the Mathematical Association and with Section A on Wednesday morning, and to have the Gibbs lecture presented before a general session of the American Association on Wednesday afternoon. The ordinary meeting of the Mathematical Association will probably fall on Thursday.

The opening number of volume 27 of the TRANSACTIONS OF THIS SOCIETY (January, 1925) contains the following papers: *On normal forms of differential equations*, by W. F. Osgood; *Congruences with constant absolute invariants*, by H. L. Olson; *On the prime divisors of the cyclotomic functions*, by C. M. Huber; *On the roots of the Riemann zeta function*, by J. I. Hutchinson; *A generalization of the Riemannian line element*, by J. L. Synge; *Elementary functions and their inverses*, by J. F. Ritt; *Analytic transformations of everywhere dense point sets*, by Philip Franklin; *An algebraic solution of the Einstein equations*, by Edward Kasner; *Electrodynamics in the general relativity theory*, by G. Y. Rainich.

At the meeting of the Division of Physical Sciences of the National Research Council in Washington on April 26, the following officers were elected for the year beginning June, 1925: Chairman, J. S. Ames; Secretary, W. E. Tisdale; Vice-Chairman, F. K. Richtmyer; Executive Committee, J. S. Ames (chairman), G. D. Birkhoff, P. D. Foote, D. C. Miller, L. Page, F. K. Richtmyer. The following persons were elected members at large: G. D. Birkhoff, D. L. Webster. New members representing scientific societies were announced as follows: for the American Astronomical Society, J. Stebbins; for the American Mathematical Society, L. P. Eisenhart; for the American Physical Society, A. J. Dempster and A. L. Foley; for the Division of Federal Relations, W. Bowie.

The British Association for the Advancement of Science will meet at Southampton, August 26–September 2, 1925. Dr. C. G. Simpson has been appointed President of Section A (mathematics and physics) and Dr. A. M. Tyndall recorder of that Section for this meeting.

The Westphalian Mathematical Society announces that it will hold a celebration in honor of Weierstrass (Weierstrass-Woche) from June 2 to June 6, 1925, at the University of Münster. Addresses will be made by G. Mittag-Leffler, D. Hilbert, O. Perron, P. Koebe, L. Bieberbach, and K. Knopp.

The Cambridge University Press has in preparation a new edition of Whitehead and Russell's *Principia Mathematica*, of which the first volume will appear shortly.

The Royal Belgian Academy has awarded its Decennial Prize for Applied Mathematics for the period 1913–1922 to Professor T. De Donder, for his works on the Einstein theory of gravitation and his *Leçons de Thermodynamique et de Chimie Physique.*

At the centenary celebration in Copenhagen of the Society for the Spread of Natural Philosophy, the Oersted gold medal of that Society was presented to Professor Neils Bohr.

The Alfred Ackermann-Teubner Memorial Prize for the promotion of mathematical sciences has been awarded to Dr. Arnold Kohlschütter, of the Astrophysical Observatory at Potsdam, for his determination of the absolute brightness of stars from the intensity relations of certain spectral lines announced in volume 40 of the ASTROPHYSICAL JOURNAL.

The Ernst Abbe Prize, founded in 1921 by Carl Zeiss for the furtherance of mathematics and physics, has been awarded to Professor Felix Klein, for his mathematical work.

Sir J. J. Thomson has been awarded the Faraday medal of the Institution of Electrical Engineers, London.

The gold medal of the Royal Astronomical Society has been awarded to Sir Frank Dyson, Astronomer Royal, for his contributions to astronomy, and in particular for his researches on the proper motions of the stars.

The Carnegie Institution has granted to Professor D. N. Lehmer, of the University of California, a preliminary sum of $ 500 with which to start the work of constructing the stencils described in his paper, *On a new method of factorization* (PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 11, No. 1).

Among the grants in aid of research announced by the American Association for the Advancement of Science for 1925 are the following:

$250 to Dr. H. W. Stager, of the University of Washington, for building a computing machine for use in constructing a list of prime numbers and a factor table; $300 to Professor A. O. Leuschner, of the University of California, for use in the study of the perturbations of minor planets.

Professor J. Hadamard, of the Collège de France, arrived in this country on April 26 and returned to France on June 17. He spent this period of two months in travel and in lecturing at the universities of Illinois, Wisconsin, Indiana, California Southern Branch, Stanford, California (Berkeley), and Rice Institute. The subjects of some of his lectures were: *Huyghens' principle, Some recent points of view on the notion of function. The second period of the work of Poincaré on differential equations, and Some recent improvements in the geometry of circles in space.*

On the occasion of the Third Pan-American Scientific Congress (see this BULLETIN, vol. 30, p. 566, and p. 290 of this number), the University of San Marcos, at Lima, conferred the degree of Honorary Doctor in the Faculty of Sciences on Professor E. V. Huntington, of Harvard University.

At the University of Catania, Dr. G. Albanese, Dr. G. Andreoli, and Dr. A. Terracini have been appointed to associate professorships, in the fields of projective and descriptive geometry, the calculus, and analytic geometry, respectively.

At the University of Cagliari, Dr. B. Caldonazzo has been appointed associate professor of rational mechanics.

At the University of Florence, Dr. F. Tricomi has been appointed associate professor of analysis.

At the University of Genoa, Dr. P. Straneo has been appointed associate professor of mathematical physics.

Professor Gustav Herglotz, of the University of Leipzig, has been appointed professor of mathematics at the University of Göttingen.

Professor B. Janet, of the University of Rennes, has been appointed professor of differential and integral calculus at the University of Caen, as successor to Professor C. Riquier, retired.

At the University of Messina, Dr. U. Crudeli has been appointed associate professor of mathematical physics.

Dr. L. Neder has been appointed to an associate professorship at the University of Leipzig.

Professor G. Pólya, of the Zurich Technical School, has been awarded a Rockefeller travelling fellowship for a year's residence at Oxford and Cambridge.

Professor H. Tietze, of the University of Erlangen, has been appointed to a professorship at the University of Munich.

Professor F. Meyer, of Königsberg, and Professor C. Runge, of Gottingen, have retired from active teaching.

The following have been admitted as privat docents: Dr. H. Behnke, at the University of Hamburg; Dr. F. Krauss, at the Technical School at Aachen; Dr. A. Walther, at the University of Gottingen; Dr. J. Wellstein, at the Technical School at Karlsruhe.

The following appointments as privat-docents in Italian universities are announced: University of Florence, Dr. E. Fermi; University of Palermo, Mrs. Margherita Piazzolla Beloch.

Professor A. A. Bennett, of the University of Texas, has been appointed head of the department of mathematics at Lehigh University.

At the University of North Carolina, Associate Professors A. W. Hobbs and J. W. Lasley, Jr., have been promoted to full professorships and Assistant Professor A. S. Winsor to an associate professorship of mathematics; Mr. J. B. Linker has been promoted to an assistant professorship of applied mathematics.

Assistant Professor V. B. Hinsch, of the Missouri School of Mines, has been promoted to an associate professorship of mathematics.

Assistant Professor H. M. Morse, of Cornell University, has been appointed associate professor of mathematics at Brown University.

Dr. H. L. Olson, of the University of Michigan, has been appointed assistant professor of mathematics at Michigan Agricultural College.

Professor G. E. F. Sherwood, of the Southern Branch of the University of California, has been granted leave of absence for the next academic year, and will go to Cambridge, England, for study.

Mr. Telesforo Tienzo has been promoted to an assistant professorship of mathematics at the University of the Philippines.

The following appointments to instructorships are announced:
Columbia University, Dr. M. H. Stone;
University of Michigan, L. M. Blumenthal, H. F. Schiefer, E. H. Wagner;
University of North Carolina, Mr. C. H. Benson;
Randolph-Macon College, S. T. Arnold and W. S. McClintic.

Professor Paul Urysohn, of Moscow, died August 17, 1924.

Professor P. Schafheitlin, of the Berlin Technical School, died October 2, 1924.

Professor G. J. Wallenberg, of the Berlin Technical School, died November 16, 1924, at the age of sixty years.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BLISS (G. A.). Calculus of variations. (Carus Monographs, No. 1.) Chicago, Open Court, 1925. 13 + 187 pp. $2.00

BURNETT (J. C.). Hyper and ornate magic squares. 15th and 16th orders. Grantham, Lincs, the Author. 1924. 36 pp.

CASTELNUOVO (G.). See HEIBERG (J. L.).

CHISINI (O.). See ENRIQUES (F.).

COOLIDGE (J. L.). An introduction to mathematical probability. Oxford, Clarendon Press, and New York, Oxford University Press, 1925. 12 + 216 pp. $5.00

ELSBACH (A. C.). Kant und Einstein. Untersuchungen über das Verhaltnis der modernen Erkenntnistheorie zur Relativitatstheorie. Berlin, de Gruyter, 1924. 8 + 374 pp.

ENRIQUES (F.). Gli elementi d'Euclide e la critica antica e moderna, editi da Federico Enriques col concorso di diversi collaboratori. Libri I–IV. (Collezione per la Storia e la Filosofia delle Matematiche, volume 1.) Roma, Stock, 1924. 10 + 323 pp.

——. Questioni riguardanti le matematiche elementari, raccolte e coordinate da F. Enriques. 3a edizione. Parte 1: Critica dei principii. Volume 1. Bologna, Zanichelli, 1924. 398 pp.

ENRIQUES (F.) e CHISINI (O.). Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche. Volume III. Bologna, Zanichelli, 1924. 593 pp.

EUCLID. See ENRIQUES (F.).

FITTING (F.). See SCHUBERT (H.).

VON GRUBER (O.). Einfache und Doppelpunkteinschaltung im Raum. Jena, Fischer, 1924.

DE GUELL (–.). L'espace; la relation et la position. Essais sur le fondement de la géométrie. Paris, Gauthier-Villars, 1924. 139 pp.

HEIBERG (J. L.). Matematiche, scienze naturali e medicina nell' antichità classica. Traduzione di Gino Castelnuovo. (Collezione per la Storia e la Filosofia delle Matematiche, volume 2.) Roma, Stock, 1924. 188 pp.

HOHMANN (C.). Abgekürztes Rechnen mit Anwendungen. Breslau, Trewendt und Granier. 1924. 46 pp.

KOWALEWSKI (G.). Einführung in die Determinantentheorie, einschliesslich der Fredholmschen Determinanten. 2te, verkurzte Auflage. Berlin, de Gruyter, 1925. 4 + 304 pp.

VON KRIES (J.). Immanuel Kant und seine Bedeutung für die Naturwissenschaften der Gegenwart. Berlin, Springer. 1924. 4 + 127 pp.

NEUBERG (J.). Bibliographie des triangles spéciaux. Liége, Librairie J. Wyckmans, 1924. 54 pp.

PÉRÈS (J.) et VOLTERRA (V.). Leçons sur la composition et les fonctions permutables. Paris, Gauthier-Villars, 1924. 184 pp.

PICARD (E.). Mélanges de mathématiques et de physique. Paris, Gauthier-Villars, 1924. 364 pp.

ROBINSON (D. S.). The principles of reasoning. An indroduction to logic and scientific method. New York and London, Appleton, 1924. 18 + 390 pp.

ROTHE (H.). Vorlesungen uber hohere Mathematik. 2te verbesserte Auflage. Wien, Seidel, 1923. 12 + 692 pp.

RUTGERS (J. G.). Meetkunde der kegelsneden (met atlas). Groningen, Noordhoff, 1924.

SAGERET (J.). La révolution philosophique et la science: Bergson, Einstein, Le Dantec, J. H. Rosny ainé. Paris, Alcan, 1924.

SALKOWSKI (E.). Der Gruppenbegriff als Ordnungsprinzip des geometrischen Unterrichts. Leipzig, Teubner, 1924. 4 + 59 pp.

SCHUH (F.). Vraagstukken over differentiaal- en integraalrekening en beschrijvende meetkunde. Derde deel. Groningen, Noordhoff, 1924. 486 pp.

SCHRUTKA (L.). Elemente der hoheren Mathematik. 3te und 4te Auflage. Wien, Deuticke, 1924. 30 + 635 pp.

SCHUBERT (H.). Mathematische Mussestunden. 4te Auflage. Neubearbeitet von F. Fitting. Berlin, de Gruyter, 1924. 245 pp.

TRIPIER (H.). Définition géométrique de la fonction exponentielle et de la fonction logarithmique. Propriétés. Paris, Vuibert, 1924. 24 pp.

VAN DER TUUK ADRIANI (N.). Über konforme Abbildung mittels elliptischer Funktionen zweiter Art. (Dissertation.) Freiburg i. Br., 1923. 47 pp.

VOLTERRA (V.). See PÉRÈS (J.).

WHETHAM (M. D.). See WHETHAM (W. C. D.).

WHETHAM (W. C. D.) and WHETHAM (M. D.). Cambridge readings in the literature of science. Being extracts from the writings of papers of science to illustrate the development of scientific thought. Cambridge, University Press, 1924. 11 + 275 pp.

PART II. APPLIED MATHEMATICS.

D'AINVILLE (J. V.). L'origine tourbillonnaire de l'atome et ses conséquences. Paris, Gauthier-Villars, 1924. 215 pp.

AUDUBERT (R.). Cours d'électrochimie. Paris, Librairie de l'Enseignement Technique, 1924. 320 pp.

BACK (E.) und LANDÉ (A.). Zeeman-effekt und Multiplettstruktur der Spektrallinien. Berlin, Springer, 1924. 160 pp.

BARR (J. H.). See KIMBALL (D. S.).

BECQUEREL (J.). La radioactivité et les transformations des éléments. Paris, Payot, 1924. 208 pp.

BELOT (E.). L'origine dualiste des mondes et la structure de notre univers. Paris, Payot, 1924.

TYCHONIS BRAHE Dani opera omnia. Edidit I. L. E. Dreyer. Tomus 7. Hauniae, Libraria Gyldendaliana, 1924. 5 + 422 pp.

BREGUET (L.). Le vol à voile dynamique des oiseaux. Paris, Gauthier-Villars, 1925. 15 + 60 pp.

BRUHAT (G.). Cours d'électricité à l'usage de l'enseignement supérieure, scientifique et technique. Paris, Masson, 1924.

BRUTZKUS (M.). Contribution à la théorie des moteurs à combustion interne. Paris, Gauthier-Villars, 1923. 78 pp.

CHAUVEAU (B.). Electricité atmosphérique. 3e fascicule. Paris, Doin, 1924. 240 pp.

CONSTAN (P.) Cours d'astronomie et de navigation. Nouvelle édition refondue. Tome 1: Astronomie. Tome 2: Navigation. Paris, Gauthier-Villars, 1924. 8 + 318 + 454 pp.

CROWTHER (J. A.). Ions, electrons and ionizing radiations. 4th edition. London, Arnold, 1924. 12 + 328 pp.

CURIE (M.). L'isotopie et les éléments isotopes. Paris, Les Presses Universitaires de France, 1924. 210 pp.

DENTON (F.). Relativity and common sense. Cambridge, University Press, 1924. 17 + 279 pp.

DESARGUES (H.). Cours de mécanique théorique et appliquée. Paris, Doin, 1925. 415 pp.

DIJKSTERHUIS (E. J.). Val en worp. Een bijdrage tot de geschiedenis der mechanica van Aristoteles tot Newton. Groningen, Noordhoff, 1924. 476 pp.

DINGLE (H.). Modern astrophysics. London, Collins, 1924. 28 + 420 pp.

DOCK (H.). Photogrammetrie und Stereophotogrammetrie. (Sammlung Goschen.) Berlin, de Gruyter, 1923. 133 pp.

DREYER (I. L. E.). See TYCHO BRAHE.

Dubs (R.). See Escher (A.).

Ebner (M.). Ausführliche Stoffauswahl fur die Lehrpläne im wissenschaftlichen Zeichnen an den hoheren Lehranstalten. Leipzig, Teubner, 1924. 5 + 20 pp.

Elderton (W. P.) and Oakley (H. J. P.). The mortality of annuitants, 1900–1920. Investigation and tables. London, C. and E. Layton, 1924. 4 + 207 pp.

Escher (R.). Die Theorie der Wasserturbinen. Ein kurzes Lehrbuch. 3te vermehrte und verbesserte Auflage, herausgegeben von R. Dubs. Berlin, Springer, 1924. 370 pp.

Franklin (W. S.) and MacNutt (B.). Mechanics. A text book for colleges and technical schools. Lancaster, Pa., 1923. 8 + 226 pp.

Freundlich (E.). The foundations of Einstein's theory of gravitation. London, Methuen, 1924. 16 + 140 pp.

Furtmayr (J.). See Haren (R.).

Haas (A.). Atomtheorie in elementarer Darstellung. Berlin, de Gruyter, 1924. 8 + 204 pp.

——. Introduction to theoretical physics. Translated from the third and fourth editions by T. Verschoyle. London, Constable, 1924. 14 + 331 pp.

Hagen (J. G.) und Stein (J.), herausgegeben von. Die veränderlichen Sterne. Band 2: Mathematisch-physikalischer Teil, von J. Stein. Freiburg i. Br., Herder, 1924. 20 + 383 pp.

Hagenbach (A.). Der elektrische Lichtbogen. 2te Auflage. Leipzig, Akademische Verlagsgesellschaft, 1924. 12 + 282 pp.

Haren (R.). Aufgabensammlung zur Festigkeitslehre mit Lösungen. 3te, vollständig neubearbeitete Auflage von J. Furtmayr. (Sammlung Goschen.) Berlin, de Gruyter, 1923. 116 pp.

Hellmuth (F.) und Wernli (F.). Neuzeitliche Vorkalkulation im Maschinenbau. Berlin, Springer, 1924. 5 + 219 pp.

Herold (K.). Finanz-Mathematik (Zinseszinsen-, Anleihe- und Kurs-Rechnung). (Mathematisch-Physikalische Bibliothek, Nr. 56.) Leipzig, Teubner, 1924. 4 + 50 pp.

von Hevesy (G.) und Paneth (F.). Lehrbuch der Radioaktivitat. Leipzig, Barth, 1923. 10 + 213 pp.

Heyl (P. R.). The common sense of the theory of relativity. Baltimore, Williams and Wilkins, 1924. 44 pp.

Hinrichs (W.). Einfuhrung in die geometrische Optik. 2te Auflage. (Sammlung Göschen.) Berlin, de Gruyter, 1924. 144 pp.

Hohn (E.). Über die Festigkeit elektrisch geschweisster Hohlkörper. Berlin, Springer, 1924. 130 pp.

KAUFMANN (W.). Statik. Teil IV: Brücken- und Ingenieurhochbau. Band 1. Berlin, Springer, 1923. 7 + 350 pp.

KIMBALL (D. S.) and BARR (J. H.). Elements of machine design. London, Chapman and Hall, 1924. 446 pp.

KISCH (B.). Fachausdrucke der physikalischen Chemie. Ein Wörterbuch. 2te, vermehrte und verbesserte Auflage. Berlin, Springer, 1923. 4 + 100 pp.

KOSSEL (W.). Valenzkrafte und Rontgenspektren. Zwei Aufsatze über das Elektronengebäude des Atoms. 2te vermehrte Auflage. Berlin, Springer, 1924. 4 + 89 pp.

KOTTCAMP (P.). Strength of materials. 2d edition. London, Chapman and Hall, 1924.

LANDÉ (A.). See BACK (E.).

LARIVIÈRE (A.). Les métamorphoses de la force. Paris, Librairie Desforges, 1924. 305 pp.

LEDUC (A.). Volumes moléculaires. Applications. Paris, Doin, 1923. 120 pp.

MACNUTT (B.). See FRANKLIN (W. S.).

MANNING (G. P.). Reinforced concrete design. London, Longmans, 1924. 500 pp.

MARCOLONGO (R.). Relatività. 2a edizione riveduta ed ampliata. Messina, Casa Editrice Giuseppe Principato, 1923. 12 + 235 pp.

NATUCCI (A.). La relatività e la fisica dell'atomo. Palermo, Sandron, 1924. 155 pp.

NORDMANN (C.). Notre maitre le temps; les astres et les heures; Einstein ou Bergson? Paris, Hachette, 1924.

OAKLEY (H. J. P.). See ELDERTON (W. P.).

OTTENHEIMER (J.). La balistique extérieure. Paris, Colin, 1924. 200 pp.

PANETH (F.). See VON HEVESY (G.).

PARTINGTON (J. R.). Chemical thermodynamics. An introduction to general thermodynamics and its applications to chemistry. London, Constable, 1924.

PHILIPPOT (H.). Expression analytique des variations de la température de l'air. Bruxelles, M. Hayez, 1924. 48 pp.

PICART (L.). Astronomie générale. Paris, Colin, 1924. 188 pp.

POSCHL (T.). Lehrbuch der Hydraulik fur Ingenieure und Physiker. Berlin, Springer, 1924. 6 + 192 pp.

RANDELL (W. L.). Michael Faraday. Boston, Small, Maynard and Company, 1924. 192 pp.

van Rijn (P. J.). Third report on the progress of the plan of selected areas, together with some remarks concerning future investigations of the plan. Groningen, Hoitsema Brothers, 1923. 4 + 96 pp.

Rinne (F.). Crystals and the fine structure of matter. Translated by W. S. Stiles. London, Methuen, 1924. 9 + 195 pp.

von Seeliger (H.). Probleme der Astronomie. Festschrift fur Hugo von Seeliger dem Forscher und Lehrer zum fünfundsiebzigsten Geburtstage. Berlin, Springer, 1924. 4 + 475 pp.

Smith (J. D. M.). Chemistry and atomic structure. London, Benn, 1924. 221 pp.

Stein (J.). See Hagen (J. G.).

Stiles (W. S.). See Rinne (F.).

Swain (G. F.). Structural engineering. Strength of materials. New York and London, McGraw-Hill, 1924. 12 + 569 pp.

Tammann (G.). Lehrbuch der heterogenen Gleichgewichte. Braunschweig, Vieweg, 1924. 12 + 358 pp.

Turpain (A.). Conférences scientifiques. Fascicule 1: Le nouveau domaine de l'électricité; l'évolution des théories électriques, Paris, Gauthier-Villars, 1924. 10 + 72 pp.

Verschoyle (T.). See Haas (A.).

Urbain (G.). L'énergétique des réactions chimiques. Paris, Doin, 1925.

Wernli (F.). See Hellmuth (F.).

Whetham (W. C. D.). Matter and change. An introduction to physical and chemical science. Cambridge, University Press, 1924. 8 + 280 pp.

Wieghardt (E.). Mathematische Geographie und Astronomie. München, Oldenbourg, 1924. 9 + 127 pp.

Williams (A. T.). Influencia de la selfinduccion y de la dilucion en la persistencia de las lineas espectrales; las lineas ultimas y la teoria quantica de las lineas opticos. Buenos Aires, Coni, 1924. 41 pp.

Wittenbauer (F.). Graphische Dynamik. Berlin, Springer, 1923. 797 pp.

Witz (A.). Thermodynamique à l'usage des ingénieurs. 4e édition. Paris, Gauthier-Villars, 1924. 334 pp.

Wyckoff (R. W. G.). The structure of crystals. New York, Chemical Catalogue Company, 1924. 462 pp.

Yule (G. U.). An introduction to the theory of statistics. 7th edition, revised. London, Griffin, 1924. 431 pp.

Zehnder (L.). Die zyklische Sonnenbahn als Ursache der Sonnenfleckenperioden. Halle, Verlag von L. Hoffstetter, 1923. 44 pp.

Book No ....
LIBRARY.

THE FORTY-FIFTH REGULAR MEETING OF THE SAN FRANCISCO SECTION

The forty-fifth regular meeting of the San Francisco Section of this Society was held at Stanford University on April 4, 1925. Professor E. R. Hedrick presided at the morning session and Professor A. F. Carpenter at the afternoon session. The total attendance was thirty-six, including the following twenty-six members of the Society:

Allardice, A. D. B. Andrews, Bernstein, Blichfeldt, Buck, Cajori, A. F. Carpenter, Daus, M. W. Haskell, E. R. Hedrick, Hoskins, Hotelling, Glenn James, Lehmer, Sophia Levy, E. W. Mc Donald, J. H. Mc Donald, W. A. Manning, Mathewson, Moreno, F. R. Morris, Noble, Pauline Sperry, Stager, A. R. Williams, Wong.

It was voted that, beginning with the next Fall meeting, the retiring Chairman deliver an address before the Section.

The dates and places of the next three regular meetings of the Section were fixed as follows: June 19, 1925, University of Oregon; October 31, 1925, University of California; April 3, 1926, Stanford University.

Titles and abstracts of papers read at this meeting follow. The papers of Dr. Barter, Dr. Mullemeister, and of Professors Bateman, Bell, Moritz were read by title.

1. Dr. John Barter: *Notes on algebraic theory. Part* I. *The linear operator.*

The present paper developes the theory of linear transformations in such directions as seem essential to application to the general algebraic theory. The theory of λ-matrices is treated with some detail from a new angle and by a method applicable to λ-matrices of any degree. The new conception of poles of different orders is made a basis for a classification of these matrices. The classic theory of elementary divisors appears as a particular case of the general theory. Automorphic transformations are treated incidentally and in an introductory manner from this standpoint.

2. Professor Harry Bateman: *An extension of Lagrange's expansion.*

When z is defined by means of the equation $z = a + x\Phi(z)$, Lagrange's theorem furnishes the expansion, in ascending powers of x, of an arbitrary function $f(z)$. The present extension of the expansion gives a corresponding series for the definite integral

$$\int_a^z F[a + x\Phi(\tau) - \tau, \tau]\, d\tau$$

where F is an arbitrary function of its two arguments. The series was suggested by a generalization of E. T. Whittaker's solution of Laplace's equation.

3. Professor E. T. Bell: *The chi functions of Glaisher and class numbers.*

The function $\chi_r(n)$ of Glaisher is the sum of the rth powers of all Gaussian primary complex numbers having norm n. This function is here expressed in terms of binary quadratic class numbers, and several new class number relations are indicated.

4. Professor B. A. Bernstein: *On the existence of fields in boolean algebras.*

The author proves that $xy' + x'y$, xy; $xy + x'y'$, $x + y$ (x' the negative of x) are the only pairs of operations for which the elements of a two-element boolean algebra are a field, and that there exist no field operations expressible in terms of addition, multiplication, and negation for the totality of the elements of any other boolean algebra.

5. Professor H. F. Blichfeldt: *On the minimum value of positive real quadratic forms in 6 variables.*

Let f represent a positive real quadratic form in n variables, having a determinant D. Then we may substitute integers, not all zero, for the variables such that the numerical value of f is not greater than $\gamma_n \cdot \sqrt[n]{|D|}$; where γ_n is a function of n only. Superior limits to γ_n have been given for any n, as well as the exact minima for $n = 2, 3, 4, 5$ (cf. TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), p. 233). In the present paper the author shows that the exact minimum for $n = 6$ is $2/\sqrt[6]{3}$.

6. Professor Florian Cajori: *American contributions to mathematical notations.*

The author makes reference to the Maya numeral notation marking the earliest regular use of a sign for zero and the principle of local value, the Peruvian knot records, the dollar mark, a sporadic notation for radicals, the sign $=\!\!\!8\!\!\!=$ for equivalence, B. Peirce's symbols for π, e and i, C. S. Peirce's notation in symbolic logic, the $\doteq$ for approach to the limit, J. W. Gibbs's vector notation, E. H. Moore's notation in general analysis. As yet, none of these have found permanent adoption in Europe, except perhaps the dollar mark.

7. Professor Florian Cajori: *Early appearance in print of the dollar mark.*

The author shows that the dollar mark originated from the superposition of the letters p and s in "pesos". The frequent statement that the first printed appearance of the sign \$ to represent dollars is in Chauncey Lee's *American Accomptant*, 1797, was shown to be inaccurate. The \$ occurred in print in the *Scholars' Arithmetic* of Daniel Adams (4th. ed., 1807) and began to be used freely in newspapers about 1810.

8. Professor A. F. Carpenter: *Two point-line correspondences determined by a ruled surface.*

By means of the two quadrics which osculate the two sheets of the flecnode surface of a ruled surface along a pair of corresponding line elements, two point-line correspondences of space are set up. These correspondences are such that all points of space are made to correspond, in two ways, to the lines of a congruence.

The paper also determines for numerous point loci of space, the corresponding line loci.

9. Professor P. H. Daus: *Certain recursion formulas connected with the solution of $x^2+y^2=z^2$.*

If

$$s_1 = (x_1, y_1, z_1)$$

is the smallest primitive solution of

$$x^2+y^2=z^2,$$

where

$$y = x + m,$$

then all solutions of these equations when m is a prime or the power of a prime are given by the formulas

$$s_{2n+1} = ps_{2n} - ps_{2n-1} + s_{2n-2};$$

$$s_{2n+2} = qs_{2n+1} - qs_{2n} + s_{2n-1};$$

where p and q depend on s_1. The solutions divide themselves into two sets, according as n of s_n is even or odd. Within each set the solutions are connected by the relation

$$s_{n+2} = 7s_{n+1} - 7s_n + s_{n-1},$$

independent of the value of m.

10. Professor M. W. Haskell: *Autopolar configurations in space.*

The method of finding multiply self-polar curves presented to the International Congress at Toronto can be extended to configurations in space. There are certain modifications to be noted, in particular with regard to the nature of the combination of a polarity with a collineation group of which the quadric surface in question is an invariant. Perspective reflexions always give rise to new polarities, while skew reflexions are of two kinds. If the invariant lines of a skew reflexion are polar to each other, then the product of the collineation and the given polarity is a second polarity; while if they are self-polar (i. e. tangent to the given quadric surface) the resulting correlation is a null-system.

For example, the Kummer surface is invariant under a group of 16 collineations of period 2 (all *skew* reflexions) and 16 correlations, of which 10 are polarities with respect to quadric surfaces and 6 are polarities with respect to null-systems.

11. Professor E. R. Hedrick: *Extensions of Picard's theorem to non-analytic functions.*

The author points out that there are a number of rather general cases in which extensions of Picard's theorem to non-analytic functions are possible, and several specific forms of the extended theorem are given. Some limitations on possible extensions are stated, for functions that are non-analytic.

12. Dr. Harold Hotelling: *Theory of mine economics.*

The equations of competition are modified in the case of exhaustible resources by the addition to each equation of a term of the form $\lambda_i e^{\delta t}$, where δ is the force of interest and the λ_i's depend upon the amounts of the resource in the hands of the various entrepreneurs. Although the the equations of competition of inexhaustible resources admit of solution when demand is assumed perfectly inelastic, this assumption in the present case leads to an absurdity. Though natural resources may in some cases be exploited too rapidly under competition, the analysis shows that in other cases the exploitation is slower than the public interest requires.

13. Dr. Harold Hotelling: *Manifolds of states of motion as multiple-sheeted spaces.*

The boundaries of a pair of plane rings which link form the branch curves of a $(2p+2)$-sheeted space homeomorphic with the manifold of states of motion on an orientable surface of genus $p>0$. For a one-sided surface of the same connectivity the representation is by branch curves of the same type in a space of $2p+4$ sheets. For the representation of the states of motion on a one-sided surface of the first kind the branch system consists of these four curves and two others, each of the latter linking doubly the other and also one of the plane rings.

14. Professor Glenn James: *On the solution of equations of degree higher than the fifth.*

This paper developes a formula for one root of the equation

$$x^n+\alpha_1 x^{n-1}+\alpha_2 x^{n-2}+\cdots+\alpha_n=0,$$

when n is odd or when n is even and α_n is negative, and presents a theoretically possible method of solving this equation for any positive integral value of n.

15. Professor D. N. Lehmer: *A new method of finding factors of numbers.*

The author's method is an improvement on the well known method which depends on the theory of quadratic residues. By means of a set of stencils the divisors which belong to a set of linear forms are made apparent at once. The method is available for numbers as high as ten billions.

16. Professor J. H. McDonald: *The equation of the elliptic cylinder.*

The solution of the linear differential equation of the second order which belongs to the elliptic cylinder is given for the general case. The equation is of a type not derivable by specialization from the hypergeometric differential equation and has a non-regular singular point at infinity. The problem is to determine the characteristic exponent and the coefficients of the series giving the solution, the coefficients being functions of the characteristic exponent.

The differential equation depends on two parameters and is first transformed into a Riccati equation which necessarily has a periodic solution. This is obtained in the form of a series of powers of one of the parameters, the coefficients being periodic functions of the independent variable. The sum of the constant terms in these coefficients is a power series in one parameter, the coefficients being rational functions of the other. This series is a branch of the characteristic exponent. The coefficients of the solution are next given explicitly by means of infinite determinants. The equation for the characteristic exponent which arises in determining the coefficients is put in a form such that all branches of the exponent can be determined.

17. Professor J. H. McDonald: *The periodic solutions of the equation of the elliptic cylinder.*

The solution of the equation of the elliptic cylinder is not periodic in general but only if a special relation holds between the parameters. The forms of a few of the periodic solutions have been determined but by a method which does not seem capable of being extended to the general case. In this paper a method is given by which they may all be obtained, and the equation between the parameters which must be satisfied for the existence of the periodic solution is completely solved. The periodic solutions are known to be the solutions of a particular integral equation. Its general solution and the characteristic numbers are accordingly determined.

18. Professor J. H. McDonald: *On the periodic solution of a certain differential equation.*

A problem in motion under a periodic disturbing force leads to the differential equation

$$\frac{d^2x}{dt^2} - mx\left(\frac{dx}{dt}\right)^2 + n^2x = a \sin pt.$$

This equation has a single periodic solution if n^2/p^2 is not equal to the square of an odd integer. The solution is of the form

$$\sum_1^\infty a_k \sin(2k-1)pt$$

and the coefficients a_k are all determined. The method used is that of solving a succession of differential equations which have the given equation as a limiting form; the solution at each step being chosen so that a succession of functions is determined which converges to a solution of the problem.

19. Professor J. H. McDonald: *The transformation of elliptic integrals.*

The theory of transformation of elliptic integrals given previously by the writer for the general case is applied to the transformation of the seventh order. The modular equation is obtained in the appropriate form and the results compared with those of Kiepert and Joubert.

20. Professor J. H. McDonald: *On the reduction of the singularities of algebraic curves.*

Halphen enunciated the theorem that an algebraic curve can be transformed birationally into one whose multiple points are double points with distinct tangents. His proof was defective but proofs have been given by Poincaré, Bertini, and others. A short proof of the theorem is given by determining a space curve whose projection is the required curve with reduced singularities.

21. Professor R. E. Moritz: *On the kinematic construction of higher plane curves.*

Professor Moritz considers the various loci which result from a composition of certain elementary motions which can readily be effected kinematically. The equations of these motions give in each case the simplest possible parametric equations of the loci under consideration. This paper will appear in the AMERICAN MATHEMATICAL MONTHLY.

22. Dr. Hermance Mullemeister: *A projection of the lines in space into pairs of lines in a plane.*

The four-dimensional line-space is projected from two centers C_1 and C_2 into pairs of lines in a plane. An axial complex on $C_1 C_2$ is projected into a pencil of lines on C_0, the intersection of $C_1 C_2$ and the picture plane α. A flat pencil is projected into two perspective pencils in α, whose vertices P_1 and P_2 are collinear with C_0 and whose axis of perspectivity is the intersection of the two planes. Two pencils P_1 and P_2 in α which are projective, but not perspective, constitute the image of a quadratic ruled system, whose axes are P_1C_1 and P_2C_2. A quadratic ruled system which does not pass through C_1 and C_2, is projected into two projective line-conics which have two common tangents through C_0.

A line-field in a plane β is projected into two perspective line-fields in α having C_0 as center of perspectivity and the intersection of α and β as axis of perspectivity. A line-sheaf on O is projected into two pencils whose vertices are the projections of O from C_1 and C_2, which are collinear with C_0. A bilinear congruence is projected into pairs of lines such that to every line-pencil (l_1) corresponds a line-conic μ_2^2, and to every line-pencil (l_2) corresponds a line-conic μ_1^2.

B. A. BERNSTEIN,
Secretary of the Section.

THE APRIL MEETING IN CHICAGO

The twenty-third Western meeting of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 10 and 11, 1925.

Ninety-six persons registered at this meeting, among whom were the following eighty-one members of the Society:

F. E. Allen, R. W. Babcock, Barnard, W. S. Beckwith, H. A. Bender, Birkhoff, Bubb, C. C. Camp, Carmichael, Carr, Chittenden, Crathorne, Curtiss, Dickson, Dostal, Dresden, Emch, Feltges, C. A. Garabedian, Garver, Gorrell, V. G. Grove, W. L. Hart, Hildebrandt, Holl, L. A. Hopkins, Mildred Hunt, Huntington, Louis Ingold, Ingraham, Dunham Jackson, Kempner, Kerékjártó, Kinney, Kouperman. Krathwohl, E. P. Lane, Laves, Libman, Logsdon, MacDuffee, W. D. MacMillan, March, William Marshall, T E. Mason, Mathews, B. I. Miller, G. A. Miller, Miser, E. H. Moore, E. J. Moulton, F. S. Nowlan, Parkinson, Reinsch, H. L. Rietz, Roever, Roth, Rowland, Runge, Schottenfels, Showman, Simmons, W. G. Simon, E. B. Skinner, Slaught, E. R. Smith, H. L. Smith, Steimley, Stouffer, Teach, E. L. Thompson, Townsend. J. S. Turner, Van Vleck, Wahlin, E. M. Weaver, K. P. Williams, Frederick Wood, F. E. Wood, J. M. Young, J. W. A. Young.

At the Council meeting the following elections to membership took place:

To sustaining membership:

Amherst College, Amherst, Mass.;
Harvard University, Cambridge, Mass.;
The Maccabees, Detroit, Mich.;
The Missouri State Life Insurance Company, St. Louis, Mo.;
The Pacific Mutual Life Insurance Company, Los Angeles, Cal.;
The Prudential Insurance Company, Newark, N. J.;
The Travelers Insurance Company, Hartford, Conn.

To ordinary membership:

Professor Ethel Beatrice Callahan, Cedar Crest College;
Mr. Claude Bernhart Dansby, Morehouse College;
Mr. Harold John Hornberger, Great Northern Life Insurance Company;
Professor Béla de Kerékjártó, University of Szeged;
Mr. Donald D. Laun, University of Chicago;
Rev. Paul Muehlman, Loyola University;
President Ellen Fitz Pendleton, Wellesley College;
Mr. Clarence Edward Rose, Arkansas Cold Storage Company;
Dr. Charles Edward St. John, Mount Wilson Observatory,
Professor Jennie Leatitia Tate, McMurry College;
Rev. George Warren Walker, Margaretville, N. Y.

Nominees of the John Hancock Life Insurance Company, Boston, Mass.:

Eleanor Alice Abbott; Harold A. Garabedian; Harold Alden Grant; L H. Howe; Earl M. Thomas.

Nominees of the Connecticut Mutual Life Insurance Company, Hartford, Conn.:

William Pond Barber, Jr.; Thomas Kilburn Dodd; Gladstone Marshall; Leslie R. Martin; H. I. B. Rice.

Nominees of the National Life Insurance Company, Montpelier, Vt.:

Andrew John Blackmore; Austin Harris Hobson; Henry Holt; Henry Hollister Jackson; Clarence Edgeston Moulton.

Nominees of the Edison Electric Illuminating Company, Boston, Mass.:

Leonard L. Elden; Robert S. Hale; Harold Cyrle Hamilton; Sidney Hosmer; Irving E. Moultrop.

The Assistant Secretary announced the following elections by mail vote of the Council:

To sustaining membership:

The Aetna Life Insurance Company, Hartford, Conn.;
The American Life Insurance Company, Detroit, Mich.;
The Connecticut Mutual Life Insurance Company, Hartford, Conn.;
The Detroit Life Insurance Company, Detroit, Mich.;
The Eastman Kodak Company, Rochester, N. Y.;
The Edison Electric Illuminating Company, Boston, Mass.,.
The Equitable Life Insurance Company of Iowa, Des Moines, Ia.;
The John Hancock Mutual Life Insurance Company, Boston, Mass.;
The Metropolitan Life Insurance Company, New York, N. Y.;
The National Life Insurance Company, Montpelier, Vt.;
The National Life Insurance Company of the United States of America, Chicago, Ill.;
The New England Life Insurance Company, Boston, Mass.

To ordinary membership:

Dr. Carl Louis Alsberg, Food Research Institute, Stanford University;
Miss Rose Lucile Anderson, Bryn Mawr College;
Mr. Frank Ayres, Agricultural and Mechanical College of Texas;
Mr. Lancelot Minor Berkeley, New York City;
Professor Durga Prasanna Bhattacharyya, Bareilly College;
Mr. Perry Aquila Caris, Philadelphia;
Mr. Edwin R. Carter, National Life Insurance Company, Chicago;
Professor William Frederick Durand, Stanford University;
Mr. James Drouie Flynn, Travelers Insurance Company, Hartford;
Mr. Thomas Henry Galbraith, Central Life Insurance Company of Illinois, Chicago;
Miss Marion Cameron Gray, Bryn Mawr College;
Mr. Leroy Francis Harza, Chicago;
Mr. Ray Nelson Haskell, Michigan Agricultural College;
Professor Mildred Hunt, Illinois Wesleyan University;
Dr. Carey M. Jensen, University of Wisconsin;
Professor Reginald Stevens Kimball, State Normal School, Worcester, Mass.;
Miss Helen Kunte, Hunter College;
Professor Anna Delia Lewis, Lake Erie College;
Professor John Hector McDonald, University of California;
Miss Laura Frances McDonough, Moravian College for Women;
Dr. Lee Horace McFarlan, University of Missouri;
Professor E. Loula McNeer, Morris Harvey College;

Professor Braxton Davis Mayo, Virginia Military Institute;
Mr. Earl Lewis Mickelson, Hamline University;
Professor Rayburn Z. Newson, Rusk College;
Mr. Frank Ordway, Phillips High School, Birmingham, Ala.;
Professor Delfin de la Paz, University of the Philippines;
Professor Carl John Rees, University of Delaware;
Mr. Charles Frederick Roos, Rice Institute;
Miss Elizabeth Thatcher Stafford, University of Texas;
Mrs. Ormelle Haines Stecker, Pennsylvania State College;
Mr. Henry J. Sternberg, Columbia University;
Professor Telesforo Tienzo, University of the Philippines;
Mr. William Arthur Watt, Pilot Life Insurance Company, Greensboro, N. C.;
Miss E. Kathryn Wyant, University of Missouri.

Nominees of Allyn and Bacon, Boston, Mass.:

Max G. Carman, University of Illinois; Alonzo Church, Princeton University, Orrin Frink, Columbia University; Raymond Garver, University of Chicago; John Haven Neelley, Yale University.

Nominees of the Connecticut General Life Insurance Company, Hartford, Conn.:

Walter Bjorn; Ward Van Buren Hart; Earl C. Henderson, Edward H. Hezlett; John Melvin Laird.

Nominees of the Eastman Kodak Company, Rochester, N. Y.:

Frederick E. Altman; F. M. Bishop; Charles Warnock Frederick, L. A. Jones; Ludwik Silberstein.

Phoenix Mutual Life Insurance Company, Hartford, Conn.:

Alden Thomson Bunyan; Henry W. Dewey; Henry N. Kaufman; Alson C. Patton; Harold Merle Springer.

Twenty-six applications for membership and thirty-two nominations by sustaining members were received.

The Council voted a resolution of thanks to Professor H. A. Perkins of Trinity College, Hartford, Conn., for the very valuable assistance he has rendered in connection with the endowment campaign, and to the Carnegie Corporation for a second contribution to the endowment fund.

The following appointments by President Birkhoff were announced:

To represent the Society at the Inauguration of Chancellor Benner at the University of Porto Rico, Professor C. G. P. Kuschke; to represent the Society at the Fiftieth Anniversary of the Founding of the George Peabody College for Teachers at Nashville, Tenn., Mr. Lewis C. Cox; to arrange for the next Josiah Willard Gibbs lecture, Professors E. R. Hedrick, Luby, E. H. Moore, Roever, and Stouffer

(Chairman); to consider the fees for colloquium lectures, Professors Archibald, H. H. Mitchell (Chairman), and Van Vleck; to consider the printing problems, Professors Cohen, Eisenhart (Chairman), Graustein, Hedrick, Slaught.

It was announced that the summer meeting and colloquium had been set for September 8–12, 1925, in Ithaca, N. Y.; that the Gibbs lecture in December, 1925, would be given by Professor J. Pierpont.

The Society decided to hold the Western Christmas meeting in Kansas City in conjunction with the meetings of the American Association for the Advancement of Science.

At the dinner held at the Del Prado Hotel on Friday evening, 67 persons were present. Professor Bliss presided and toasts were responded to by Professors E. H. Moore, Huntington, Kerékjártó, Stouffer, Birkhoff and Dresden. Professor Kerékjártó called attention to a new journal established in Hungary, to which further reference is made in the Notes in this issue.

On Saturday a resolution was passed thanking the Department of Mathematics of the University of Chicago for their hospitable reception.

The papers read at the meeting are listed below. Not mentioned in this list is the symposium address delivered on Friday afternoon by Professor W. D. MacMillan on *Some mathematical aspects of cosmology.* On Friday forenoon the Society met in two sections; the one for geometry, before which the papers numbered 1, 2, 3, 4, 5, 26, 27, and 30 were presented, was presided over by President Birkhoff; the other for algebra and theory of numbers, which heard the papers numbered 6, 7, 8, 9, 10, 11, 12, 13, 28, and 29, was presided over by Vice-President Hildebrandt. The latter also presided on Friday afternoon, while President Birkhoff took the chair during the closing session on Saturday forenoon.

The papers by Bennett, Caris, Reilly, Shohat, Wilson, and Ingraham's second paper were read by title; Mr. Jenkins was introduced to the Society by Professor J. W. Glover, and Mr. Robinson by Professor E. P. Lane.

1. Professor R. M. Mathews: *Cubic curves and desmic surfaces.*

The twelve points of contact of the four tangents from each of three collinear points on a cubic of the sixth class form a Hessian $(12_4, 16_3)$ configuration on the curve. The sides of the three quadrangles of the points meet by threes in a similar configuration on a conjugate cubic which meets the given one in the nine vertices of the diagonal triangles of the quadrangles. A desmic surface is a quartic surface with twelve nodes which group into three tetrahedra each pair of which is perspective from each vertex of the third. These twelve nodes determine a counter-set of twelve nodes and for each point P of space there are two desmic surfaces, one on each set of nodes. The projection from P of the space configuration upon an arbitrary plane gives the configuration of the cubics. Various properties of the surfaces and curves are derived, among them relations with a 16_6 configuration and with a Poncelet hexagon and associated Steiner lines.

2. Professor Rufus Crane: *Another poristic system of triangles.*

This paper discusses the loci of points of a variable triangle having incircle and nine-point circle fixed. The loci of the centroid, circumcenter, orthocenter, and Nagel point are circles. The locus of the excenters is a limaçon; that of vertices of the triangle is a bicircular quartic. The relations of certain fixed points to these loci are also discussed.

3. Professor V. G. Grove: *A general theory of nets on a surface.*

In this paper, we first refer the sustaining surface to the asymptotic net, and then to an arbitrary non-conjugate net. We thus find that all of the projective properties of the net are expressible in terms of quantities characterizing the surface, and two arbitrary functions determining the most general non-conjugate net on the surface. From the forms of the formulas involved, one can tell at a glance those properties of the net which are really properties of the net, those which are properties of either of the two one-parameter families of curves forming the net, and those which are properties entirely of the surface.

The R-reciprocal congruences associated with a net are Green reciprocals if and only if the tangents to the curves of the net form with the asymptotic tangents a constant cross ratio. There exists one and only one pair of Green reciprocal congruences that are in relation R with respect to a non-conjugate, non-asymptotic net.

4. Professor Arnold Emch: *On surfaces and curves which are invariant under involutory Cremona transformations.*

The present paper extends to projective spaces of n dimensions the investigations of the curves which are invariant under involutary Cremona transformations of two superposed planes begun in some papers in the TOHÔKU MATHEMATICAL JOURNAL.

If the transformation is given the form $\varrho x_i' = \varphi_i$, $i = 1, 2, \cdots, n+1$; and $\sigma x_i = \varphi_i(x')$, we form the Plückerian coordinates $\varrho p_{ik} = x_i\varphi_k - x_k\varphi_i$ of the line joining two corresponding points P and P'. These, as well as $\sigma q_{ik} = x_i\varphi_k + x_k\varphi_i$ are invariant under the transformation. By means of the identities $p_{ik}p_{jl} = q_{il}q_{jk} - q_{ij}q_{kl}$, it is found that in every involutory transformation in S_n there are two types of invariant surfaces (curves in S_2) representable by the equations $K(p_{12}, \cdots p_{ik} \cdots) = 0$, $L(q_{11}, q_{12}, \cdots, q_{ik}, \cdots, q_{ii}, \cdots) = 0$. When K is of even degree in the p_{ik}'s, then it may be classed with the form L. When K is of odd degree, then its square only belongs to the class L. Thus every invariant hypersurface, or its square, belongs to class L. Accordingly, invariant hypersurfaces belong to two classes: Those which belong to L but are not squares of K; those whose squares only belong to L. The second class is connected with the contact problem. As applications are given the quadratic transformation and its connection with the Cayley cubic surface, the Geiser transformation, transformations in S_3, the Schur sextic curve, the Enriques sextic surface, etc.

5. Dr. E. E. Libman: *Mean curvature curves on quadric surfaces.*

The name "mean curvature curve" is given to a curve (on a surface) at every point of which the mean curvature of the surface is the same. This paper is an investigation into the properties of such curves on quadric surfaces. Some of the more interesting results follow. The mean curvature curves on a quadric are algebraic curves of order twelve. They touch both cyclic sections along a conic, the central section by a plane normal to the cyclic sections. Along a mean curvature curve the tangent of the angle at which two generators intersect upon the curve is proportional to the square of the distance of the tangent plane from the center. The mean curvature curve on a central quadric along which the mean curvature of the quadric is zero is the locus of the intersections of perpendicular pairs of generators. It is a sphero-conic upon the director sphere. The zero mean curvature curve on a paraboloid is a conic section, the intersection of a plane normal to the direction of the infinite center.

6. Professor J. F. Reilly: *Interpolation formulas containing parameters.*

An interpolation formula which gives satisfactory results throughout one portion of a table may yield rather inaccurate values throughout other portions of the same table. In this paper formulas are developed containing one or two parameters, which may be adjusted to fit any portion of the table. Sprague's third order differences osculatory interpolation formula is obtained by specializing the parameters. The use of such formulas is illustrated by showing how to adjust the parameters in order to obtain the best interpolated values in the least squares sense.

7. Professor A. A. Bennett: *On sets of three consecutive integers which are quadratic residues of primes.*

This paper appears in full in the present number of this BULLETIN.

8. Mr. P. A. Caris: *A solution of the quadratic congruence, modulo p, $p = 8n+1$, n odd.*

Explicit solutions of the quadratic congruence $x^2 \equiv a \pmod{p}$, where p is prime and of the form $4q+3$ or of the form $8n+5$ have been obtained by Legendre and others. The present paper contains explicit solutions for $a = -1, 2$ and n when $p = 8n+1$ and for a in general when n is odd.

9. Professor C. C. MacDuffee: *The nullity of a matrix relative to a field.*

The row-nullity ϱ of a matrix x relative to a field F is defined as the number of linearly independent linear relations with coefficients in F among the rows of x; the column nullity $\varkappa$ is similarly defined. If a and b are two non-singular matrices with elements in F, then ϱ and $\varkappa$ are invariants of x under the transformation $x' = axb$.

Certain relative projective invariants of algebraic forms written as determinants give rise to invariant matrices, e. g., the hessian matrix $H \equiv (f_{x_i x_j})$ of f_n is transformed according to the equation $H' = \breve{a}Ha$ by the projective transformation of matrix a. Thus the row-nullity of H relative to the field of the coefficients is an arithmetic invariant h of the form f_n itself. The complete solution of Hesse's problem is stated very simply in terms of h. Moreover, if the rank of H is 1, $h = n - 1$, and f_n is a constant times a perfect nth power of a linear function.

10. Dr. H. A. Bender: *On groups of order p^m which contain an abelian subgroup of order p^{m-1}.*

In every group of order p^m, p being an odd prime number, which contains an abelian subgroup of order p^{m-1} the abelian subgroup must be transformed in the same manner as it is transformed by an isomorphism of order p. In this paper we establish the necessary conditions that an isomorphism be of order p, and with this we proceed to show the possible ways a central and a commutator subgroup may be selected so as to give groups of order p^m and to determine the number of distinct groups there are for a given central and a given commutator subgroup.

11. Dr. H A. Bender: *On orders of operators in the group of isomorphisms of prime power abelian groups.*

If $Q_n(x) = 0$ be the algebraic equation whose roots are the primitive nth roots of unity without repetition, then the group of isomorphisms of the abelian group of order p^m, p being a prime-number, and type $(1, 1, 1, \cdots)$ contains operators of orders $p^\alpha \Pi_d Q_d(p)$ where d varies through all the distinct divisors of $\alpha_1, \alpha_2, \cdots, a_n$. The α's may be distinct and are to assume all possible values which satisfy the relation $\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n=\beta\leqq m\ (\beta=0,1,2,\cdots,m)$ $(p^{\alpha-1}<m-\beta\leqq p^{\alpha})$.

12. Professor G. A. Miller: *Imprimitive substitution groups.*

Various possible definitions of imprimitive substitution groups are observed, including the following: A necessary and sufficient condition that a transitive group G is imprimitive is that some co-set of G with respect to a subgroup G_1 composed of all the substitutions of G which omit a fixed letter, generates an intransitive group, or that such a co-set generates a proper subgroup of G. It is also necessary and sufficient that G contains a set of letters which does not include all its letters and is transformed into itself by all the substitutions of G which transform a fixed letter of G into some other fixed letter. To every proper subgroup generated by a co-set of G with respect to G_1, or by more than one such co-set, there corresponds a system of imprimitivity of G, and vice versa.

The number of possible sets of systems of imprimitivity of a transitive group G each of which involves two letters is always odd whenever there is a least set of such systems, and there is an infinite number of imprimitive groups which have the common property that each of them involves any arbitrary given odd number of different such systems of imprimitivity. In a transitive group whose order is of the form p^m, p being an odd prime, the number of the different systems of imprimitivity each of which involves exactly p letters is always of the form $kp+1$. This is not necessarily the case when the order of G is not of the form p^m. In fact, for every value of p it is possible to construct imprimitive groups which have exactly two different systems with just p letters in one system.

13. Professor G. E. Wahlin: *On a problem in Diophantine analysis.*

The author develops a method for obtaining the solutions of a Diophantine equation in which one member is a quadratic form in n variables and rational integral coefficients; and the other member a product of two or more unknown factors. It is based on the construction of an algebra over the field of rational numbers and consideration of the ideals of certain quadratic fields which are contained in the algebra.

14. Dr. C. C. Camp: *Expansions in terms of solutions of partial differential equations.* Third paper, involving partial differential equations possessing a fundamental set of solutions.

There is a striking analogy between the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{2\partial^2 u}{\partial x \partial y}+\frac{\partial^2 u}{\partial y^2}+a_1(x, y)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)+a_0(x, y) u=0,$$

where the a's are analytic near $(0, 0)$, and an ordinary linear differential equation of the second order. By employing the operator $D=\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$ as the analog of ordinary differentiation, one obtains a fundamental set of independent solutions such that the general solution may be written $u=\varphi_1 u_1+\varphi_2 u_2$. Here the φ's are functions of $x-y$. The initial conditions $u(x, 0)=p(x)$, and $Du(x, 0)=q(x)$, and analytic, determine a unique analytic solution. The theory extends to equations of higher order as well as to those of the first order. For the latter by Lagrange's method of solution one obtains a relation $y=f(x)-h$. With suitable boundary conditions the problem of expansion of an arbitrary function $f(x, y)$ involves a Birkhoff expansion along this curve and the coefficients are obtained by line integrals. The series converges to the mean value as illustrated by the expansion of xy in the case of the equation $Du+\lambda u=0$.

15. Professor K. P. Williams: *Non-synchronized relative invariant integrals.*

The paper deals with relative linear invariant integrals of the system $dz_i/dt=Z_i(z_1, \cdots, z_m)$. Such integrals are taken along closed paths drawn on a tube of trajectories. If the path passes through points that correspond to the same values of t we have a synchronized invariant $J=\int \sum L_i \delta z_i$. If it passes through points corresponding to different values of t we have a non-synchronized invariant $I=\int \sum L_i \delta z_i+K \delta t$. The paper develops a criterion on K so as to render I invariant if J is invariant.

16. Mr. J. A. Shohat: *On some properties of polynomials.*

In this paper the author considers integrals of the type $\int_a^b(x-a)^{\alpha-1}(b-x)^{\beta-1} q(x)\, M(x)\, dx\, (\alpha, \beta>0,\ q(x) \geqq q$ in

(a, b), $M(x)$ a polynomial; (a, b) finite). Using formulas previously established by the author in the theory of orthogonal Tchebycheff polynomials, he obtains some inequalities involving the maximum and minimum of $M(x)$, $M'(x)$ in (a, b); for example, a precise form of the law of the mean for polynomials. These results are applied, in particular, to polynomials which are monotonic in (a, b). A formula similar to one developed in this paper was given by Tchebycheff, using the theory of polynomials deviating the least from zero.

17. Professor H. L. Smith: *On functions of closest approximation.*

In the first part of this paper the notion of linear independence of n functions $p_1(x), \cdots, p_n(x)$, bounded in (ab), relative to a function $u(x)$ of limited variation on (ab) is defined. The existence of a linear combination $(c)(p) = c_1p_1 + \cdots + c_np_n$ for which the Stieltjes integral $\int_a^b |f-(c)(p)|^m du$ has its minimum value is then proved where f is a fixed function integrable (u) and m a fixed number $\geqq 1$. The minimizing function is proved to be unique when $m > 1$. A theorem is developed concerning the sequence of approximating functions corresponding to a convergent sequence of functions u and the theorem is applied to the study of the minimizing of finite sums.

In the second part the functions $p_1, \cdots, p_n$ are replaced by $(2n+1)$ trigonometric functions, the convergence of the approximating function to f as n approaches infinity is studied and the results are applied to the case of finite sums.

18. Professor Dunham Jackson: *On the convergence of certain processes of closest approximation over an infinite interval.*

This paper is concerned with polynomials affording the closest approximation to a given function over an infinite interval, according to the criterion of least squares (or least mth powers), with a weight function of appropriate character. Mr. Shohat (BULLETIN, Nov.–Dec., 1924) has discussed some of the problems relating to such polynomials of approximation. It is shown in the present paper that under specified hypotheses the approximating polynomial will converge to the value of the given function, as the degree of the polynomial is indefinitely increased. The analysis applies in particular to the series of Hermite's polynomials used in the theory of statistics under the name of the Gram-Charlier series.

19. Professor Dunham Jackson: *On vector analysis in function space.* Preliminary communication.

This paper contains a further development of certain ideas which were presented for consideration at an earlier meeting of the Society (October, 1924). In particular, there is a discussion of a functional operator analogous to the divergence of a vector point function in ordinary space.

20. Professor W. H. Wilson: *Two related functional equations.*

The equation $f(x+y)=f(x)g(y)+g(x)f(y)+2\lambda f(x)f(y)$, where λ is a constant which may be zero, is discussed in part I of this paper and the functions, $f(x)$ and $g(x)$, satisfying this equation are expressed in terms of the functions $\varphi(x)$ and $\psi(x)$ whose addition theorems are $\varphi(x+y)=\varphi(x)+\varphi(y)$ and $\psi(x+y)=\psi(x)\,\psi(y)$, respectively. In part II the equation $g(x+y)=g(x)g(y)+\mu^2 f(x)f(y)$, where μ is a constant, is discussed and the functions, $f(x)$ and $g(x)$, satisfying this equation are expressed in terms of $\varphi(x)$ and $\psi(x)$.

21. Dr. R. W. Babcock: *On thermal convection.*

The differential equations which apply to slow steady convective motion of a viscous fluid are the standard equations of hydrodynamical theory, combined with equations expressing the law of thermal interchanges in a moving fluid. In the present paper approximate solutions are found following the method outlined by Oberbeck (ANNALEN DER PHYSIK, 1879, p. 271). A comparative study of the exact solution of a special system of non-homogeneous ordinary equations and its Oberbeck approximate solution shows that the magnitude of the constants in the system controls the rate of convergence of the approximation, and often produces divergence. Solutions with graphs of temperature and velocity fields have been computed for special cases of plane and cylindrical boundaries with sinusoidal impressed boundary temperatures. If this temperature has a vertical gradient only, the Oberbeck method produces a null solution, corresponding to equilibrium of the fluid in stratified layers. Arithmetical solutions for common liquids at ordinary temperatures show that the Oberbeck solutions diverge unless the gradient is much too small to be easily applied in laboratory practice.

22. Professor H. W. March: *The deflection of a rectangular plate with two opposite edges supported and two edges free.*

Solutions of this problem for the case of a uniform load and of other problems relating to the rectangular plate with two opposite edges supported were given by Estanave (ANNALES DE L'ECOLE NORMALE, 1900) using the method of Lévy. This paper gives a simpler solution of the problem for the case of a uniform load and also a solution for the case of a load which is a function of x alone, x being measured in the direction perpendicular to the supported edges. The simplification effected results from first choosing a solution of the non-homogeneous differential equation which satisfies part of the boundary conditions and then adding to this a suitably chosen solution of the corresponding homogeneous equation.

23. Professor M. H. Ingraham: *Solution of certain functional equations relative to a general number system.*

Relative to any normally-orderable associative division number system $\mathfrak{A}$, there exists a range P and a field F, either the field of rationals or a field of integers modulo a prime, such that $\mathfrak{A}$ is isomorphic with the set V of all finitely non-zero functions of P to F. Relative to functions on such a system V to V it is shown that any solution of the functional equation

$$\sum_{i}^{0,n+1} (-1)^{i}_{n+i}C_{i}f(v_0+iv_1)=0 \qquad (v_0, v_1)$$

is, if F is rational or has a modulus $n+1$, a polynomial of degree equal to or less than n in the coordinates of the elements of V with coefficients in V and conversely. The trivial case in which F has a modulus $m \leqq n+1$ is dealt with. The paper is an extension of the results of Hamel for the equation $f(x+y)=f(x)+f(y)$ and the real number system. Hamel made use of the Zermelo "axiom of selection".

24. Professor E. V. Huntington: *Postulates for order on a closed line: I. Reversible order (separation of point-pairs).*

The theory of reversible order on a closed line (also known as the theory of separation of point-pairs), which is of fundamental importance in the development of projective geometry, was first systematically studied by Pasch in 1882. The first set of postulates for this theory was given by Vailati in 1895, and has been substantially followed by later writers. The present paper gives a more exhaustive analysis of the properties of this type of order (expressed in terms of a tetradic relation *ABCD*), and leads to the following new set of six postulates (of which the first two serve only to exclude obviously trivial cases):

0. There is at least one true tetrad, say *XYZW*.
00. If *ABCD* is true, then the elements *A, B, C, D* are distinct.
I. If *ABCD* is true, then *BCDA* is true. (Cyclicality.)
II. If *ABCD* is true, then *ABDC* must be false. (Homogeneity.)
III. If *ABCD* is true, and *X* is any fifth element, then at least one of the relations *AXCD* and *ABCX* is true. (Connexity.)
IV. If *ABCD* is true, then *DCBA* is true. (Reversibility.)

This paper (which extends and completes an unpublished paper presented to the Society in December, 1916) will be followed by a second paper on irreversible order, in which Postulate IV is replaced by its opposite.

25. Professor J. B. Shaw: *On the classification of linear algebras.*

The classification of linear algebras, whether associative or not, may be made from two very different standpoints. In one the basis is that of the properties of any number, the general number, considered as held fixed, while all other numbers are multiplied by it. This would correspond geometrically to a collineation. The classification on this basis I have carried out in previous papers. A different basis is that initiated by Cayley in his paper on double algebras, and is based upon the numbers that are invariant in the algebra in the sense of being their own squares, or are zero. That is they are idempotent or nilpotent. The latter basis is geometrically that of quadratic transformations. In both cases there are invariants that must count double, triple, etc., and the number of these furnishes the basis for the divisions. In the paper a study is made of the conditions this multiplicity imposes upon the system. It shows up in the formation of certain sub-algebras. Non-associative algebras may be classified from either standpoint, the property of associativity becoming relatively unimportant.

26. Professor Louis Ingold: *Associated types of linear connection.*

In this paper the transformation $\overline{\Gamma}^s_{i\lambda} = A^s_\alpha\left(\frac{\partial a^\alpha_i}{\partial u^\lambda} + a^r_i \Gamma^\alpha_{r\lambda}\right)$ on the functions Γ^k_{ij}, which are used to define the neighborhood connections in a geometry, is associated with an arbitrary matrix $||a^i_j||$. The elements a^i_j are functions of the coordinates of the space and $A^s_\alpha \cdot |a^i_j|$ is the cofactor of a^s_α. It is shown that the resultant of the transformations associated with two matrices is again a transformation of the same type,—in fact, the one associated with the resultant matrix. The geometries corresponding to different sets of functions Γ related in this way are called associated geometries. The effect of the transformations on certain fundamental tensors is considered and a series of properties is found which hold in all associated geometries.

27. Professor Ingold: *The geometry of a set of n vectors.*

A set of n independent vectors is taken as the basis of a geometry. For convenience they are represented in a function space by the functions $\theta_i(x; u^1, u^2, \cdots, u^n), i=1,2,\cdots,n$. Quantities analogous to the fundamental quantities and Christoffel symbols of differential geometry are formed from the functions θ_i and their derivatives with respect to the coordinates u^i. These serve to define the neighborhood connections. If instead of the functions θ_i, a set of n independent functions φ_j are used as basis, where $\varphi_j = a^i_j\,\theta_i$, the coefficients Γ^k_{ij} of the linear connection undergo the transformation discussed in the previous paper.

28. Mr. W. A. Jenkins: *On a central difference summation formula.*

Three finite difference formulas are discussed each of which interpolates a certain number of values between each of a given series of equidistant ordinates and sums approximately the interpolated along with the given values, in terms of the original ordinates and their differences. Two of these, due to Lubbock and De Morgan, are proved inferior in convergency to the third, a central difference formula derived by Woolhouse in the JOURNAL OF THE INSTITUTE OF ACTUARIES, volume 11, page 307. An approximate test of accuracy is developed.

29. Professor J. S. Turner: *Note on prime factors.*

This note presents a short and direct proof of the theorem "If a prime p is a factor of a product ab, where a, b are positive integers and a is prime to p, then p is a factor of b."

30. Mr. P. G. Robinson: *Surfaces with constant absolute invariants.*

Consider a surface such that the homogeneous coordinates of any point of it are analytic functions of two parameters. Let the asymptotic tangents of any point of the surface be distinct, and let neither have more than second order contact with the surface at any point. Wilczynski has shown that under these conditions and by a proper choice of the tetrahedron of reference, the surface may be represented by a development of the form

$$z = xy + \frac{1}{6}(x^3 + y^3) + \frac{1}{24}(Ix^4 + Jy^4) + \cdots$$

where I and J are absolute invariants of the surface.

This paper shows first that there are no surfaces for which either I or J is equal to zero and the other equal to a constant different from zero, at all points which satisfy the conditions formulated above. Then the case is considered where both I and J are equal to non-vanishing constants. In this case the differential equations of the surfaces are obtained and integrated. From these results are obtained properties of the surfaces.

31. Professor M. H. Ingraham: *A general theory of linear sets.*

In this paper the author, using the postulational method, considers classes of vectors, on a finite range, whose elements belong to a general division algebra, and gives a general basis preliminary to the more intensive study of classes of vectors on a general range and of number systems over a division number system. This paper has appeared in the April, 1925, number of the TRANSACTIONS OF THIS SOCIETY.

ARNOLD DRESDEN,
Assistant Secretary.

CONCERNING THE COMPLEMENTARY INTERVALS OF COUNTABLE CLOSED SETS*

BY J. R. KLINE

Professor R. L. Moore† recently gave an example showing that the following statement due to Hobson‡ is incorrect: "*A non-dense closed set is enumerable if its complementary intervals are such that every one of them abuts on another one at each of its ends.*"

In considering this example, we notice that while each of the complementary intervals of the original set of Moore's example are such that each one of them abuts on another one at each of its ends, the same is not true if we consider the complementary intervals of G', the first derived set§ of G. This leads us to the following theorem:

THEOREM A. *A non-dense closed set G is enumerable, if G and every derived set of G has the property that every one of its complementary intervals is such that it abuts on another one at each of its ends.*

PROOF: Let us suppose that G is not enumerable. Then there exists a number β of the first or second class such that the derived set $G^{\beta+1}$ is identical with the derived set G^β and G^β is perfect.|| It follows that G^β has no

---

* Presented to the Society, October 25, 1924.

† See R. L. Moore, *An uncountable, closed, and non-dense point set, each of whose complementary intervals abuts on another one at each of its ends*, this BULLETIN, vol. 29, (1923), pp. 49–50.

‡ See E. W. Hobson, *The Theory of Functions of a Real Variable* 1st edition (1907), p. 92; and 2d edition (1921), p. 113.

§ If G is a point set, then the set G' consisting of all limit points of G is called the *derived set* or the derivative of G.

|| See E. W. Hobson, loc. cit., p. 115. In his proof of the theorem that every non-dense linear closed set is, in general, made up of an enumerable set and a perfect set, Hobson here shows that unless the non-dense linear closed set G is enumerable, there is a number β of the first or second class such that G^β is perfect.

isolated points. As G is non-dense and closed, it follows that G^β is non-dense. Select a complementary interval of G^β. As this interval abuts on another complementary interval of G^β at both its ends, it follows that the supposition that G is not enumerable has lead to a contradiction.

As Hobson points out, it is not necessary that each complementary interval abut on another at each of its ends in order that a set be enumerable. But as no subset of a closed countable set is perfect,* then there must be some definite number β of the first or second class such that $G^{\beta+1}$ vanishes. Thus G^β consists only of isolated points. It follows that its complementary intervals abut at both their ends. Thus we are enabled to state the following theorem.

THEOREM B. *In order that a linear closed set be enumerable it is both necessary and sufficient that there exist some number* β *of the first or second class such that* G^β *and all the derived sets of* G *after* G^β *have the property that each of its complementary intervals abuts on another one at both its ends.*

UNIVERSITY OF PENNSYLVANIA

---

* See R. L. Moore, *An extension of the theorem that no countable point set is perfect,* PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 10, pp. 168-170.

ON SETS OF THREE CONSECUTIVE INTEGERS WHICH ARE QUADRATIC RESIDUES OF PRIMES*

BY A. A. BENNETT

In this paper we shall prove the following theorems.

THEOREM I. *For each prime, p, for which there are as many as three incongruent squares, there is a set of three consecutive residues (admitting zero and negative numbers as residues) which are squares, modulo p.*

THEOREM II. *For $p = 11$, and for each prime p greater than 17, (and for no other primes), there is a set of three consecutive least positive (non-zero) residues which are squares, modulo p.*

The problem† of finding three consecutive integers which are quadratic residues of a prime, p, is equivalent to the formally more general problem of finding two quantities, x, y, $(y \neq 0)$, such that x, y, $x+y$, $x-y$, are proportional to squares in the domain,‡ since we then have $(x/y)-1$, x/y, $(x/y)+1$ as consecutive squares in the domain. We may show that for residues with respect to a modulus the condition is equivalent to the existence of a square of the form§ $u\,v\,(u+v)\,(u-v)$. By taking $u = x$, $v = y$, we see that the condition is necessary.

---

* Presented to the Society, April 10, 1925.

† For references, compare article of similar title by H. S. Vandiver, this BULLETIN, vol. 31 (1925), p. 33.

‡ That, in the system of natural numbers, it is impossible to have distinct quantities, x, y, such that x, y, $x+y$, $x-y$ are all proportional to squares was proved by Fermat by his celebrated method of "infinite descent". See Carmichael, *Theory of Numbers*, p. 86.

§ It is of interest to note that in the case of natural numbers we may take $u = x$ and $v = y$ for this relation. Indeed, if x, y, $x+y$, $x-y$ were proportional to squares, certainly their product would be a square. Conversely, suppose that their product were a square. Then either x, y, $x+y$, $x-y$ would all be relatively prime, or if

That it is also sufficient may be shown as follows. Take $x=(u^2+v^2)^2$ and $y=4uv(u^2-v^2)$. Now x may also be written in the form $4u^2v^2+(u^2-v^2)^2$. Hence $x+y=[2uv+(u^2-v^2)]^2$, and $x-y=[2uv-(u^2-v^2)]^2$. Thus if $uv(u^2-v^2)$ is a perfect square, so also are x, y, $x+y$, and $x-y$, when these are related in this manner.

The expression $uv(u+v)(u-v)$ takes on the values $6^2\cdot 5$, $2^2\cdot 6$, $2^2\cdot 30$, for the choices of (u,v) as (5,4), (3,1), (5,1), respectively. But at least one of the three numbers 5, 6, 30 is a quadratic residue of the prime p no matter how p is chosen. Hence there is always a choice of $uv(u+v)(u-v)$ which is a non-zero quadratic residue for each prime p greater than 5. The corresponding solutions of the original problem are (1/4, 5/4, 9/4), (1/24, 25/24, 49/24), (49/120, 169/120, 289/120). These numbers in turn are all different from zero for $p=11$, or for $p>17$, but not otherwise. Now every three consecutive residues no one of which is congruent to zero are congruent to a set of three consecutive least positive (non-zero) residues. Thus we establish the theorems announced.

There is no difficulty in obtaining linear forms, the primes within which are such that for each of these choices of p, a preassigned number of consecutive residues shall be squares. Indeed, we have merely to select assigned numbers to be quadratic residues. Thus for p of the form $24k+1$ or $24k+23$, each of the numbers 1, 2, 3, 4 is a square. By choosing a form for which 2, 3, 5, 7, -1 are all quadratic residues, and dropping the condition of positiveness, we have always the following twenty-one consecutive residues as squares, -10, -9, -8, $\cdots$, 9, 10.

THE UNIVERSITY OF TEXAS

---

two had a common factor, d, this would be a factor throughout. Write $x=dx'$, $y=dy'$; then $x'y'(x'+y')(x'-y')$ also would be a square. Its four relatively prime factors would then be individually squares and x, y, $x+y$, $x-y$ would therefore be proportional to squares. This method of argument is inapplicable to the case of residues.

GROUPS IN WHICH THE NORMALISER OF EVERY ELEMENT EXCEPT IDENTITY IS ABELIAN*

BY LOUIS WEISNER

1. *Introduction.* Certain groups, of which the tetrahedral and icosahedral groups are non-trivial examples, possess the property that the normaliser of every element except identity is abelian. In this paper we shall investigate the properties of such groups.

2. *Abelian Subgroups.* An abelian subgroup of a group will be called a *maximal* abelian subgroup if it is not contained in any larger abelian subgroup of the group.

If two abelian subgroups of a group G, in which the normaliser of every element except identity is abelian, have an element besides identity in common, they generate an abelian group; for an element common to the abelian subgroups H and I is invariant under (H, I), which must therefore be abelian. It follows that *the maximal abelian subgroups of G are independent*; that is, no two maximal abelian subgroups of G have an element in common besides identity.

Since every prime-power group possesses an invariant element besides identity, *the Sylow subgroups of G are abelian.* Moreover, *the Sylow subgroups of G are independent.* For if two Sylow subgroups H and I (necessarily of the same order p^a) had an element in common besides identity, (H, I) would be an abelian subgroup of G of order $p^b > p^a$, which contradicts the fact that p^a is the highest power of p that divides the order of G.

Let H be a maximal abelian subgroup of G, I a Sylow subgroup of H, and J that Sylow subgroup of G which includes I. Since H and J have I in common, (H, J) is

---

[*] Presented to the Society, September 8, 1925.

an abelian group. But H is a *maximal* abelian subgroup of G. Hence H contains J. Therefore *the Sylow subgroups of a maximal abelian subgroup of G are Sylow subgroups of G.* It follows that *the orders of two non-conjugate maximal abelian subgroups of G are relatively prime.* If G possesses exactly r conjugate sets of maximal abelian subgroups, of which $H_1, H_2, \cdots, H_r$, of orders $h_1, h_2, \cdots, h_r$ respectively, are representatives, then every element of G except identity is included in one and only one conjugate of one of the H's, and the order of G is $h_1 h_2 \cdots h_r$, $h_1, h_2, \cdots, h_r$ being relatively prime.

If a maximal abelian subgroup H, of order h, has y conjugates under G, the total number of elements of G whose orders divide h is $y(h-1)+1$; for every element whose order divides h is included in a conjugate of H, and the conjugates of H are independent. By Frobenius' theorem,

$$y(h-1)+1 \equiv 0, \pmod{h},$$

whence

$$y \equiv 1, \pmod{h}.$$

Therefore, *the number of conjugates of a maximal abelian subgroup of order h is of the form $1+xh$.*

3. *Solvable Groups.* Let G be a solvable group in which the normaliser of every element except identity is abelian. A minimal invariant subgroup I is an abelian group. The maximal abelian subgroup H which contains I must be invariant under G; otherwise I would be included in H and a conjugate of H, whereas the maximal abelian subgroups of G are independent.

Arrange G in cosets as regards H. Every element of G transforms H into itself, thus establishing an isomorphism of H. Elements in the same coset establish the same isomorphism, while elements in different cosets establish different isomorphisms. Since no element of H except identity is commutative with an element of G not in H, G/H is simply isomorphic with a group of isomorphisms of H, each of which leaves no element of H unchanged except identity. It follows* that G/H is either a cyclic group or a dicyclic group. If G/H is dicyclic, it contains a non-abelian Sylow subgroup of even order and G contains a subgroup simply isomorphic with this Sylow subgroup, since the order of H is prime to the order of G/H. But the Sylow subgroups of G are abelian. Hence G/H is a cyclic group.

Thus *a solvable group in which the normaliser of every element except identity is abelian contains an invariant maximal abelian subgroup, and the corresponding quotient group is cyclic.*

4. *Insolvable Groups.* An insolvable group G in which the normaliser of every element except identity is abelian must contain a minimal invariant subgroup H, which is a simple group of composite order. If G is larger than H, let I be a subgroup of G which contains H, such that H is of prime index p in I.

Suppose that no element of H is commutative with an element of I which is not in H. A maximal abelian subgroup J of H must also be a maximal abelian subgroup of I; otherwise, H would contain two non-conjugate abelian subgroups of the same order, one of which is a maximal abelian subgroup, which conflicts with a theorem proved in § 2. Therefore the normaliser K of J in I is larger than, and includes, the normaliser L of J in H. By § 3, K/J is a cyclic group. Since J is a maximal abelian subgroup of K, the order of J is prime to the order of K/J. Hence K contains an element t whose order n equals the order of K/J, and $K = (t, J)$. Since L is a subgroup of K which includes J, $L = (t^a, J)$, where a is a divisor of n. Since t^a is an element of H which is commutative with t, $t^a = 1$, and $L = J$. It follows that J is the normaliser in H of every Sylow subgroup of J. Since the Sylow subgroups of J are Sylow subgroups of H, H is not a simple group,† contrary to supposition.

---

* Burnside, *Theory of Groups*, 2d edition, p. 336, Theorem V.

† Burnside, p. 327, Theorem II.

It follows that I contains an element t not in H, which is commutative with some element of H other than identity. Let J be the normaliser of t in I, and K the subgroup common to J and H. Since the maximal abelian subgroups of H and of I are independent, while J and K are maximal abelian subgroups of I and H respectively, the number of conjugates of J under I equals the number of conjugates of K under H. Let this number be x, and let h, i, j, k denote the orders of H, I, J, K respectively. The elements of H whose orders divide $j=kp$ are included in the conjugates of K, while the elements of I whose orders divide kp are included in the conjugates of J. Observing that the order of every element of I not in H divides kp, we have

$$x\,(kp-1)+1=x\,(k-1)+h\,(p-1),$$

from which $x=h/k$. Therefore K is the normaliser of K in H. It follows as in the preceding paragraph that H is not a simple group.

Since the assumption that G is larger than H leads to a contradiction, it follows that *an insolvable group in which the normaliser of every element except identity is abelian is a simple group.*

The linear fractional group of order $2^n\,(2^{2n}-1)$, $(n>2)$ is an example of a simple group in which the normaliser of every element except identity is abelian. That this group actually possesses the property in question follows from its analysis.*

UNIVERSITY OF ROCHESTER

---

* Dickson, *Linear Groups*, pp. 262–265.

NOTE ON GIBBS' PHENOMENON

BY C. N. MOORE

In my review of the third edition of volume I of Picard's *Traité d'Analyse*,[*] I took exception to Picard's claim that Du Bois-Reymond had discovered Gibbs' phenomenon. Concerning the question of priority in this discovery, Professor G. N. Watson has kindly called my attention to an article by H. Wilbraham.[†]

The phraseology of this paper indicates that Wilbraham was not quite clear in his own mind as to the distinction between the curve which is the limit curve of the approximation curves, $y = S_n(x)$, and the curve $y = f(x)$, which represents the limit function. However, his discussion applies to the former curve and furnishes a valid proof of the central features of Gibbs' phenomenon in the case of the particular series he discusses. He also points out that a similar discussion will establish an analogous behavior in the case of another special series. As Professor Watson has said in his letter to me: "It is a remarkable paper for so early a date."

At first thought it seems rather surprising that Wilbraham's paper, dealing with so interesting a property of Fourier's series, should have remained virtually unnoticed for a period of more than half a century, during which the theory of these series was being greatly enlarged. There is, however, a rather natural reason for this, the same reason why Gibbs' paper in NATURE at first attracted no general attention. Both Wilbraham and Gibbs restricted their discussion to particular series, and failed to point out that the property in question characterized the behavior of the Fourier's series corresponding to a very broad class of functions. It was Bôcher's wide generalization* of Gibbs' brief remarks, as well as his lucid exposition of the details of Gibbs' phenomenon, that first aroused general interest in this remarkable property of Fourier's series.

---

[*] This BULLETIN, (2), vol. 30 (1924), pp. 554–556.

[†] CAMBRIDGE AND DUBLIN MATHEMATICAL JOURNAL, vol 3 (1848), pp. 198–201.

Subsequent to writing the preceding paragraphs I received a letter from Professor H. S. Carslaw, calling my attention to Wilbraham's paper, and I also had the privilege of seeing a copy of the manuscript of Professor Carslaw's historical note, *Gibbs' phenomenon in Fourier's series and integrals.*† Since this note goes into some detail concerning Du Bois-Reymond's paper, it seems to me desirable to amplify somewhat my own position regarding this paper, as previously indicated in the review to which reference has been made.

The formulas which Professor Carslaw quotes from Du Bois-Reymond's paper are the ones to which I referred when I made the statement that "there are certain formulas in the article from which Gibbs' phenomenon might have been deduced". However, if one goes back to page 244 of Du Bois-Reymond's article, the paragraph beginning at the bottom of this page reveals the fact that he had decided on the basis of physical intuition as to how the approximation curves ought to behave. His analysis seems to have been carried out with a view to substantiating this prejudgment on his part, and that fact in my opinion explains his error as to the behavior of the integral $\int_0^{nx}[(\sin \alpha)/\alpha]\,d\alpha$. If he had examined the behavior of this integral on its own merits, it does not seem likely that he would have made such an error. But the extreme limits $\pm(\pi/2)$ fitted in with his preconceived ideas, so he apparently jumped to the conclusion that they were correct, having in mind doubtless the relation $\int_0^{\pm\infty}[(\sin \alpha)/\alpha]\,d\alpha = \pm(\pi/2)$.

It seems to the writer that Gibbs' phenomenon is most decidedly one of those facts that can only be discovered by the methods of analysis. The approximation curves

---

* ANNALS OF MATHEMATICS, (2), vol. 7 (1905-06), pp. 123-132.

† See page 419 of the present number of this BULLETIN.

$y = S_n(x)$ are obviously continuous curves, and either geometric or physical intuition would naturally suggest that they might behave as Du Bois-Reymond supposed. It is only the more delicate considerations of analysis that would reveal the true facts. That so competent an analyst as Du Bois-Reymond should have overlooked these facts, when he had formulas before him from which they could readily have been deduced, seems to me all the more reason for not associating his name with the phenomenon in question. It is quite natural and proper to refer to his work in giving the historical background of Gibbs' phenomenon, but in my opinion Professor Picard's designation of the phenomenon as "Phénomène de Du Bois-Reymond et Gibbs" was without adequate justification, even if Wilbraham's article had not existed.

THE UNIVERSITY OF CINCINNATI

A HISTORICAL NOTE ON GIBBS' PHENOMENON IN FOURIER'S SERIES AND INTEGRALS

BY H. S CARSLAW

In 1899, Gibbs called attention* to the fact that for large values of n the approximation curves

$$y = S_n(x) = 2\sum_1^n (-1)^{r-1}\frac{\sin rx}{r},$$

for the Fourier's series which represents $f(x) = x$ in the interval $-\pi < x < \pi$, fall from the point $(-\pi, 0)$ at a steep gradient to a point very nearly at a depth $2\int_0^\pi [(\sin \alpha)/\alpha]d\alpha$ below the axis of x, then oscillate above and below $y = x$ close to this line until x approaches π, when they rise to a point very nearly at a height $2\int_0^\pi [(\sin\alpha)/\alpha]d\alpha$ above the axis, and then fall rapidly to $(\pi, 0)$.

At the point of discontinuity, where $x = \pi$, in the series $2\sum_1^\infty (-1)^{r-1}(\sin rx)/r$ the approximation curves thus tend to coincide, not with the segment joining the points (π, π) and $(\pi, -\pi)$, but with the straight line whose ends are the points

$$\left(\pi, \pi + \frac{D}{\pi}\int_\pi^\infty \frac{\sin\alpha}{\alpha}\,d\alpha\right)$$

and

$$\left(\pi, -\pi - \frac{D}{\pi}\int_\pi^\infty \frac{\sin\alpha}{\alpha}\,d\alpha\right),$$

where $D = f(\pi+0) - f(\pi-0)$, the amount of the "jump" in the sum of the series at that point.

In 1906, Bôcher showed† that the same phenomenon occurred in general in the Fourier's Series for the arbitrary function $f(x)$ in the neighborhood of a point of discontinuity whose abscissa is a, when $f(a \pm 0)$ exist. The approximation curves approach a limiting curve in which the vertical portion is the line joining the points $(a, f(a-0)$ and $(a, f(a+0))$; *but this vertical line has to be produced beyond these points by an amount that bears a definite ratio to the magnitude of the jump.*

---

* NATURE, vol. 59 (1899), p. 606.

† ANNALS OF MATHEMATICS, (2), vol. 7, (1906).

Bôcher gave the name *Gibbs' phenomenon* to this property of the approximation curves of the Fourier's series. Both Bôcher and Gibbs were under the impression that the property had remained unsuspected till Gibbs discovered it for the series

$$2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$

in 1899.

In my book on *Fourier's Series and Integrals,*[*] I repeat this statement, and in the review of the third edition of Picard's *Traité d'Analyse* by C. N. Moore,[†] it is again to be found in print. *However, it is now known that Gibbs's phenomenon had been observed about* 50 *years earlier than* 1899. Indeed, in 1848, Wilbraham,[‡] a B. A. of Trinity College, Cambridge, discussed the equation

$$y = \cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \cdots .$$

His object was to show that this equation represents "a locus composed of separate straight lines, of which each is equal to π, parallel to the axis of x, and at a distance $\pm \frac{1}{4}\pi$ alternately above and below it, joined by perpendiculars which are themselves part of the locus" but extend beyond the ends $\pm \frac{1}{4}\pi$ by a distance $\frac{1}{2} \left| \int_\pi^\infty [(\sin \alpha)/\alpha]d\alpha \right|$ at either extremity.

---

[*] Carslaw, *Fourier's Series and Integrals,* 2d ed., 1921, p. 266.

[†] THIS BULLETIN, vol. 30 (1924), p. 555.

[‡] *On a certain periodic function,* CAMBRIDGE AND DUBLIN MATHEMATICAL JOURNAL, vol. 3 (1848), p. 198.

It could not be expected that Wilbraham in 1848 would have a clear idea of what is meant by the sum of the series for the value x_0: i. e., in our notation $\lim_{n \to \infty} S_n(x_0)$. In dealing with this sum he allows x to be a function of n which tends to x_0 as $n \to \infty$; he takes the sum $S_n(x)$ for the positive integer n and the corresponding value of x_0 and he then lets n tend to infinity. But, as his diagrams show, he saw quite clearly the behavior of the approximation curves of his series of cosines, and he remarked that "a similar investigation of the equation

$$y = \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \cdots$$

would lead to an analogous result".

This second series, it will be seen, is the same as that which Gibbs used in 1899.

Wilbraham's paper seems to have attracted little attention and to have been forgotten until Burkhardt quoted it in his article on *Trigonometrische Reihen und Integrale* (*bis etwa* 1850).* The fact that Gibbs' phenomenon had been observed earlier is mentioned also by Hilb and Riesz in their recently published article *Neuere Untersuchungen über trigonometrische Reihen.*†

We may still call this property of Fourier's series (and certain other series) Gibbs' phenomenon; but we must no longer claim that the property was first discovered by Gibbs.

In the review of Picard's *Traité d'Analyse* mentioned above, Moore comments on the title which Picard gives to the section in which Gibbs' phenomenon is discussed, namely, "*Phénomène de Du Bois-Reymond et Gibbs*". He concludes that it is wrong to associate Du Bois-Reymond's name with this discovery. In this I think the reviewer is right. But it is certainly true that, if Du Bois-Reymond had not made a curious slip as to the values which the integral $\int_0^{nx}[(\sin\alpha)/\alpha]\,d\alpha$ can take when $n\to\infty$ and $x\to 0$ simultaneously, he would have enunciated Gibbs' phenomenon both for Fourier's series and Fourier's integral in 1874.

---

* ENCYKLOPADIE DER MATHEMATISCHEN WISSENSCHAFTEN, Bd. II, Teil I, 2. Halfte, p. 1049, 1914.

† ENCYKLOPADIE, Bd. II, Teil 3, p. 1203, 1924.

The memoir* to which Picard refers is well known. In it he distinguishes carefully between

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} f(x\pm\varepsilon, n),$$

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} f(x\pm\varepsilon, n),$$

and the case when n tends to infinity and ε to zero simultaneously: e. g. $\varepsilon=\varphi_1(t)$ and $1/n=\varphi_2(t)$, while as t tends to its limit, $\varphi_1(t)$ and $\varphi_2(t)$ tend to zero.

He uses the symbol Lim for such an arbitrary simultaneous approach of n to infinity and ε to zero, and his work is based on the second theorem of mean value. But even when his results are correct, the argument is not always clear or rigorous.

Yet he finds (loc. cit. § 9) that in the case of Fourier's integral, when $x\to x_1$ and $h\to\infty$, we have

$$\begin{aligned}\text{Lim}\ \frac{1}{\pi}\int_0^h d\alpha\int_{-\infty}^{\infty} d\beta\, f(\beta)\cos\alpha\,(\beta-x) &= \frac{1}{2}\,\{f(x_1+0)+f(x_1-0)\}\\ &+\left\{\frac{f(x_1+0)-f(x_1-0)}{\pi}\right\}\ \text{Lim}\int_0^{h(x-x_1)}\frac{\sin\alpha}{\alpha}\,d\alpha,\end{aligned}$$

while for Fourier's series, when $x\to x_1$, and $n\to\infty$, we have

$$\begin{aligned}\text{Lim}\ S_n(x) &= {}^1/_2\,\{f(x_1+0)+f(x_1-0)\}\\ &+\left\{\frac{f(x_1+0)-f(x_1-0)}{\pi}\right\}\ \text{Lim}\int_0^{n(x-x_1)}\frac{\sin\alpha}{\alpha}\,d\alpha.\end{aligned}$$

---

* *Uber die sprungweisen Werthänderungen analytischer Functionen*, MATHEMATISCHE ANNALEN, vol. 7 (1874), p. 241.

But he goes on to say that the integrals

$$\int_0^{h(x-x_1)} \frac{\sin \alpha}{\alpha}\, d\alpha \text{ and } \int_0^{n(x-x_1)} \frac{\sin \alpha}{\alpha}\, d\alpha,$$

which occur in these results, can take every value between $\pm \frac{1}{2}\pi$ — of course this should have been $\pm \int_0^{\pi} [(\sin \alpha)/\alpha]\, d\alpha$ — according to the way in which h and $(x-x_1)$, or n and $(x-x_1)$, tend to infinity and zero respectively. He thus concludes that the Fourier's integral and series at a point of ordinary discontinuity, in this simultaneous limit, represent all the values between $f(x_1-0)$ and $f(x_1+0)$, instead of extending beyond $f(x_1 \pm 0)$ by the amount

$$\left| \left\{ \frac{f(x_1+0)-f(x_1-0)}{\pi} \right\} \int_{\pi}^{\infty} \frac{\sin \alpha}{\alpha}\, d\alpha \right| .$$

Further, Du Bois-Reymond seems to have been the first to bring up the question of the behavior of Fourier's integral in this connection. And with a corresponding emendation his results are correct.

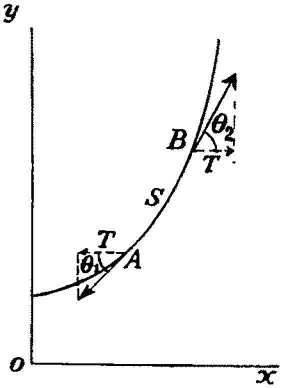
Early in 1925, I sent to the QUARTERLY JOURNAL OF MATHEMATICS a paper on Gibbs' phenomenon in one of the early sections of which I refer to the work of Wilbraham and Du Bois-Reymond. But the mistake about the discovery of Gibbs' phenomenon is so common that it seems proper to make this correction also in this BULLETIN.

SYDNEY, AUSTRALIA

A GENERAL FORM OF THE SUSPENSION BRIDGE CATENARY

BY IRA FREEMAN

1. *Introduction.* The form of the curve assumed by a uniform chain or string hanging freely between two supports was first investigated by Galileo, who erroneously determined it to be a parabola; Jungius detected Galileo's error, but the true form was not discovered until 1691, when James Bernoulli published it as a problem in the ACTA ERUDITORUM. He also considered the cases when the chain was (1) of variable density, (2) extensible, (3) acted upon at each point by a force directed to a fixed center. These curves attracted much attention, and were discussed by John Bernoulli, Leibnitz, Huygens, David Gregory and others. A review of the literature shows, however, that in the problem of the inextensible string acted upon by gravitational forces, only two special cases are considered:—the case where there is simply a string of uni-form linear density, and the case of a string of inappreciable weight supporting a uniform horizontal load. Moreover, the modern text-books in the subject appear to shun a more general case, and content themselves with presenting only the two very special cases mentioned.



The object of the present paper is to consider what happens when a cable of appreciable uniform linear density supports a uniform horizontal load, conditions that are very nearly realized in the suspension-bridge. The texts solve the special cases by the solution of two first-order differential equations, the first of which gives the intrinsic equation of the curve. The following method uses one second-order differential equation, and does not involve the intrinsic equation as an intermediate step.

2. *Densities Different and not Zero. Both the cable and the bridge platform have uniform linear densities which are appreciable, and, in general, different.*

Let α be the weight per unit length of the cable, and let β represent that of the bridge platform. If any arc s is in equilibrium, the horizontal component of the tensions, at the extremities of the arc, will be the same constant which we may call T. The difference of the vertical components of the extreme tensions will equal the total weight on the arc; that is, we shall have

$$T(\tan\theta_2-\tan\theta_1) = \alpha s+\beta x.$$

If s is small, this becomes

$$T\frac{d^2y}{dx^2} = \alpha\frac{ds}{dx}+\beta, \tag{1}$$

since

$$\tan\theta_2-\tan\theta_1 = \frac{d}{dx}\left(\frac{dy}{dx}\right)dx.$$

But since

$$\left(\frac{ds}{dx}\right)^2 = \left(\frac{dy}{dx}\right)^2+1,$$

we may write

$$T\frac{d^2y}{dx^2} = \alpha\sqrt{1+\left(\frac{dy}{dx}\right)^2}+\beta.$$

If $p = dy/dx$, we have

$$\frac{dp}{a\sqrt{1+p^2}+b} = dx, \tag{2}$$

where $a = \alpha/T$ and $b = \beta/T$. The equation (2) is the differential equation of the curve. To integrate it, put $p = \tan\theta$; then

$$\frac{\sec^2\theta d\theta}{b+a\sec\theta} = dx, \qquad \text{or} \qquad \frac{d\theta}{\cos\theta(a+b\cos\theta)} = dx.$$

By the method of partial fractions, we may write

$$(3) \qquad \frac{d\theta}{\cos\theta} - \frac{b\,d\theta}{a+b\cos\theta} = a\,dx.$$

Integrating, we find

$$(A) \qquad \log(\sec\theta+\tan\theta) - \frac{b}{\sqrt{b^2-a^2}}\log\left[\frac{b+a\cos\theta+\sqrt{b^2-a^2}\sin\theta}{a+b\cos\theta}\right] = ax,$$

the constant of integration being zero if we let $x=0$ when $\theta=0$. Dividing (2) by dy, we may write

$$(4) \qquad \frac{\frac{dp}{dy}}{a\sqrt{1+p^2}+b} = \frac{dx}{dy} = \frac{1}{p}; \quad \frac{p\,dp}{a\sqrt{1+p^2}+b} = dy.$$

To integrate this put, as before $p=\tan\theta$. Then

$$\frac{\tan\theta\sec^2\theta\,d\theta}{a\sec\theta+b} = dy, \qquad \text{or} \qquad \frac{\sec\theta[d(\sec\theta)]}{a\sec\theta+b} = dy.$$

Integrating, we find

$$b+a\sec\theta-b\log(b+a\sec\theta) = a^2y+K.$$

If $y=y_0$ when $\theta=0$, $K=b+a-b\log(a+b)-a^2y_0$, and hence

$$(B) \qquad a(\sec\theta-1)+b\log\left[\frac{a+b}{b+a\sec\theta}\right] = a^2(y-y_0).$$

The equations (A) and (B) are the parametric equations of the curve, the parameter being θ, the angle of inclination.

3. *Alternative Forms. If* $b<a$, *equation* (A) *in its present form will involve square roots of negative quantities.* But in this case, there is a form of (A) which is real, and which is therefore better suited for computation. This form will now be developed.

The second term of the left member of (A) is now

$$z = \frac{b}{\sqrt{b^2-a^2}}\log\left[\frac{b+a\cos\theta+\sqrt{b^2-a^2}\sin\theta}{a+b\cos\theta}\right],$$

that is,

$$z = \frac{b}{i\sqrt{a^2-b^2}} \log\left[\frac{b+a\cos\theta+i\sqrt{a^2-b^2}\sin\theta}{a+b\cos\theta}\right],$$

or

$$e^{\frac{iz\sqrt{a^2-b^2}}{b}} = \frac{b+a\cos\theta+i\sqrt{a^2-b^2}\sin\theta}{a+b\cos\theta}$$

or

$$\cos\left(\frac{z\sqrt{a^2-b^2}}{b}\right)+i\sin\left(\frac{z\sqrt{a^2-b^2}}{b}\right) = \frac{b+a\cos\theta+i\sqrt{a^2-b^2}\sin\theta}{a+b\cos\theta}.$$

Equating the imaginary parts,

$$\sin\left(\frac{z\sqrt{a^2-b^2}}{b}\right) = \frac{\sqrt{a^2-b^2}\sin\theta}{a+b\cos\theta};$$

$$z = \frac{b}{\sqrt{a^2-b^2}}\sin^{-1}\left[\frac{\sqrt{a^2-b^2}\sin\theta}{a+b\cos\theta}\right]$$

so that (A) may be replaced by the equation

$$\text{(C)} \qquad \log(\sec\theta+\tan\theta) - \frac{b}{\sqrt{a^2-b^2}}\sin^{-1}\left[\frac{\sqrt{a^2-b^2}\sin\theta}{a+b\cos\theta}\right] = ax.$$

4. *Bridge Density Zero. If the linear density of the bridge becomes zero, the equations reduce to those of the common catenary.* Here $b=0$, so (A) takes the simple form $\log(\sec\theta+\tan\theta) = ax$, since the second term reduces to zero. Hence we may write

$$e^{ax} = \sec\theta+\sqrt{\sec^2\theta-1}, \qquad \text{or} \qquad \sec\theta = \frac{1}{2}(e^{ax}+e^{-ax}).$$

The equation (B) becomes $\sec\theta-1 = a(y-y_0)$, since the argument of the logarithm never becomes infinite. Combining these results, we have

$$\frac{1}{2}(e^{ax}+e^{-ax}) = 1+a(y-y_0),$$

which is the equation of the common catenary.

5. *Cable Density Zero. When we put the linear density of the cable equal to zero, a becomes zero, and this makes* (A) *and* (B) *reduce to identities. We can, however, obtain the equations which should result:* If we divide (A) throughout by a and then put $a = 0$, the result is an indeterminate form, of the type zero divided by zero. This is evaluated by the rule of de L'Hospital:

$$\{b/(a^2-b^2)\}\Big\{[\sqrt{b^2-a^2}(b+a\cos\theta+\sqrt{b^2-a^2}\sin\theta)$$
$$-(a+b\cos\theta)(\sqrt{b^2-a^2}\cos\theta-a\sin\theta)]\div$$
$$[(b+a\cos\theta+\sqrt{b^2-a^2}\sin\theta)(a+b\cos\theta)]+$$
$$\frac{a}{\sqrt{b^2-a^2}}\log\left[\frac{b+a\cos\theta+\sqrt{b^2-a^2}\sin\theta}{a+b\cos\theta}\right]\Big\}.$$

When we now put $a = 0$, this reduces to $(1/b)\tan\theta$, and (A) takes the form

$$(A^1) \qquad \tan\theta = bx.$$

Similarly, when we divide (B) through by a^2 and take the quotient of the derivatives, we have

$$\left.\frac{(\sec\theta-1)\left(1-\dfrac{b^2}{(a+b)(b+a\sec\theta)}\right)}{2a}\right|_{a=0} = \frac{0}{0}.$$

Applying the rule a second time,

$$\left.\frac{b^2}{2}(\sec\theta-1)\frac{b(1+\sec\theta)+2a\sec\theta}{(a+b)^2(b+a\sec\theta)^2}\right|_{a=0} = \frac{1}{2b}\tan^2\theta,$$

so that (B) takes the form

$$(B^1) \qquad \frac{1}{2b}\tan^2\theta = y-y_0.$$

Combining (A^1) and (B^1), we may write

$$y-y_0 = \frac{b}{2}x^2,$$

which is the equation of a parabola.

THE UNIVERSITY OF CHICAGO

ON THE SOLUTION OF DIOPHANTINE EQUATIONS BY MEANS OF IDEALS*

BY G. E. WAHLIN

1. *Introduction.* In a previous article† I considered the application of the theory of ideals to diophantine analysis. At suggestions from Professors Dickson and Hedrick I have undertaken to make a study of the applications of the theory to problems of less general type than the one considered in that article. In this paper we shall consider diophantine equations in which one member is a binary quadratic form.

Since the publication of the previous article, I have noticed that, with a slight modification in the development, we could have obtained a theory for the solution of equations in which the left hand member is a more general decomposable form. This generalization would not involve any additional difficulty. In the following pages I have made use of this fact and have thus gained the advantage that, in the equations here considered, the only restriction imposed upon the binary quadratic form, used as the left hand member of the equation, is that its discriminant shall not be a perfect square.

In making the modification I have found it necessary to change the definition of a ring ideal from the one previously used in which we had the condition that a ring ideal is always relatively prime to the conductor. This restriction has been removed. Those ring ideals which are relatively prime to the conductor shall be called *regular* ring ideals. Since the properties of ring ideals as developed in Bachmann‡ apply only to regular ring ideals, it has been found necessary to insert the proof of a property of ring ideals in general. In speaking of equivalent ring ideals and classes of ring ideals, let it be understood that this refers only to regular ring ideals.

---

* Presented to the Society December 26, 1924.

† This BULLETIN, vol. 30, Nos. 3–4 (Mch.-Apr., 1924), p. 140. We shall refer to this article by the symbol (W).

‡ *Allgemeine Arithmetik der Zahlenkörper.*

The theorems of §§ 2, 3, 4 of (W) refer only to regular ring ideals. Article 2 of the present paper is devoted to a general property of ring ideals. It is a modification of the development on pp. 152–3 of (W), and is used to simplify the following development.* As in (W) $\{\gamma\}$ is a principal field ideal and $[\gamma]$ is a principal ring ideal. Δ is used to denote the index and f the conductor of a ring.

2. *A General Property of Ring Ideals.* In this section I shall give a proof of the following theorem.

THEOREM I. *Let $\mathfrak{M}$ be an ideal of a ring R in an algebraic number field of degree n and γ an integer of $\mathfrak{M}$. Let $\{\gamma\} = T\cdot P_k$ where T is a field ideal relatively prime to $\mathfrak{M}\cdot f$, and P_k a field ideal whose prime ideal divisors are all divisors of $\mathfrak{M}\cdot f$. Then, if c is any rational integer divisible by $\mathfrak{M}$ and I a field ideal relatively prime to $\{c\Delta\}$ and belonging to the class reciprocal to that of T, we have $N(I)\cdot P_k/I = \{\gamma^{(k)}\}$ where $\gamma^{(k)}$ is an integer belonging to $\mathfrak{M}$.*

This theorem is obviously true if P_k is divisible by f, but requires a proof if this condition is not fulfilled. The proof does not, however, preclude this condition.

We shall denote by $\mathfrak{G}$ the unit ideal of the field. Since $\mathfrak{G}\mathfrak{M}\cdot f = \mathfrak{M}\mathfrak{G}f = \mathfrak{M}\cdot f$, $\mathfrak{M}f$ is a field ideal. As stated above let γ be any integer of $\mathfrak{M}$ and $\{\gamma\} = T\cdot P_k$ where T and P_k are field ideals satisfying the given condition. Since T is relatively prime to $\mathfrak{M}f$, it is relatively prime to f and hence there is a corresponding regular ring ideal $T^{(R)}$. From the ring class reciprocal to that of $T^{(R)}$ select an ideal $I^{(R)}$ such that the corresponding field ideal I is relatively prime to $\{c\Delta\}$. The possibility of such a choice of $I^{(R)}$ was shown in § 3, (W). $I^{(R)}$ is then relatively prime to $\mathfrak{M}f$ which is a divisor of $\{c\Delta\}$ and is also a ring ideal.

---

* By making use of the theorem of § 2 below, the development of (W) can be made without the introduction of the ideal I and the rational integer M_2.

Denote the norm of I by M. Since M and $c\Delta$ are rational integers and I is relatively prime to $\{c\Delta\}$, M and $c\Delta$ are relatively prime, because if M and $c\Delta$ have a common factor let p be a rational prime which divides both M and $c\Delta$, some ideal prime factor of p must be a dividor of I and hence a common factor of I and $\{c\Delta\}$ contrary to our assumption regarding I. Hence $I_k = \{M\}/I$ is also relatively prime to $\{c\Delta\}$ and hence also to $\mathfrak{M}f$ and the corresponding ring ideal $I_k^{(R)}$ is regular.

Since

$$I_k \cdot I = \{M\}, \qquad I_k^{(R)} \cdot I^{(R)} = [M];$$

and since $I^{(R)}$ and $T^{(R)}$ belong to reciprocal classes in R, $I^{(R)} \cdot T^{(R)} = [\alpha]$, where $[\alpha]$ is regular. Multiplying both members of the last equation by $\mathfrak{G}$, we have $T \cdot I = \{\alpha\}$; and since T and I are both relatively prime to $\mathfrak{M}f$, we have

$$\mathfrak{M}f + \{\alpha\} = \{1\} = \mathfrak{G}.$$

Hence $\mathfrak{M}f + [\alpha]$ is the unit ideal of R. The ideals I_k and T belong to the same class, namely the one reciprocal to that of I; hence I_k and P_k belong to reciprocal classes and $I_k \cdot P_k = \{\gamma^{(k)}\}$.

We shall next see that with I_k as defined, the integer $\gamma^{(k)}$ may be chosen from $\mathfrak{M}$. $T \cdot P_k = \{\gamma\}$ where γ belongs to $\mathfrak{M}$. Since M is a rational integer $M\gamma$ belongs to $\mathfrak{M}$. If we then write

$$I_k P_k = \frac{\{M\}}{I} P_k = \frac{\{M\} T \cdot P_k}{IT} = \left\{ \frac{M\gamma}{\alpha} \right\} = \{\gamma^{(k)}\},$$

we may choose $\gamma^{(k)} = M\gamma / a$; hence $\alpha\gamma^{(k)} = M\gamma$, and therefore $\alpha\gamma^{(k)}$ belongs to $\mathfrak{M}$. Therefore if a be any integer from $[\alpha]$, $a\gamma^{(k)}$ belongs to $\mathfrak{M}$. If b be any integer from $\mathfrak{M}f$ then $b\gamma^{(k)}$ belongs to $\mathfrak{M}f$ and hence to $\mathfrak{M}$. From this it follows that $(a+b)\gamma^{(k)}$ belongs to $\mathfrak{M}$. But we have seen that $\mathfrak{M}f + [\alpha] = [1]$; hence a and b may be so chosen that $a + b = 1$ and therefore $\gamma^{(k)}$ belongs to $\mathfrak{M}$.

3. *Preliminary Computations for Rings in a Quadratic Field.* From now on we shall be concerned only with quadratic fields. The development of this article is that corresponding to § 2 of (W).

Let us consider a ring R in a quadratic number field $k(\sqrt{\mu})$. Let $(1, \varrho)$ be a fundamental system of R, so that all its numbers may be written in the form $x+y\varrho$, where x and y are rational integers. As in § 2 of (W), we can then write

$$\varrho^t = C_1^{(t)} + C_2^{(t)}\varrho,$$

and if ϱ is a root of the equation $x^2-mx-n=0$ we have

$$\varrho^2 = n + m\varrho. \tag{1}$$

We therefore see that

$$\begin{aligned} C_1^{(0)} &= 1, & C_1^{(1)} &= 0, & C_1^{(2)} &= n, \\ C_2^{(0)} &= 0, & C_2^{(1)} &= 1, & C_2^{(2)} &= m. \end{aligned}$$

Multiplying both members of

$$\varrho^{t-1} = C_1^{(t-1)} + C_2^{(t-1)}\varrho$$

by ϱ and substituting for ϱ^2 its value from (1), we have

$$\varrho^t = C_2^{(t-1)} \cdot n + (C_1^{(t-1)} + m \cdot C_2^{(t-1)})\varrho,$$

whence we have the recursion formulas

$$C_1^{(t)} = n \cdot C_2^{(t-1)}, \qquad C_2^{(t)} = C_1^{(t-1)} + m \cdot C_2^{(t-1)}. \tag{2}$$

From this, if $\alpha^{(i)} = a_{1i} + a_{2i}\varrho$, we obtain

$$\begin{aligned} \alpha^{(1)} \cdot \alpha^{(2)} \cdots \alpha^{(k-1)} &= B_0^{(k-1)} + B_1^{(k-1)}\varrho + \cdots + B_{k-1}^{(k-1)}\varrho^{k-1} \\ &= A_1^{(k-1)} + A_2^{(k-1)}\varrho, \end{aligned}$$

where

$$A_1^{(k-1)} = \sum_{i=0}^{k-1} C_1^{(i)} B_i; \qquad A_2^{(k-1)} = \sum_{i=0}^{k-1} C_2^{(i)} B_i.$$

We note also that

$$\begin{aligned} (A_1^{(k-1)} + A_2^{(k-1)}\varrho)(a_{1k}a + a_{2k}\varrho) &= (a_{1k}A_1^{(k-1)} \cdot a + a_{2k}A_2^{(k-1)} \cdot n) \\ &\quad + (a_{1k}A_2^{(k-1)} \cdot a + a_{2k}A_1^{(k-1)} + a_{2k}A_2^{(k-1)} \cdot m)\varrho. \end{aligned}$$

4. *Quadratic Forms.* We shall next consider the binary quadratic form $ax^2+bxy+cy^2$ in which a, b, c are rational integers, $a>0$ and $b^2-4ac = g^2\mu$, where $\mu \neq 0$ or 1 and has no square factor. This form we shall denote by $Q(x, y)$, so that

$$a\cdot Q(x, y) = (ax+\varrho y)(ax+\varrho' y), \tag{3}$$

where $\varrho = (b+g\sqrt{\mu})/2$ and $\varrho' = (b-g\sqrt{\mu})/2$.

The numbers ϱ and ϱ' are the roots of the equation

$$x^2-bx+ac = 0. \tag{4}$$

Let us consider all numbers of the form $x+y\varrho$, where x and y are rational integers. Since the sum or difference of two such numbers is a number of the same form, they constitute a modul. Moreover it is easily shown that the product of two of these numbers is a number of the same form, and since for $x = 1$, $y = 0$ we get the number 1, we see that the modul is a ring R of which $(1, \varrho)$ is a fundamental system.

If we next consider all numbers of the form $ax+\varrho y$, they also constitute a modul $\mathfrak{M}$ in R and

$$\begin{aligned}(ax_1&+\varrho y_1)(x_2+\varrho y_2)\\ &= ax_1x_2+(ax_1y_2+x_2y_1)\varrho+y_1y_2\varrho^2\\ &= a(x_1x_2-cy_1y_2)+\varrho(ax_1y_2+x_2y_1+by_1y_2).\end{aligned}$$

Hence the product of any number of $\mathfrak{M}$ by a number of R is a number of $\mathfrak{M}$, and $\mathfrak{M}$ is therefore an ideal in R. The ideal $\mathfrak{M}$ has the fundamental system (a, ϱ).

Since

$$a = a\cdot 1+0\cdot\varrho, \qquad \varrho = 0\cdot 1+1\cdot\varrho,$$

$$\begin{vmatrix} a & 0\\ 0 & 1\end{vmatrix} = a$$

is the norm of the ideal $\mathfrak{M}$.

We shall next determine the index and conductor of R. Let $(1, \theta)$ be a fundamental system of $k(\sqrt{\mu})$. Then $\theta = \sqrt{\mu}$ if $\mu \equiv 2$ or 3 (mod 4) and $\theta = (1+\sqrt{\mu})/2$ if $\mu \equiv 1$ (mod 4). Let us consider the case when $\mu \equiv 2$ or 3 (mod 4). Then

$$(5)\qquad 1 = 1\cdot 1+0\cdot\theta,\qquad \varrho = \frac{b}{2}\cdot 1+\frac{g}{2}\cdot\theta.$$

Now $g^2\mu \equiv 0$ or μ (mod 4) and $b^2-4ac \equiv 0$ or 1 (mod 4). But $g^2\mu = b^2-4ac$, and hence if $\mu \not\equiv 1$ (mod 4), $b^2-4ac \not\equiv 1$ (mod 4), so that $g^2\mu \equiv 0$ (mod 4) and g is even; hence also b is even and $b/2$ and $g/2$ are integers. From (5) we see that $\Delta = g/2$. In the case where $\mu \equiv 1$ (mod 4), if $g^2 \equiv 0$ (mod 4) also $b^2 \equiv 0$ (mod 4), and if $b^2 \equiv 0$ (mod 4) also $g^2 \equiv 0$ (mod 4); hence b and g are simultaneously even or odd. We may then write

$$(6)\qquad 1 = 1\cdot 1+0\cdot\theta,\qquad \varrho = \frac{b-g}{2}+g\theta$$

and see that in this case $\Delta = g$. The conductor f of R is the totality of all integers of $k(\sqrt{\mu})$ such that the product of any integer of $k(\sqrt{\mu})$ by any integer of f belongs to R. Any integer of the field can be written in the form

$$(7)\qquad x+\theta y = x-\frac{b}{g}y+\frac{2y}{g}\varrho = \frac{gx-by}{g}+\frac{2y}{g}\varrho,$$

when $\mu \equiv 2$ or 3 (mod 4), and

$$(8)\quad x+\theta y = x-\frac{b-g}{2g}y+\frac{y}{g}\varrho = \frac{2gx-by+gy}{2g}+\frac{y}{g}\varrho$$

when $\mu \equiv 1$ (mod 4).

The question then remains to determine the nature of an integer α if $\alpha(x+y\theta) = \xi+\varrho\eta$, where ξ and η are rational integers. Since f is contained in R, α must be of the form $u+v\varrho$; hence when $\mu \equiv 2$ or 3 (mod 4), we have

$$\alpha(x+y\theta) = u\cdot\frac{gx-by}{g}+\frac{2uy+gvx-bvy}{g}\varrho+\frac{2vy}{g}\varrho^2$$
$$= \frac{u(gx-by)-2vyac}{g}+\frac{v(gx-by)+2uy+2bvy}{g}\varrho.$$

It follows that u and v must be such that

$$u(gx-by)-2vyac \equiv 0 \qquad (\text{mod } g),$$
$$v(gx-by)+2uy+2bvy \equiv 0 \qquad (\text{mod } g);$$

and since b and g are both even, this reduces to

$$(9)\quad \frac{b}{2}u+acvy\equiv 0,\qquad yu+\frac{b}{2}vy\equiv 0,\qquad (\text{mod } g/2).$$

Since y is arbitrary and hence in general not divisible by a factor of $g/2$, it may be divided out, and we have

$$\frac{b}{2}u+acv\equiv 0,\qquad u+\frac{b}{2}v\equiv 0,\qquad (\text{mod } g/2).$$

Since $b^2/4\equiv ac$ (mod $(g/2)^2$), we see that the first congruence is obtained from the second by multiplying it by $b/2$ and hence any values of u and v which satisfy the second congruence also satisfy the first. We may therefore write $u=w\cdot g/2-v\cdot b/2$, and hence

$$u+v\varrho=w\cdot\frac{g}{2}-v\cdot\frac{b}{2}+v\cdot\frac{b}{2}+v\cdot\frac{g}{2}\sqrt{\mu}=\frac{g}{2}(w+v\sqrt{\mu});$$

hence α must be a multiple of $g/2$. Moreover as is easily seen from (7), any integer of the field times $g/2$ is an integer of R. Therefore f is contained in $\{g/2\}$, and $\{g/2\}$ is contained in f, so that $f=\{g/2\}$.

In the case where $\mu\equiv 1$ (mod 4), (8) shows that g, and hence also $\{g\}$, is contained in f. Again, let $u+v\varrho$ be any integer of f. Then

$$(u+v\varrho)(x+\theta y)=\frac{(2gx-by+gy)u-2acvy}{2g}+\frac{(2gx-by+gy)v+2uy+2bvy}{2g}\varrho,$$

and, as above, this leads to the congruences

$$(b-g)u+2acv\equiv 0,\quad 2u+(b+g)v\equiv 0,\quad (\text{mod } 2g);$$

or, since b and g are simultaneously odd or even,

$$(10)\quad \frac{b-g}{2}u+acv\equiv 0,\qquad u+\frac{b+g}{2}v\equiv 0,\quad (\text{mod } g).$$

Multiplying the second congruence by $(b-g)/2$, we have

$$\frac{b-g}{2}\cdot u+\frac{b^2-g^2}{4}v\equiv 0\qquad (\text{mod } g).$$

But $b^2-4ac = g^2\mu$ and hence $b^2-g^2 = 4ac+g^2(\mu-1) \equiv 4ac (\text{mod } 4g^2)$ since $\mu-1 \equiv 0 \ (\text{mod } 4)$; and therefore $(b^2-g^2)/4 \equiv ac \ (\text{mod } g)$. Again we see that any pair of values of u and v which satisfy the second congruence will also satisfy the first, and we may therefore write

$$u = wg - \frac{b+g}{2}\cdot v$$

and

$$\begin{aligned} u+v\varrho &= wg - \frac{b+g}{2}\cdot v + \frac{b}{2}\cdot v + \frac{vg}{2}\sqrt{\mu} \\ &= g\left(w+v\frac{1+\sqrt{\mu}}{2}\right) = g(w+v\theta). \end{aligned}$$

Hence f is contained in $\{g\}$; and, as above, we conclude that $f = \{g\}$.

EXAMPLE. As an example of the foregoing, let us consider the form $Q(x, y) = 11x^2+5xy+2y^2$. The discriminant is -63; hence $g = 3$, $\mu = -7$, $\mu \equiv 1 \ (\text{mod } 4)$, $\varDelta = 3$, $f = \{3\}$, $\varrho = (5+3\sqrt{-7})/2$, $R = (1, \varrho)$ and $\mathfrak{M} = (11, (5+3\sqrt{-7})/2)$. The number of classes in $k(\sqrt{-7})$ is one and in R it is four. If we denote the principal class of R by C_0 and the others by C_1, C_2, C_3, the following table gives a representative ideal for each class, the reciprocal class, and a binary quadratic form from the corresponding class of forms. To save space, the computations have been omitted.

Class	Representative ideal	Corresponding form	Reciprocal class	Representative ideal
C_0	$\left[1, \frac{5+3\sqrt{-7}}{2}\right]$	$x^2+5xy+22y^2$	C_0	$\left[1, \frac{5+3\sqrt{-7}}{2}\right]$
C_1	$\left[2, \frac{5+3\sqrt{-7}}{2}\right]$	$2x^2+5xy+11y^2$	C_2	$\left[2, \frac{5-3\sqrt{-7}}{2}\right]$
C_2	$\left[2, \frac{5-3\sqrt{-7}}{2}\right]$	$2x^2+5xy+11y^2$	C_1	$\left[2, \frac{5+3\sqrt{-7}}{2}\right]$
C_3	$\left[7, \frac{7+3\sqrt{-7}}{2}\right]$	$7x^2+7xy+4y^2$	C_3	$\left[7, \frac{7+3\sqrt{-7}}{2}\right]$

5. *Application to Diophantine Equations.* We shall now turn our attention to the study of the diophantine equation

$$Q(x, y) = u_1 u_2 \cdots u_{k-2}, \tag{11}$$

where it is required to determine the rational integral values of $x, y, u_1, u_2, \cdots u_{k-2}$ which satisfy the equation. Obviously the equation may be written in the form

$$N(ax+\varrho y) = au_1 u_2 \cdots u_{k-2}. \tag{12}$$

Following the procedure in (W), with the modification that $P_1, P_2, \cdots, P_{k-2}$ are constructed from prime ideals relatively prime to $\mathfrak{M}f$, we can show in the same manner that $u_i = \varepsilon_i \mu_i' \cdot \mu_i'' F_i(e_1^{(i)}, e_2^{(i)})$.

Before writing down the expressions for x and y, I shall make some observations regarding μ_i', and μ_i''. $|\mu_i' \cdots \mu_{k-2}'| = N(P_{k-1})$ where P_{k-1} is relatively prime to $\mathfrak{M}f$, but no μ_i' is itself the norm of an ideal. Hence each μ_i' contains only the first powers of such rational primes as are primes in $k(\sqrt{\mu})$ and every such prime must occur in an even number of the μ_i'. But since these primes are relatively prime to f and hence also to Δ, $x+\varrho y$ cannot contain such factors unless the x and y are both divisible by the prime, because with respect to such a prime as a modulus ϱ cannot satisfy a congruence of lower than the second degree. But such a solution can always be obtained from one in which the x and y do not have this common factor by multiplying it into x and y and any two of μ_i'. There is therefore no loss in assuming that $P_{k-1} = \{1\}$ and $I_{k-1} = \{1\}$; and hence $\alpha^{(k-1)} = +1$. This makes it possible to make $\mu_i' = +1$, $(i = 1, 2, \cdots, k-2)$, and in so doing we have in no way restricted the solution.

In accordance with what was stated above $P_1, P_2, \cdots, P_{k-2}$ are all relatively prime to $\mathfrak{M}f$ and P_k contains no ideal prime factors except such as are factors of $\mathfrak{M}f$. In writing the equation in the form (12) the factor a was introduced on the right hand side of the equation and now

$$N(P_k) = |a\mu_1'' \cdot \mu_2'' \cdots \mu_{k-2}''|, \qquad \text{where } a = N_R(\mathfrak{M}).$$

If we now use the method of (W) and the formulas in § 3 above, when we put $\gamma^{(k)} = a_{1k}\alpha + a_{2k}\varrho$, we have the following formulas for the solutions of (11):

$$(13) \quad \begin{cases} x = \dfrac{1}{M}\left(a_{1k}A_1^{(k-2)} - a_{2k}A_2^{(k-2)}\cdot c\right), \\ y = \dfrac{1}{M}\left(a_{1k}A_2^{(k-2)}\cdot a + a_{2k}A_1^{(k-2)} + a_{2k}A_2^{(k-2)}\cdot b\right), \end{cases}$$

where $M = N(I_1 \cdot I_2 \cdots I_{k-2})$ as in (W);

$$(14) \qquad \mu_i = \varepsilon_i \mu_i'' F_i(e_1^{(i)}, e_2^{(i)}), \quad (i = 1, 2, \cdots, k-2),$$

$$\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{k-2} = +1.$$

In these formulas the upper index on the A_i is $k-2$ in place of $k-1$ due to the fact that $\alpha^{(k-1)}$ was chosen equal to $+1$.

If $(\beta_1^{(i)}, \beta_2^{(i)})$ is a fundamental system of the ring ideal $I_i^{(R)}$ and

$$\beta_1^{(i)} = b_{11}^{(i)} + b_{12}^{(i)}\varrho, \qquad \beta_2^{(i)} = b_{21}^{(i)} + b_{22}^{(i)}\varrho,$$

we have, if $\alpha^{(i)} = a_{1i} + a_{2i}\varrho = e_1^{(i)}\beta_1 + e_2^{(i)}\beta_2$ is any integer of $I_i^{(R)}$,

$$(15) \quad a_{1i} = b_{11}^{(i)}e_1^{(i)} + b_{21}^{(i)}e_2^{(i)}, \qquad a_{2i} = b_{12}^{(i)}e_1^{(i)} + b_{22}^{(i)}e_2^{(i)}.$$

By means of (15) for $i = 1, 2, \cdots, k-2$ the expressions (13) for x and y may be obtained in terms of the parameters $e_1^{(i)}, e_2^{(i)}, i = 1, 2, \cdots, k-2$. The a_{1k} and a_{2k} are also parameters which enter into the expressions for the μ_i in the factors μ_i''.

The complete development of the foregoing part of this section is identical with that given in (W) except that here the $P_1, P_2, \cdots, P_{k-2}$ are all relatively prime to $\mathfrak{M}f$ and due to the development of § 2 above, the introduction of the ideal J and the rational integer M_2 is not needed. This is also true for (W) but I did not see it at the time.

The results can now be summarized in the following statement of a method for obtaining the solutions of (11). Select any integer γ from $\mathfrak{M} = (\alpha, \varrho)$, and separate $\{\gamma\}$ into the product of two field ideals P_k and T, where T is relatively prime to $\mathfrak{M}f$ and P_k contains no prime factors except such as are divisors of $\mathfrak{M}f$. Separate T into the product of $k-2$ field ideals $P_1, P_2, \cdots, P_{k-2}$ in any way and let $P_i^{(R)}$ be the ring ideal corresponding to P_i. From the ring class reciprocal to that of $P_i^{(R)}$ select an ideal $I_i^{(R)}$ which is such that the corresponding field ideal I_i is relatively prime to $\{a\Delta\}$ and put $I = I_1 \cdot I_2 \cdots I_{k-2}$. Let $N(I) = M$ and $I_k = \{M\}/I$ and $I_k \cdot P_k = \{\gamma^{(k)}\} = a_{1k} + a_{2k}\varrho$ by § 2. Let $F_i(x, y)$ be the binary quadratic form corresponding to the base $(\beta_1^{(i)}, \beta_2^{(i)})$ of $I_i^{(R)}$. Then the formulas (13) and (14) by means of relations (15) for $\varepsilon_1 \cdot \varepsilon_2 \cdots \varepsilon_{k-2} = +1$ give solutions for (11) in terms of the parameters $e_1^{(i)}, e_2^{(i)}$ with the signs of the μ_i'' so chosen that the sign of their product is the same as that of $N(\gamma^{(k)})$.

By a variation in the separation of T into factors or by a variation in the choice of γ so that different classes are represented by $P_1^{(R)}, P_2^{(R)}, \cdots, P_{k-2}^{(R)}$ we can obtain all solutions of the equation. The values given to the parameters are rational integral and all solutions so obtained are rational integers.

EXAMPLE. Let us consider the equation

$$11x^2 + 5xy + 2y^2 = u_1 \cdot u_2 \cdot u_3.$$

Here $\mathfrak{M} = (11, (5+3\sqrt{-7})/2)$. Select, from $\mathfrak{M}$, $\gamma = 11 + 2(5+3\sqrt{-7})/2 = 16+3\sqrt{-7}$;

$$\{\gamma\} = \{2-\sqrt{-7}\}\{1+2\sqrt{-7}\},\ P_5 = \{2-\sqrt{-7}\},$$
$$T = \{1+2\sqrt{-7}\}.$$

T is a prime ideal and hence its separation into $P_1 \cdot P_2 \cdot P_3$ gives $P_1 = \{1+2\sqrt{-7}\}$, $P_2 = \{1\}$, $P_3 = \{1\}$,

$$P_1^{(R)} = \left(29, \frac{13-3\sqrt{-7}}{2}\right),\ P_2^{(R)} = \left(1, \frac{5+3\sqrt{-7}}{2}\right),$$
$$P_3^{(R)} = \left(1, \frac{5+3\sqrt{-7}}{2}\right).$$

$P_1^{(R)}$ belongs to the class C_1 (see table in § 4). $P_2^{(R)}$ and $P_3^{(R)}$ belong to class C_0.

$$
\begin{aligned}
I_1^{(R)} &= \left(2, \frac{5-3\sqrt{-7}}{2}\right), & F_1(x, y) &= 2x^2+5xy+11y^2, \\
I_2^{(R)} &= \left(1, \frac{5+3\sqrt{-7}}{2}\right), & F_2(x, y) &= x^2+5xy+22y^2, \\
I_3^{(R)} &= \left(1, \frac{5+3\sqrt{-7}}{2}\right), & F_3(x, y) &= x^2+5xy+22y^2, \\
I_1 &= \left\{\frac{1+\sqrt{-7}}{2}\right\}, & I_2 &= \{1\}, \quad I_3 = \{1\}, \\
I &= I_1 \cdot I_2 \cdot I_3 = I_1, & M &= N(I) = 2,
\end{aligned}
$$

$$I_5 = \{M\}/I = I_1' = \frac{1-\sqrt{-7}}{2},$$

$$I_5 \cdot P_5 = \left\{\frac{1-\sqrt{-7}}{2}\right\}\{2-\sqrt{-7}\} = \left\{-\frac{5+3\sqrt{-7}}{2}\right\} = \{\gamma^{(5)}\};$$

$\gamma^{(5)}$ is an integer of $\mathfrak{M}$. The sign may be neglected.

The linear forms used in computing (13) and (14), after application of (15), become

$$
\begin{aligned}
\alpha' &= 2e_1^{(1)} + \frac{5-3\sqrt{-7}}{2} e_2^{(1)}, \\
\alpha'' &= e_1^{(2)} + \frac{5+3\sqrt{-6}}{2} e_2^{(2)}, \\
\alpha''' &= e_1^{(3)} + \frac{5+3\sqrt{-7}}{2} e_2^{(3)}, \\
\alpha' \cdot \alpha'' \cdot \alpha''' \gamma^{(5)} &= 11 \cdot 2x + \varrho \cdot 2y,
\end{aligned}
$$

from which we have

$$
\begin{aligned}
x = {} & e_2^{(1)} e_1^{(2)} e_1^{(3)} - 2e_1^{(1)} e_2^{(2)} e_1^{(3)} - 2e_1^{(1)} e_1^{(2)} e_2^{(3)} \\
& - 10e_1^{(1)} e_2^{(2)} e_2^{(3)} - 22e_2^{(1)} e_2^{(2)} e_2^{(3)}, \\
y = {} & e_1^{(1)} e_1^{(2)} e_1^{(3)} + 5e_1^{(1)} e_2^{(2)} e_1^{(3)} + 11e_2^{(1)} e_2^{(2)} e_1^{(3)} + 5e_1^{(1)} e_1^{(2)} e_2^{(3)} \\
& + 11e_2^{(1)} e_1^{(2)} e_2^{(3)} + 3e_1^{(1)} e_2^{(2)} e_2^{(3)} + 55e_2^{(1)} e_2^{(2)} e_2^{(3)}.
\end{aligned}
$$

Since $N(P_5) = a = 11$, we find $\mu_1'' = \mu_2'' = \mu_3'' = 1$; and hence

$$
\begin{aligned}
u_1 &= \pm 2(e_1^{(1)})^2 + 5(e_1^{(1)} e_2^{(1)}) + 11(e_2^{(1)})^2, \\
u_2 &= \pm (e_1^{(2)})^2 + 5(e_1^{(2)} e_2^{(2)}) + 22(e_2^{(2)})^2, \\
u_3 &= \pm (e_1^{(3)})^2 + 5(e_1^{(3)} e_2^{(3)}) + 22(e_2^{(3)})^2.
\end{aligned}
$$

For $e_1^{(1)} = 2$, $e_2^{(1)} = 1$, $e_1^{(2)} = -1$, $e_2^{(2)} = 2$, $e_1^{(3)} = -2$, $e_2^{(3)} = -1$, we have $x = 98$, $y = -181$, $u_1 = \varepsilon_1 29$, $u_2 = \varepsilon_2 79$, $u_3 = \varepsilon_3 36$, $\varepsilon_1\varepsilon_2\varepsilon_3 = +1$. For $e_1^{(1)} = 1$, $e_2^{(1)} = 1$, $e_1^{(2)} = -1$, $e_2^{(2)} = 1$, $e_1^{(3)} = 2$, $e_2^{(3)} = -1$, $x = 24$, $y = -12$, $u_1 = \varepsilon_1 18$, $u_2 = \varepsilon_2 18$, $u_3 = \varepsilon_3 16$, $\varepsilon_1\varepsilon_2\varepsilon_3 = +1$.

For all $e_i^{(i)} = +1$, $x = -35$, $y = 91$, $u_1 = \varepsilon_1 18$, $u_2 = \varepsilon_2 28$, $u_3 = \varepsilon_3 28$, $\varepsilon_1\varepsilon_2\varepsilon_3 = +1$.

In order to obtain a different class of solutions, it would be necessary to make another choice of γ so that it can be separated into factors representing different classes from C_1, C_0, C_0.

6. *The Case* $k = 4$. When the diophantine equation to be solved is $Q(x, y) = u_1 \cdot u_2$, the same method as that used above will give a solution which may be easily written down

$$\begin{aligned} \alpha' &= e_1^{(1)}\beta_1^{(2)} + e_2^{(2)}\beta_2^{(2)}, \\ \alpha'' &= e_1^{(2)}\beta_1^{(2)} + e_2^{(2)}\beta_2^{(2)}, \\ \alpha^{(4)} &= a_{14}a + a_{24}\varrho, \end{aligned}$$

where $(\beta_1^{(1)}, \beta_2^{(1)})$ and $(\beta_1^{(2)}, \beta_2^{(2)})$ are bases of the two ideals I_1 and I_2. Let $F_1(x, y)$ and $F_2(x, y)$ be the corresponding binary quadratic forms. Then the solutions are given by the following relations:

$$(16) \qquad \left\{\begin{aligned} x &= \frac{a_{14}A_{12} - a_{24}A_{22}c}{M}, \\ y &= \frac{A_{14}A_2^{(2)} + a_{24}A_1^{(2)} + a_{24}A_2^{(2)}b}{M}, \\ u_1 &= \varepsilon_1\mu_1'' F_1(e_1^{(1)}, e_2^{(1)}), \\ u_2 &= \varepsilon_2\mu_2'' F_2(e_1^{(2)}, e_2^{(2)}), \end{aligned}\right.$$

where

$$(17) \qquad A_1^{(2)} = \sum_{i=0}^{2} C_1^{(i)}B_i, \qquad A_2^{(2)} = \sum_{i=0}^{2} C_2^{(i)}B_i,$$

$$(18) \qquad \left\{\begin{aligned} C_1^{(0)} &= 1, & C_1^{(1)} &= 0, & C_1^{(2)} &= -ac, \\ C_2^{(1)} &= 0, & C_2^{(1)} &= 1, & C_2^{(2)} &= b, \end{aligned}\right.$$

$$(19)\quad B_0 = a_{11}a_{12}, \quad B_1 = a_{11}a_{22} + a_{12}a_{21}, \quad B_2 = a_{21}a_{22},$$

$$(20)\qquad \begin{cases} a_{11} = b_{11}^{(1)}e_1^{(1)} + b_{21}^{(1)}e_2^{(1)}, \\ a_{21} = b_{12}^{(1)}e_1^{(1)} + b_{22}^{(2)}e_2^{(2)}, \\ a_{12} = b_{11}^{(2)}e_1^{(2)} + b_{21}^{(2)}e_2^{(2)}, \\ a_{22} = b_{12}^{(2)}e_1^{(2)} + b_{22}^{(2)}e_2^{(2)}, \end{cases}$$

where the $b_{ij}^{(k)}$ are constants depending on the ideals I_1 and I_2.

Let us now suppose that the ring R is the totality of all integers in $k(\sqrt{\mu})$. Then $(1, \varrho)$ is a fundamental system of the field; hence when $\mu \equiv 2$ or $3 \pmod 4$, $b = 0$ and when $\mu \equiv 1 \pmod 4$, $b = 1$. In either case $f = \{1\}$ and $\Delta = \pm 1$. No simplification would occur in the formulas (16) to (20) except that the b would be replaced by its value.

If, however, we make the additional restriction that $\mathfrak{M} = \mathfrak{G} = \{1\}$, we have $a = 1$, $P_k = \{1\}$; hence I_1 and I_2 belong to reciprocal classes, and in this case we may make $I_2 = I_1'$ and $F_1(x, y) = F_2(x, y)$. Moreover, since $P_k = \{1\}$ and $\mathfrak{M} = \{1\}$, we have $\mu_1'' = \mu_2'' = 1$, and the last two formulas of (16) become

$$u_1 = \varepsilon_1 F\left(e_1^{(1)}, e_2^{(1)}\right), \quad u_2 = \varepsilon_2 F\left(e_1^{(2)}, e_2^{(2)}\right), \qquad \varepsilon_1\varepsilon_2 = +1.$$

Since I_2 is the conjugate of I_1, we have $I_1I_2 = \{e\}$, where $N(I_1) = e$; hence $N(I) = e^2 = M$ and $\{M\}/I = I_4 = \{e\}$. Then $I_4P_4 = \{e\}\{1\} = \{e\}$. Hence $\gamma^{(k)} = e$ and $a_{14} = e_1$, $a_{24} = 0$. The first two of the formulas (16) now become

$$x = A_1^{(2)}/e = \frac{a_{11}a_{12} - a_{21}a_{22}c}{e},$$

$$y = A_2^{(2)}/e = \frac{a_{11}a_{22} + a_{21}a_{12} + a_{21}a_{22}b}{e}.$$

Since I_1 and I_2 are conjugate ideals and their norm is e, we can write the base of I_1 in the form $(e, f + \varrho)$ and that of I_2 in the form $(e, f + \varrho')$, where $f^2 + bf + c \equiv 0 \pmod e$. Hence we have

$$b_{11}^{(1)} = e, \quad b_{21}^{(1)} = f, \quad b_{11}^{(2)} = e, \quad b_{12}^{(1)} = 0, \quad b_{22}^{(1)} = 1, \quad b_{12}^{(2)} = 0,$$

for all μ; and

$$b_{21}^{(2)} = f, \qquad b_{22}^{(2)} = -1,$$

when $\mu \equiv 2$ or 3 (mod 4); and

$$b_{21}^{(2)} = f+1, \quad b_{22}^{(2)} = -1,$$

when $\mu \equiv 1$ (mod 4).

Using these values for the $b_{ij}^{(k)}$ and combining the relations (16) to (20), we have, when $\mu \equiv 2$ or 3 (mod 4),

$$(21) \qquad \begin{cases} x = ee_1^{(1)}e_1^{(2)} + f(e_1^{(1)}e_2^{(2)} + e_2^{(1)}e_1^{(2)}) + ge_2^{(1)}e_2^{(2)}, \\ y = e_1^{(2)}e_2^{(1)} - e_1^{(1)}e_2^{(2)}, \end{cases}$$

where $g = \dfrac{f^2+c}{e}$. When $\mu \equiv 1$ (mod 4), we have

$$(22) \qquad \begin{cases} x = ee_1^{(1)}e_1^{(2)} + fe_2^{(1)}e_1^{(2)} + (f+1)e_1^{(1)}e_2^{(2)} + ge_2^{(1)}e_2^{(2)}, \\ y = e_1^{(2)}e_2^{(1)} - e_1^{(1)}e_2^{(2)}, \end{cases}$$

where $g = (f^2+f+c)/e$.

The binary quadratic form $F(x, y)$ may be chosen as the form corresponding to the base of I_1 as given above, and hence

$$F(x, y) = ex^2 + 2fxy + gy^2, \quad \text{when} \quad \mu \equiv 2 \text{ or } 3 \pmod 4;$$
$$F(x, y) = ex^2 + (2f+1)xy + gy^2, \quad \text{when} \quad \mu \equiv 1 \pmod 4.$$

The formulas (16) as thus reduced are those given by Dickson* if we put

$$e_1^{(1)} = e, \quad e_2^{(1)} = g, \quad e_1^{(2)} = u, \quad e_2^{(2)} = -2.$$

A shorter proof of these formulas was published later by Dickson†. The proof as here given is by reduction from the general case by the following specializations: (1) $k = 4$; (2) $R = \mathfrak{G}$, the ring of all integers in $k(\sqrt{\mu})$; and (3) $\mathfrak{M} = \mathfrak{G}$.

THE UNIVERSITY OF MISSOURI

---

* This BULLETIN, vol. 27, No. 8 (May, 1921), p. 353.

† This BULLETIN, vol. 29, No. 10 (Dec., 1923), p. 464.

NÖRLUND ON CALCULUS OF DIFFERENCES

Vorlesungen über Differenzenrechnung. By Niels Erik Norlund. Berlin, Springer, 1924. ix + 551 pages.

This is the first book to develop the theory of the difference calculus from the function-theoretic point of view and to include a significant part of the recent researches having to do with the analytic and asymptotic character of the solutions of linear difference equations. As such it is an important contribution to the mathematical literature and will render a service not procurable from any of its predecessors. The author presents a connected account of what appears to him to be the most important and the best developed domains of the difference calculus The book is intended to give a preliminary view of the field and to facilitate the reading of the original memoirs. With this purpose in view the author sometimes omits proofs (too frequently, we think) leaving the reader to find them in the indicated memoirs. There is unfortunately little attempt to make clear the historical aspects of the subject, with the result that the casual reader of the book will get an erroneous impression of the actual historical development. It is the reviewer's opinion that this danger is not sufficiently safeguarded by the reference to Norlund's *Neuere Untersuchungen über Differenzengleichungen* in the ENZYKLOPADIE DER MATHEMATISCHEN WISSENSCHAFTEN, Band II C 7.

The most important earlier books on the difference calculus are those of Lacroix, Boole, Markoff, Pascal, Pincherle, and Wallenberg and Guldberg. They proceed principally from the algebraic standpoint (with a partial exception in the case of the last) and often employ symbolic methods. In Norlund's book the point of view is function-theoretic throughout. This marks well the shift of interest on the part of those who have developed the difference calculus. The more formal aspects, elementary solutions, interpolations and such things engaged the attention of the founders, among whom were Newton, Taylor, Stirling, Laplace and Gauss. "Dann kam eine Zeit," as Norlund says (in his introduction, page 2), "in der das Gebiet fast ganz brach lag und keine nennenswerten Fortschritte erzielt wurden. Heute jedoch zeigt sich wieder frisches Leben; unter neuen Gesichtspunkten und mit besseren Hilfsmitteln als früher hat man das Studium der zum Teil bis in die Anfange der Differenzrechnung zurückreichenden Probleme wieder aufgenommen, und in den letzten Jahren ist eine ziemlich umfangreiche Literatur emporgewachsen, aus der als besonders wertvoll einige von Pincherle, Birkhoff, Carmichael, Galbrun und Perron herrührende Arbeiten zu nennen sind."

To this list of recent investigators the name of Nörlund himself is to be added and is to be given a place of large importance. We believe that the name of Poincaré should also be mentioned here as one who made an important contribution to the difference calculus in 1885 and thus gave the first intimation of the possible rejuvenescence of the theory of difference equations. Norlund himself (p. 272) refers to this work of Poincaré as opening up for the first time a real theory of linear difference equations by means of a (now celebrated) theorem on the asymptotic behavior of the solutions of such equations.

The main part of Norlund's text covers 453 pages. About one-third of the whole (making up the larger part of the first 200 pages) is given to the problem of finite integration and its congeners and generalizations. By a finite integral of a function $\varphi(x)$ we mean a function $f(x)$ satisfying the equation $[f(x+\omega)-f(x)]/\omega = \varphi(x)$. A related problem is that of determining a function $g(x)$ which satisfies the equation $[g(x+\omega)+g(x)]/2 = \varphi(x)$. In each of these equations ω is an arbitrary complex number. For many purposes ω may be given the value unity without loss of generality. But, for the theory of finite integration, as developed by Norlund, it plays the role of an independent variable. The generalization of finite integration in the direction of what may be called repeated finite integration and a similar generalization of the related problem associated with the second of the foregoing equations are treated at length. As an example of the first we have the solution of the second order equation

$$\frac{f(x+\omega_1+\omega_2)-f(x+\omega_1)-f(x+\omega_2)+f(x)}{\omega_1\,\omega_2} = \varphi(x)$$

for $f(x)$ in terms of the given function $\varphi(x)$. Special cases of these problems are also treated at length; and, in particular, the determination of the finite integral of $\nu x^{\nu-1}$ when $\omega = 1$, together with the solution of related special problems. An elegant treatment is given of the theory of the usual and the higher Bernoulli and Euler polynomials.

The chief attention of the first seven chapters (except for a treatment of gamma functions and related functions in the twenty-one pages of Chapter V) is given to the various aspects of the problem of finite integration, or, as the author names it, that of finding the "sum" of a given function. The general finite integral of the given function $\varphi(x)$ involves an arbitrary periodic function of x of period ω. One principal purpose of Norlund is to develop a direct algorithm by means of which finite integrals having characteristic properties can be selected from the totality of finite integrals. This purpose he achieves by means of appropriate definitions of summability applied to the direct formal sums

$$-\omega \sum_{s=0}^{\infty} \varphi(x+s\omega), \qquad \omega \sum_{s=1}^{\infty} \varphi(x-s\omega)$$

as formal finite integrals of the given function $\varphi(x)$. By this means he is able to set up appropriate definitions of finite integrals of such sort as to characterize them in a very satisfactory way and to render them determinate except for an additive constant. Thus in this book and in the relevant parts of his preceding memoirs the author brings the problem of finite integration to a stage of development comparable to that of integration in the infinitesimal calculus. Here his treatment has much of novelty and importance. One can only wish that he had more effectively brought out the relation of his own methods and results to those of other (and especially earlier) investigators who have had occasion to treat finite integration either for its own sake or as incidental to some larger problem in which it occurs.

Chapter eight (with 58 pages) is given to various types of interpolation series, with special attention to those of Stirling and Newton and to the character of functions defined as sums of such series.

An all-too-short development of the theory of factorial series is given in the ninth chapter, attention being confined to the most important results and especially to those which will be needed in later chapters.

Chapters ten to thirteen inclusive (pages 272–386) are devoted to the general theory of linear homogeneous difference equations. The first one opens with a general existence theorem developed by a method due to Norlund. Then comes Ostrowski's recent proof of Holder's theorem to the effect that the gamma function can not satisfy an algebraic differential equation, thus bringing out the fact that linear difference equations introduce us to a class of transcendental functions not brought to light by the study of differential equations. The generalizations or extensions of Holder's theorem are not treated. A few elementary matters then follow and the chapter closes with a final section (of 13 pages) on Poincaré's theorem concerning the asymptotic character of solutions of difference equations.

The eleventh chapter (pages 314–352) is devoted to homogeneous linear difference equations with rational coefficients. It gives a rather extended development of a considerable part of the beautiful theory of these equations. The leading reference (page 314) is to Norlund's 1915 paper in the ACTA MATHEMATICA and the methods employed in the chapter are due principally to him. In another and later part of the volume, namely on page 379 in the thirteenth chapter, is the statement that Birkhoff obtained (for these equations) many interesting and beautiful results "zu denen wir in Kapitel 11 unter anderen Gesichtspunkten gelangt sind." When it is remembered that Birkhoff's work was published in 1911, four years before that of Norlund, it is seen to be desirable that Chapter eleven should contain a brief analysis of the relation between the work of the two authors with a clear indication of priority, especially if it is true (as I believe it to be) that the most fundamental known properties of these equations are first developed in Birkhoff's memoir.

The twelfth chapter (pages 353–378) contains a development of the theory of those linear homogeneous difference equations whose coefficients can be expressed by means of factorial series. It contains an important contribution due to Norlund himself. The method is one of the most pleasing yet known for treating the theory of linear difference equations; and I expect to see it developed by generalization so as to be still more effective. Factorial series afford one of the most important tools of analysis; and I believe that both they and certain of their generalizations have an importance beyond that which is generally recognized.

The treatment of linear homogeneous difference equations is brought to a close in the thirteenth chapter (pages 379–386) with a very short and imperfect account of the researches of Birkhoff in 1911. With 107 pages devoted to other contributions to the theory of homogeneous linear difference equations, it is not satisfying to see this important memoir passed over so briefly. If the material contained in these 107 pages requires that space for its proper presentation, then, it appears to me that another hundred pages is needed to present with proper distribution of emphasis the matter which is omitted. This would be taken mostly from the work of three out of the five recent investigators singled out by Norlund for special mention in the passage which we have quoted from his introduction. In the absence of such a modified distribution of emphasis the exposition fails to give a correct impression of the historical order of development of the subject and of the relative importance of the results so far attained.

In the fourteenth chapter (pages 387–414) we have a treatment of the non-homogeneous linear difference equation. Section one contains an elementary account of Lagrange's method of variation of parameters. Equations with constant coefficients are treated in Section two. An account of the researches of E. Hilb in 1922 is given in the final Section three. It appears to me that the most important memoir on the solutions of non-homogeneous linear difference equations is that of K. P. Williams (TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), pp. 209–240). Nörlund nowhere in his book utilizes the results of this memoir by Williams, and there is indeed no reference to Williams in the index though his papers are listed in the bibliography. The failure to utilize Williams's researches leaves the chapter inadequate to represent the actual state of the theory of non-homogeneous equations.

The last chapter (pages 415–455) is devoted to reciprocal differences and continued fractions. It is followed by a few pages of tables, an extensive bibliography and an index. In the bibliography the author has included all works known to him and dealing with the general theory of difference equations. From the very extensive special literatures of Bernoulli polynomials and gamma functions he has listed only the most important works. The bibliography is a very useful one. It is hardly to be expected that it should be complete. In fact I have found a considerable number of omissions by checking it against the partial bibliography which I have collected in an incidental way during the past fifteen years.

It is natural to expect that an exposition of a general subject should involve an important element depending on the personal interests of the author; and this is particularly true in the case of a book which is essentially the first in its field. But in the present book this element appears to me to have played too large a role in determining the distribution of emphasis and the selection of material. Much of the work in the first two hundred pages might well have been given with less fullness and the space so gained have been utilized in the presentation of some of the important matters which are omitted.

While this book will probably stand for some time as the best book in its field, and as such is therefore of great importance, it can not be regarded as having come near to being a definitive treatise on the difference calculus, even in its present state of development. Whatever one may think of the distribution of emphasis and selection of material in this volume there is still a definite need for another book with a quite different distribution and selection — one in which the personal equation of the author does not play so large a role.

R. D. CARMICHAEL

ENRIQUES ON ALGEBRAIC GEOMETRY

Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche. Vol. I and vol. II. By F. Enriques. Bologna, O. Chisini, 1915, 1918.

The reviewer, far from being a specialist in algebraic geometry, commences this short review with the misgiving that in the last two years of reading "at" the work of this famous author he has set his wisdom teeth into a sticky mouthful. It may however be said at the outset that the paucity, however unfair, of references to living workers in American universities,—a general reference to Osgood's *Funktionentheorie* and particular ones to Scott (C. A. Scott) and "Angas" ("Ch. Angas Scott"(!)),—indicates that the field is one in which few of us are expert, and therefore not only that the point of view of the reviewer will be that of most of his readers, but also that the treatise itself unbars a field which some of us might well explore. The reviewer tried the experiment of lecturing from it to capable, and patient, advanced students during the past academic year.

In the first place, although much elementary material is given, the prerequisites in the way of geometry and algebra are not meagre. Since many particular theorems of projective geometry are used and since the connection between algebra and geometry is obtained mainly by means of projective coordinates, which are independent of metrical axioms, the book should be preceded by a good study of projective geometry, such as Enriques's own, to which many references are given. The reviewer found to be even better adapted for this purpose the simpler logical structure of Veblen and Young, volume I and parts of volume II, of which he gave a three months analysis in the lectures just mentioned. On this basis, for example, the theory of linear systems and abstract elements in Book I is much simplified. As to prerequisites in algebra, one may assume that for advanced students such things as linear dependence and some ideas about substitutions and groups will be familiar; but can one say the same about topics like resultants and discriminants of functions of more than one variable and the Galois theory? The reader will save time by knowing something of these in advance. Perhaps finally, for good measure, he might have at hand Miss Scott's book as an introduction to the second volume and a general indication of the sort of problem to be considered.

The general plan of the treatise, in the author's words, is to investigate all those methods and concepts suggested originally by a study of plane algebraic curves, and to apply them in such a way as to constitute a qualitative account of algebraic equations and functions. So far, four books, in two volumes, have been published, viz., (I) a general introduction, (II) correspondences, (III) the theory of plane curves based on polars, (IV) the singularities of curves. A third volume is planned, in order to deal with the properties of curves invariant under birational transformations; perhaps in this the function-theoretic analysis will receive greater emphasis. Many methods are used for the analysis of the same problem, some for heuristic, some for rigoristic purposes and some for additional insight into the various ramifications of the problem. The various books stand as far as practicable on their own feet and may to that extent be read independently, and many of the "notes" or complements may be omitted in a first reading. Historical notes receive considerable emphasis throughout. What will much delight and instruct the reader is the frequent discussion of paradoxes and sophisms, which clarify the assumptions on which theorems rest.

Algebraic geometry is necessarily of the complex domain, as distinguished from that which seeks merely to illustrate the infinitesimal analysis of real functions, and equally naturally is, as far as achievable, a geometry *in extenso.* The methods which are developed may also be generalized and extended to give information about hyperspaces and algebraic varieties in an arbitrary number of dimensions.

Book I starts with an analysis of groups of points on a line and formations invariant under projective transformations, with special reference to groups of four points, involutions and the equations of the third and fourth degrees. Then the equation $f(x, y) = 0$ is analyzed, both as curve and as correspondence between projective forms of the first grade, and quadratic transformations are introduced to replace "infinitely near" for abstract geometry. The desirability of using the geometry of space of an arbitrary number of dimensions is seen not merely in the possibility of such generalizations as that of Pluckerian coordinates to hyperspaces immersed in an n-space, or linear systems as abstract geometries, but also in relation to the essential logical structure of the subject.

Consider in fact a question relating to the correspondence of algebraic varieties, in connection with Netto's theorem (Book I, Ch. III). Suppose that between the algebraic varieties V_n, V_m, themselves irreducible, there exists an irreducible correspondence such that to a generic point of V_n correspond ∞^r points of $V_m (r \geqq 0)$, and a generic point of V_m corresponds to ∞^s points of V_n. We can imagine this correspondence established by a V_q in a V_{n+m},—in the same way as a correspondence between points x and points y on a line is defined by an $f(x, y) = 0$ in S_2,—and the V_q so formed will be irreducible. Netto's theorem, of which the author gives a special proof for algebraic varieties, then gives the result: $q = n + r = m + s$ or $n = m + s - r$. For simple applications of this theorem, for example to the finding of the dimension of the space of lines in S_3, where V_m is S_3 and the correspondence is of line to bundle, there is no difficulty; but in more complicated cases it would seem that unless one took care actually to find the algebraic coordinates of the varieties involved, one's road would be tangled with questions of reducibility, and "sofismi" and "paradossi" would beset it.

Netto's theorem is thus made the basis for the meaning of "in general", e. g., the difference between "in general" for a correspondence $[m, n]$ and a curve of degree $m + n$, the counting of constants, the theory of dimensions, and the theory of compatibility of algebraic equations, leading to the principle of Plücker-Clebsch. The latter principle may be briefly stated in this form: if for generic values of parameters a set of algebraic equations is incompatible, then when it does for particular values become consistent it also becomes indeterminate; more exactly, if a system of $m = n - r$ equations in n unknowns $(r \geqq 0)$ is consistent and admits ∞^r, not ∞^{r+1}, solutions for some particular values of the parameters it is compatible (admitting ∞^r solutions) for generic values of the parameters.

In Book II the emphasis is on correspondences and the "principle of correspondence". The latter deals with the $m + n$ fixed points of an $[m, n]$ correspondence of points on a line. The relations between these and the multiple points are given by Zeuthen's rule. The book contains a general theory of involutions of order higher than two, an elementary theory of plane curves and a chapter on the real representation of the imaginary and Riemann surfaces. Needless to say, the author's philosophical predilections make such general discussions extremely interesting.

The independence of the various books makes it possible, if one so desires, to read the next part, Book III, in large measure before Book II. In fact to many a reader the theory of plane curves based on polars, therein developed, will be the most familiar, although he will perhaps be a little startled by the discussion of one-sided (or odd) and two sided (or even) branches of curves, in the analysis of the real quartic in the projective plane. An algebraic curve without double points cannot have more than one odd branch, and if of even order cannot have any. The section on numerative geometry is extended to twisted curves and hyperspaces.

The last book starts again from elementary considerations and by means of Puiseux developments, quadratic transformations and differential methods, develops the theory of singularities of algebraic curves. The singularities of twisted curves and algebraic varieties are analyzed by means of projections; i. e.,— a curve in S_r is projected on a plane by means of the variable S_{r-2} which joins its generic point with a fixed S_{r-3}. An arbitrary plane curve, with arbitrary singularities, can arise in this way from a twisted curve in hyperspace which has no double points. Similarly, an arbitrary surface in S_3 is transformable into a surface F of a convenient hyperspace ($r \geqq 5$) without singularities, a fact by means of which is established the theorem that the surface can also by a birational transformation be transformed into another surface of S_3 with a double nodal curve having merely triple points (triple at the same time for the curve and for the surface).

For a person whose geometrical background is not more extensive than that of the reviewer some of the proofs may seem to be clouded by ambiguous wording or lack of sufficient detail, and some of the fundamentals, like the theory of quadratic transformations, not to be developed with sufficient broadness. The discussion of the relation of the equation of the fifth degree to the finite groups of projectivities is decidedly incomplete. Yet one has only to compare the book in quantity of material with Salmon, or in point of view and range of subject with almost any book in English on algebraic curves, to realize that it is the work of a rare scholar as well as a notable contributor to mathematics (and epistemology). Writers of minor and major theses are recommended to it.

G. C. EVANS

THREE REPRINTS FROM THE ENCYKLOPÄDIE

Neuere Untersuchungen über Funktionen reeller Veränderlichen, nach den Referaten von L. Zoretti, P. Montel, und M. Fréchet. By A. Rosenthal. Leipzig, Teubner, 1924. 351 pp.

Neuere Untersuchungen über trigonometrische Reihen. By E. Hilb and M. Riesz. Leipzig, Teubner, 1924. 40 pp.

Neuere Entwicklung der Theorie partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus. By L. Lichtenstein. Leipzig, Teubner, 1924. 58 pp.

The first article is a revision of the corresponding article in the French Encyclopedia, prepared under the direction of Professor Borel by Professors Zoretti, Montel and Fréchet. The original plan was to translate the French article as it stood, but a desire for greater completeness and accuracy led to an arrangement for its revision instead. As a result of the interruption in the work caused by the war, a ten year interval elapsed between the conclusion of this arrangement and the publication of the article. Hence the new material added in the revision took on the proportions of an entirely new article, the present version being about two and a half times the size of the original one. The additions are all indicated by the symbols $_* \cdots *$, in accordance with the agreement made with Professor Borel and the usage in the French revision of articles appearing first in the German Encyclopedia.

The first part of the article, which forms the revision of the portion of the original article due to Zoretti, deals with the theory of point sets, and is based both on the literature prior to 1912 (the date of the French edition) and subsequent to that date. In the chapter entitled "Verallgemeinerungen" the author gives a brief account of work in General Analysis, to which he refers as "die weitgehenden Untersuchungen von E. H. Moore und seinen Schulern." He stresses particularly the very great generality of the results attained. A detailed account of these results is not given, since such an account would belong more properly in the article by Pincherle on functional operations and equations (ENCYCLOPÄDIE, II A 11), which was completed before the publication of any of the work in General Analysis.

The second part of the article forms the revision of the portion of the original version contributed by Montel, and deals with integration and differentiation. Among the noteworthy additions to be found in this part may be mentioned work on integration due to Pierpont, F. Riesz, Denjoy, Hellinger, and Perron, and the notion of the approximate derivative due to Khintchine and Denjoy.

The third part of the present article deals with sequences of functions and forms the revision of the first half of that portion of the original article which is due to Fréchet. The second half of Fréchet's contribution, dealing with trigonometric series, is omitted here, since that topic is completely covered by the article of Hilb and Riesz, reviewed below, and the previous article by Burkhardt (ENCYCLOPÄDIE, II A 12) on trigonometric series and integrals prior to 1850.

Among the important additions in this part to the topics treated in the original version may be mentioned the following: Osgood's fundamental results regarding the distribution of points in the neighborhood of which the convergence of a series is respectively uniform or non-uniform, together with generalizations of these results, the conception of relative uniform convergence due to E. H. Moore, the relation between the classification of Borel assemblages and Baire's classification of functions, the relations between measurable functions and Baire's classes, and the recent results of Blumberg relative to properties of entirely arbitrary functions.

The second article gives an excellent bird's-eye view of the extensive domain of mathematical theory with which it deals. It includes the more important results of all the modern researches in this field, beginning with Riemann's fundamental memoir, and gives occasional indications of methods of proof that have been particularly fruitful. In addition the numerous references to the literature found in the footnotes will serve to orient the reader in a more detailed study of any particular topic which he may select.

In one of the early sections the high lights in the development of the theory from the historical standpoint are indicated by the authors. These include Riemann's noteworthy contribution, in which the introduction of a general theory of integration greatly enlarged the scope of Fourier series, while at the same time the foundations were laid for a study of trigonometric series in general. Closely related to Riemann's work is Cantor's study of the uniqueness of the development, and Du Bois-Reymond's theorem that a convergent trigonometric series, representing a function with a Riemann integral, is necessarily a Fourier series, as well as the latter's examples of continuous functions with divergent Fourier series. Fejér's investigation of the summation of Fourier's series by Cesàro's means of the first order revealed the significance of the methods of divergent series in the field of trigonometric series. Finally Lebesgue's fundamental definition of the integral paved the way for developments that have added much to the unity and completeness of the whole theory.

After this brief sketch of the development of the theory, the logical rather than the historical order is used by the writers in their detailed accounts of methods and results. In dealing with Fourier series these are grouped with respect to certain central topics, such as Fourier coefficients, general convergence theory, conjugate series, uniform and absolute convergence, special features of convergence and divergence, summation processes, Parseval's theorem, and the Riesz-Fischer theorem. In the presentation of the general theory of trigonometric series a summary of Riemann's work is followed by an account of the developments and generalizations of his ideas, due to various later writers. An appendix is devoted to the topics of multiple Fourier and trigonometric series and the approximate representation of continuous functions by means of trigonometric polynomials. The discussion of multiple series is extremely brief, it being pointed out by the authors that a general view of the present status of the theory may be gained by an examination of two papers dealing with this field, one by W. H. Young (PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY, vol. 11 (1912)), and the other by H. Geiringer (MONATSHEFTE FUR MATHEMATIK UND PHYSIK, vol. 29 (1918)). The discussion of approximation theory is considerably more complete and includes the salient features of recent work in that field.

The third article connects up with the article by A. Sommerfeld (ENCYKLOPÄDIE, II A 7c) on boundary value problems in the theory of partial differential equations. The latter article was dated 1900 and gave an account of the developments in the field under consideration up to that date. Professor Lichtenstein's article deals with the additional developments since that time in the more restricted field indicated in his title.

As the author points out, the introduction of the methods of integral equations into the study of boundary value problems gave a very considerable impulse to the growth of the theory, and that portion of it dealing with linear differential equations of elliptic type may now be regarded as having reached a certain state of finality. A brief but comprehensive account of this portion of the theory, with complete references to the literature, is given in the present article. Even a casual examination of these references will readily reveal what an extensive portion of the recent developments in this field is due to Professor Lichtenstein himself.

The last two sections of the article are concerned with boundary value problems in the case of non-linear differential equations of elliptic type. Here, of course, the theory is much less developed, but many important additions have been made to it in the past quarter of a century. A systematic account of these researches, with references to the literature, is given in the sections mentioned.

C. N. MOORE

COURANT AND HILBERT ON MATHEMATICAL PHYSICS

Methoden der mathematischen Physik. Bd. I. By R. Courant and D. Hilbert. Berlin, Julius Springer, 1924. xiii + 450 pp.

The book under review is the twelfth volume of the series *Die Grundlagen der mathematischen Wissenschaften in Einzeldarstellungen.* It is the second book of this series to be devoted to mathematical physics, it being preceded by volume IV, Madelung's *Die mathematischen Hilfsmittel des Physikers.** This earlier volume covers very extensive ground but, necessarily, in a rather cursory manner. The present volume, on the contrary, centers around one single physical problem, the oscillation problem, with its mathematical equivalents, the boundary value and expansion problems.

These are the main problems. Incidentally the reader will pick up a good deal about methods which are applicable to other problems of mathematical physics, but he will have to supply the applications himself. However, in these days of *Morbus relativitus* the information might be welcome that the word tensor appears on page 3 of the book and disappears on page 30, and it is not frequently used.

A few words regarding the joint authorship should be appropriate. The book is obviously and avowedly written by Courant. It is true that most of the subject matter originated directly or indirectly with Hilbert, whose spirit hovers over almost every page of the book. But the reader can easily verify, by looking up the several references, that a considerable portion of the book is based upon Courant's own investigations. This is especially the case with Chapter VI. Otherwise, the simple choice of methods, the fondness of heuristic considerations and a certain delicate touch of the pen, sometimes a bit vague but always elegant, betray the writer if nothing else does. All these qualities make the book easy and enjoyable reading.

We have already mentioned that the book deals with the oscillation problems of mathematical physics. This theory culminates in Chapters V and VI of the book, the former giving the equivalent boundary value and expansion problems, the latter the properties of the characteristic values and functions. The existence of the solutions is, occasionally postulated, many existence proofs being postponed to the second volume of the book which will appear later. The first four chapters lay a foundation for the theory; they deal with linear transformations and quadratic forms, expansions in terms of orthogonal functions, linear integral equations and the elements of the calculus of variations. In the seventh and last chapter some special functions defined by boundary value problems are dealt with in more detail.

---

* Review by B. O. Koopman, this BULLETIN, vol. 30, Nos. 5–6 (May–June, 1924), p. 272.

The methods employed in the book are mostly very simple. Equicontinuity and Bessel's inequality are the only fairly novel tools which are used extensively. The integrals occurring are seldom interpreted in the sense of Lebesgue. Stieltjes and Hellinger integrals are completely avoided. In order to be able to place themselves on such an elementary basis the authors purposely restrict the discussion to functions which, together with the necessary number of derivatives, are continuous in adjacent intervals (stuckweise stetig).

The leading idea in the greater part of the book is that the characteristic values and the characteristic functions* which belong to a given oscillation problem, satisfy certain *maximum-minimum conditions.* Thus the nth characteristic value is the upper limit for the minimum of a particular quadratic functional $F(\varphi)$ associated with the problem when the function φ is properly normalized and satisfies $(n-1)$ variable conditions of orthogonality. The function φ_n, supposed to exist, for which $\operatorname{Min} F(\varphi)$ reaches its upper limit, is the nth characteristic function; the corresponding $(n-1)$ orthogonal functions are the preceding characteristic functions. This *independent* definition of the nth characteristic value is due to Courant, though it seems to have been expressed for finite quadratic forms earlier by E. Fischer; the classical *recurrent* definition goes back to H. Weber. This simple principle is manipulated with great dexterity and yields surprisingly rich results.

The reader meets this guiding notion as early as on page 11 in the first chapter where the principal axes of a central hyper-quadric are found. It is clearly expressed for quadratic forms on page 17. The whole first chapter is essentially an algebraic analog from the theory of quadratic forms to the oscillation theory for continuous bodies developed later. It is the well known Hilbert theory oriented and adapted for a particular purpose. The second chapter opens with a general discussion of orthogonal functions, followed by a discourse on equicontinuous functions. The reader should notice the correction on page xiii referring to the discussion on page 40. In § 3 we meet the notions of *measure of independence* m and *asymptotic dimension number* r for a set of functions $f_i(x)$. The former is defined as the minimum of the quadratic form

$$\sum_{i,j=1}^{n} f_{ij} t_i t_j \quad \text{for} \quad \sum_{i=1}^{n} t_i^2 = 1, \quad \text{where} \quad f_{ij} = \int_a^b f_i f_j \, dx.$$

---

* Eigenwerte und Eigenfunktionen. The reader should notice that the authors use the term characteristic numbers (charakteristische Zahlen) for the reciprocals of the characteristic values.

If the functions $f_1, \cdots, f_n$ are linearly dependent, $m = 0$; if they are normalized and orthogonal, $m = 1$. The authors do not mention that this is the maximum value of m for normalized functions. Similarly the Gramian is discussed at various places, but the authors fail to notice that $\det(f_{ij}) \leqq f_{11}f_{22}\cdots f_{nn}$, where the equality sign holds if and only if the functions are orthogonal to each other.* The asymptotic dimension number is defined as the least integer r such that, if $s > r$, the measure of independence of the functions $f_{n_1}, f_{n_2}, \cdots, f_{n_s}$ converges to zero when $n_1, n_2, \cdots, n_s \rightarrow \infty$. The remainder of the second chapter is devoted to special orthogonal systems and to the approximation theorem of Weierstrass, Fejér's proof for trigonometric polynomials, and Landau's for ordinary ones.

In the third chapter we find an outline of the theory of integral equations. In addition to the classical theory of such equations, we find two new methods for the existence proofs. Both are based upon uniform approximation of the kernel by degenerate kernels, but the method of extracting a convergent sequence from the approximate solutions differ. One method employs the properties of equi-continuous functions, the other the asymptotic dimension number. For the symmetric case, the maximum-minimum properties of the characteristic values and functions play a fundamental role in the discussion. We miss a reference to Heywood and Fréchet's treatise in the bibliography appended to the third chapter.

The introduction is the most interesting part of Chapter IV, the elements of the calculus of variations. Here we are granted a foretaste of the direct methods of solving variational problems by means of minimal sequences which subject will be one of the main features of the second volume. In § 9 Hamilton's principle is introduced, with the aid of which the differential equations of vibrating masses are derived. The other parts of the chapter are more useful than exciting.

These differential equations form the main object of Chapter V. Here the boundary value problem is reduced to the solving of a symmetric integral equation by means of the corresponding Green's function. The existence of such a function is proved merely in the linear case; for two or more dimensions it is postulated and the proof will be given in the second volume. A solution of the expansion problem is given which is improved upon in the following chapter by reducing the restrictions. Various simple boundary value problems are treated explicitly.

---

* For a proof see O. Dunkel, *Integral equalities with applications to the calculus of variations*, AMERICAN MATHEMATICAL MONTHLY, vol. 31 (1924), pp. 326–337. This interesting paper unfortunately does not give any references to the literature.

In Chapter VI we reach the climax. The maximum-minimum principle is given full play and is used for a very interesting qualitative discussion of the dependence of the characteristic values upon the data of the problem, namely the coefficients of the equation, the basic region and the conditions on the boundary. This discussion is facilitated by the following simple observation: The minimum of the functional $F(\varphi)$, when φ ranges over a certain field of functions, is not decreased by restricting the field, and not increased by enlarging the field. The same principle gives a simple determination of the asymptotic distribution of the characteristic values, the basic region under consideration being approximated by rectangles for which the characteristic values are known. An application to the black-body problem is given.

The seventh chapter is not so well done as the rest of the book. To be sure, there are brilliant points: the interpretation of Laplace integrals at the beginning of the chapter and the presentation of the saddle-point method at the end are gems. But the long and involved treatment of Bessel's functions seems scarcely justifiable except on the basis that contour-integration is a method of mathematical physics. It would have been simpler to start directly with the differential equation for $J_\lambda(x)/x^\lambda$ on page 399 and leave out most of the preceding discussion for which the reader could have been referred to the corresponding parts of Watson's standard treatise. A graver remark must be directed against the discussion of the zeros of Bessel's functions on pages 412–414. The formula (33) on page 413 becomes illusory for $\lambda<-1$, and the conclusions drawn in this case concerning the complex zeros of $J_\lambda(x)$ are erroneous. On page 419 Legendre's function of the second kind, $Q_\nu(x)$, is defined by an integral for $\Re(\nu)>-1$ and by the relation $Q_\nu(x)=Q_{-1-\nu}(x)$ for $\Re(\nu)<0$. This implies a contradiction in the common strip which could be avoided by making $\Re(\nu)=-\frac{1}{2}$ the dividing line between the two definitions. Why should the symbol $Q_\nu(x)$ represent different analytic functions of ν in different parts of the ν-plane? Why not use the original definition by a contour-integral which is valid for every value of ν, not a negative integer?

This finishes our survey of the different chapters. If our labor has not been in vain, it ought to be clear to the reader of this review that the book, in spite of its restricted scope, is rich in material and in points of view which are either novel or little known. The book — as most human work — is not perfect, but the imperfections are mostly on side-issues. It was obviously not meant as an opiate, but intended to stimulate interest, discussion and research in a field which still belongs to the richest in mathematical physics. We look forward to the appearance of Volume II with eager expectation.

EINAR HILLE

SHORTER NOTICES

Hydrodynamics. By Horace Lamb. Fifth edition. Cambridge, University Press, 1924. xvi + 687 pp.

This work reached its fifth edition in the forty-fifth year after its original publication in 1879, the successive editions having appeared at intervals of sixteen, eleven, ten, and eight years. In passing from the first edition to the second it was largely remodeled and extended, but since then there has been no change in general plan and arrangement. As it now appears it has again been carefully revised, several passages having been rewritten and some considerable additions having been made. It has, however, not undergone extensive enlargement. In his preface the author observes that "the work has less pretensions than ever to be regarded as a complete account of the science with which it deals," owing partly to the difficulty of doing justice to the growing literature. He adds: "Some memoirs deal chiefly with questions of mathematical method and so fall outside the scope of this book; others though physically important hardly admit of a condensed analysis; others, again, owing to the multiplicity of publications, may unfortunately have been overlooked. And there is, I am afraid, the inevitable personal equation of the author, which leads him to take a greater interest in some branches of the subject than in others." Though the book is extensive and important it is clear from these remarks that there are important aspects of the subject which it does not adequately treat — a fact which is stated without any implication of criticism, since it did not fall within the author's purpose to treat them and since the book probably serves better a large majority of its readers on account of the omission of certain subjects and the consequent fuller treatment of others.

The principal single increase in the scope of this work occurred in passing from the first to the second edition. The interpolations and additions incorporated in the third edition amounted to about one-fifth of the whole and thus required a renumbering of the sections. There was considerable increase in size in the fourth edition even though a few investigations of secondary interest were condensed or omitted. There are no omissions of consequence in passing from the fourth edition to the fifth; and there are several additions. Of the latter some are of the nature of interpolations (mostly short) inserted in the sections which are reproduced from the earlier edition, while the others treat new topics in the form of additional articles of which there are fourteen scattered through the volume. Throughout the whole work there is repeated evidence of careful revision both in the addition of new matter and even in the smaller details of phraseology. The printing in the fifth edition, at least in the case of formulas, is a little more compact than in the fourth, and there is a considerable increase in the amount of matter in smaller type so that there is a reduction of twenty-one pages in size of the volume while there is an actual increase in matter.

As is to be expected in a fifth edition, the author adheres closely to the terminology which he had previously adopted. The most considerable change which came to the reviewer's attention is in the usage of the words "stream-line" and "stream-tube" as set forth in Sections 19 and 21; and even this is not of great importance.

In its new edition the book will continue to hold, and a little more effectively than ever before, the important place which it has now held for a generation.

R. D. CARMICHAEL

Geschichte der Elementarmathematik. By Johannes Tropfke. Bd. 7: *Stereometrie, Verzeichnisse.* Berlin and Leipzig, Walter de Gruyter & Co., 1924. iv + 128 pp

The seventh and last volume of Tropfke's revised *History of Elementary Mathematics* maintains the high standard of the preceding volumes, and completes a work that is invaluable to the teacher of elementary mathematics, as well as to the student of the history of the subject. That the revision has been thoroughgoing is shown by the fact that the 35 pages of the first edition on solid geometry have been increased to 52 pages, while the number of references to the literature has been increased from 145 to 309. The very complete index covers 74 pages, and is arranged in two alphabets, under the headings "Names and Writings" and "Subject-matter".

Two items may be singled out for special mention, among the numerous improvements over the first edition. First, the additional information given concerning the history of technical terms is of great interest and value, especially for teachers, who will find such facts as are here to be found adding materially to the interest of many students in the subject-matter itself. Secondly, the history of the regular solids is enlarged and brought into line with the latest historical researches. The fact may be of general interest that the dodecahedron was known to the ancient Etruscans, who found it in natural form, in the crystals of iron pyrites that abound in northern Italy.

One error was noted. It is stated on page 32 that the theorem, "Sections of an oblique circular cone parallel to the base are circles", although known in antiquity, is first found explicitly proved in the *Book of the Three Brothers* in the 9th century. As a matter of fact, the proof occurs in Apollonius, along with that of the theorem that the subcontrary sections are also circles.* On pages 40 and 41 occurs a passage where the revision was not edited so carefully as usual: Kepler is still, as in the first edition, credited with the cubature of the torus, whereas in the preceding paragraph (an addition in the new edition) this result is described as found in Heron of Alexandria. In this connection "the so-called theorem of Guldin" is mentioned. (See also p. 8.) Is it not time that Guldin's name was entirely dropped, and the theorem referred to exclusively as "the theorem of Pappus"? The reasons for doing this are clearly stated by Tropfke himself in the passage in question.

The index has evidently been very carefully compiled, which is of course an absolute necessity where the number of references is so great. The reviewer has checked a large number of these, and has noted the following corrections: p. 77, Kahun, I 134 should read, II 134; p. 101, Addition, II 65 should be I 65. Two other misprints should be mentioned: p. 51, l. 4, "Oktaeders" should read "Dodekaeders;" and in vol. IV, p. 139, l. 17, "4000 v. Chr." should read "2000 v. Chr."

Two topics that the reviewer believes should have been included have been entirely omitted: the *regula falsi*, and the "Russian peasant method of multiplication". Each is of historical interest and not without significance as furnishing supplementary material for class-room work or for outside reference, as in mathematics club programs.

We must end as we began, that this work is unique and of the highest excellence in its field, and a monument to the skill and industry of its author.

R. B. McCLENON

Binomial Factorizations, Giving Extensive Congruence Tables and Factorization Tables. Vols. II, III, and VI. By Lt. Col. Allan Cunningham. London, Francis Hodgson.

Of the seven volumes of this work the reviewer has in hand Volumes I, II, III, IV and VI. The content and general make-up of these tables has been considered in a previous review in the BULLETIN (vol. 31, pp. 180—181). By means of certain congruence-tables, giving the smallest roots, y, of the congruence

$$\frac{y^n \pm 1}{y \pm 1} \equiv 0 \qquad (\text{mod}\, p).$$

the author has built up factorization tables for numbers of certain special forms.

The value of such tables depends entirely upon their accuracy. An examination of certain of the tables, selected at random, leaves one in a state of doubt as to the value of the whole. Thus on page 193 of Volume II are two little tables purporting to give the factors of what he calls "Trin-Aurifeuillian Quindecimans" and "Quint-Aurifeuillian Quindecimans", which, in plain English are factorizations of numbers of the forms $x^{15} \pm y^{15}$ after the algebraic divisors have been removed. Certain of the entries are left undecomposed, but at the head of the tables we are informed "All divisors less than 1, 000 cast out". Nevertheless the following numbers on this page are left with a question mark:

---

* T. L. Heath, *Apollonius of Perga*, Cambridge, 1896, pp. 2, 3.

$$17,780,401 = 151 \times 117,751$$
$$17,354,461 = 181 \times 95,881$$
$$15,679,621 = 211 \times 74,311.$$

The last number in the first table is 10, 545, 971 which is an error for 10, 545, 991 = 151×211×331. Also the number 25, 437, 261 should be 125, 437, 261. Moreover the complete examination of all the numbers left doubtful on this page ought not to occupy a skilful computer many hours. One questions the usefulness or importance of publishing a table of factors of "Trinomial Dimorph Sextans", of eight entries, containing only five complete factorizations, and leaving five factors undecomposed two of which, 12, 419, 509 = 2029×6121, and 13, 401, 901 = 1297×10,333, would have yielded without much effort.

In spite of the fact that the author had access (p. 171, vol. I.) to the recently published factor-tables he leaves doubtful, in Volume I such numbers as

9, 705, 193, p. 171, vol. I,
9, 670, 849, p. 211, vol. I,
9, 843, 601, p. 217, vol. I,

and in Volume II we are given (p. 189) a question mark after 4, 144, 741.

These inaccuracies are, of course, not important in themselves, but show pretty clearly that these tables cannot be used in important work without careful checking.

D. N. LEHMER

Einführung in die Projective Geometrie Mehrdimensionaler Räume. By E. Bertini. Translated from the second Italian edition by A. Duschek. Wien, Seidel & Sohn, 1924. 480 pp.

In 1860 Cremona, then thirty years of age, became Professor of Higher Geometry at the ancient University of Bologna. About thirty years later Italy led the world in research in geometry. Due chiefly to Cremona's influence, the field of projective geometry in hyperspace was developed with great vigor. Today this subject can be said to be almost wholly Italian. Much good work on the subject has been done outside of Italy but nowhere else has there been such an army of the ablest mathematicians working at it. Professor Bertini of the University of Pisa has contributed his share in this work and it is, therefore, fitting that the classical work on this subject should appear from his pen. Bertini has been lecturing for many years on n-dimensional projective geometry at Pisa and in 1899 a litographed edition of his lectures was published. This was revised and enlarged and finally appeared in print in 1907. The second edition appeared in 1923. It is this second edition which Dr. Duschek has translated. It has now been so many years since the appearance of the first edition that it will be unnecessary to discuss the contents. I will, therefore, only mention the added material which this second edition contains. The aim has been to bring the work up to date and new theorems are added in many places. One new chapter has been inserted, Chapter XII, Moduli, representation of a form by linear combinations of others.

Let $F_1 = 0, F_2 = 0, \cdots, F_h = 0$ be h algebraic hypersurfaces (of order $n_1, n_2, \cdots, n_h$) in n-space, which intersect in a variety of $n-h$ dimensions.

$F_1, F_2, \cdots, F_h$ are called *forms*, and $(F_1, F_2, \cdots, F_h)$ is called the *modulus*. A form F of order r is said to belong to the modulus if

$$F = A_1F_1 + A_2F_2 + \cdots + A_hF_h,$$

where the A_i are forms of order $r - n_i$. This is expressed by writing

$$F = 0, \qquad \text{mod } (F_1, F_2, \cdots, F_h).$$

The fact that the difference of two forms $F - F'$ belongs to the modulus is expressed by

$$F' = F, \qquad \text{mod } (F_1, F_2, \cdots, F_h).$$

A Schnitt of the modulus $(F_1, F_2, \cdots, F_h)$ is formed by connecting the coordinates by one or more linear relations.

With this introduction the problem is considered of finding the geometric conditions which the hypersurface $F = 0$ must satisfy if F can be written in the form

$$F = 0, \qquad \text{mod } (F_1, F_2, \cdots, F_h).$$

To Chapter XIV (XIII of the first edition) has been added a discussion of the Segre variety which is defined as follows: Let

$$\sum_{i=1}^{i=m} \lambda_i u_{ki} = 0, \qquad (k = 1, 2, \cdots, t),$$

where the λ's are parameters and u's linear functions, be the equations of t bundles of hyperplanes. The locus of points which will make the matrix of the u's of rank $m-h$ is the Segre variety and has many interesting properties.

The book has lost none of its attractiveness at the hands of the translator.

C. L. E. MOORE

Elements of Analytic Geometry. Vol. I. By Neilos Sakellarios. Athens, E. and I. Blazoudake, 1924. vi + 288 pp. (In Greek.)

This elementary text for undergraduates is decidedly more reminiscent of Weber and Wellstein than of Tanner and Allen. Admirable in its clarity, in the logical sequence of definitions and theorems, and in the frequent and abundant exercises provided for the student, there is throughout the book an exhaustive Teutonic precision and insistence on detail more properly characteristic of the treatise than of the elementary text, though it is only fair to add that this "classical" style of textbook writing has always been orthodox on the Continent. Examples of it are the rigorous distinction of line and sect, circumference and circle, geometric sum and difference from algebraic sum and difference; the systematic use of determinants, the inclusion of poles and polars for all conics, of the power of a point with respect to a circle and with respect to a sphere, and of the theorems of Menelaus and of Ceva.

From the outset, the analytic geometry of space is developed along with the analytic geometry of the plane. After a preface which traces the history of analytic geometry from Archimedes and Appolonius down to Plucker and Klein, the work is divided into five chapters. Chapters one and two, on completing the customary themes of coordinate axes, coordinates of a point, the various equations of the straight line, angles, and triangles, pass at once to the analogous discussion of the equations of the plane and of the line in space. Chapter three beginning with metrical properties in the plane, such as distance, perpendicularity, angle bisection, area of the triangle, leads likewise to the treatment of metrical properties in space. Similarly, chapter four makes use of the analysis of the circle to introduce the analysis of the sphere. The last chapter treats fully the ellipse, the hyperbola, and the parabola, in the order named, and concludes with a short comparison of curves of the second degree, algebraically, as loci of the general equation of the second degree, and geometrically, as sections of the cone. The analogous solid-analytic material of the last chapter, the treatment of quadric surfaces, is evidently reserved for the projected second volume, as is no doubt also the consideration of higher plane curves. The table of formulas at the end, as well as the printing of the statement of each theorem in heavy type, deserves praise. There are few typographical errors.

Any good book on geometry in the language of the race which gave geometry to the world should be of interest to mathematicians conversant with that language, especially so finely wrought a text as this is, in spite of the fact that the method of approach is not the ordinary one.

G. MORGENSTERN

Complementi di Geometria Descrittiva—visibilita—ombre—chiaroscuro—prospettiva lineare. By Gino Loria. Milano, Ulrico Hoepli, 1924. xii+191.

In his preface the author states that this will probably be the concluding volume of his series of books on descriptive geometry published by this firm. (The earlier books were (1) *Metodi della Geometria Descrittiva,* 1st ed., 1909, 2d ed., 1919; (2) *Poliedri, Curve e Superfici, secondo i Metodi della Geometria Descrittiva,* 1912; (3) *Storia della Geometria Descrittiva della Origini Sino ai Giorni Nostri.* The two latter are reviewed in this BULLETIN, vol. 20, p. 415, and vol. 28, p. 414.)

The eight chapters are grouped into three "books". The first, whose title is the same as the principal title of the volume, has chapters on visibility, on surfaces of the second order, and on ruled surfaces. Book II (shadows and chiaroscuro) has chapters entitled shadows cast by points, lines, and plane polygons; shadows on surfaces; ideas concerning chiaroscuro; and the illumination of surfaces. These two books are of the same length; but book III (linear perspective) is much briefer (18 pages) and attempts nothing beyond "definitions, constructions, and formulas".

In writing this brief introduction to the applications of descriptive geometry, the author had in mind the student with a good geometrical background rather than the draftsman who wants merely a set of "working rules" for making perspective drawings and putting in shades and shadows.

E. B. COWLEY

Traité du Calcul des Probabilités et de ses Applications. By E. Borel, Tome III, *Les Applications de la Théorie des Probabilités aux Sciences Economiques et Biologiques.* Fascicule I. *Assurances sur la Vie. Calcul des Primes,* by Henri Galbrun. Paris, Gauthier-Villars, 1924. 311 pp.

This volume, the first part of a work on the mathematical side of life insurance, is just the sort of a book that we, here in America, would expect to be written by a professor of mathematics who was interested in life insurance. The best known writers in English on actuarial mathematics are actuaries who write for actuaries and who ever keep the practical applications in view. For them mathematics is simply a means to an end. The author is a French actuary and his book well illustrates the somewhat different view points of practical actuaries in this country and in France. Some allowance must, however, be taken for the fact that this volume is a part of a monumental work on the theory of probability and its applications.

Galbrun starts with the law of large numbers and proceeds in a straightforward way through the topics Bernoulli's theorem, Tchebycheff's theorem, interest, tables of mortality, Makeham's law, interpolation, summation, approximate integration, pure endowments, life annuities, premiums on various kinds of life insurance, joint life and survivorship problems, loading of premiums. This is all done in much the usual way, but more attention is paid to the elegance of the mathematical arguments and the calculus is used more freely than in many books on actuarial mathematics. A feature of the book is the discussion of the errors involved in the calculation of endowments, annuities, policy premiums, and so on. Along with average rates he computes a sort of safe rate which will cover all but one percent of the possible cases which may arise. In places, the methods used in the book will lead to much more arithmetic drudgery then is necessary, but on the whole it is a thoroughly readable and scholarly book on that part of actuarial mathematics which it covers.

A. R. CRATHORNE

Leitfaden zum Graphischen Rechnen. By Rudolf Mehmke. 2d edition. Leipzig and Vienna, Franz Deuticke, 1924. viii + 183 pp.

Whether one does or does not believe in the practical value of graphic methods one must take seriously this guide to the subject. It is a mine of ingenious devices. Here are not only the usual graphic constructions for expressions of the form $\sum ab/c$ together with solutions of linear equations in two, three, four and more unknowns, with logarithmic scales for the more complicated functions, but exhaustive discussion of methods for integrating differential equations and the determination of moments of various sorts. In connection with logarithmic methods is a description of a "logarithmic compass" devised by Professor E. A. Brauer which has three points, the distance between two of which is a function of the distance between any other pair. Interesting use is made of this device.

As to the practical use of graphical methods one may be permitted to have serious doubts. Professor E. T. Whittaker, in his preface to his *Calculus of Observations*, remarks: "When the Edinburgh Laboratory was established in 1913 a trial was made, as far as possible, of every method which had been proposed for the solution of the problems under consideration, and many of these methods were graphic. During the ten years which have elapsed since then, the graphic methods have almost all been abandoned, as their inferiority has become evident; and at the present time the work of the Laboratory is almost exclusively arithmetic. A rough sketch on squared paper is often useful, but (except in descriptive geometry) graphic work performed carefully with instruments on a drawing-board is generally less rapid and less accurate than the arithmetic solution of the same problem."

With this judgement those who have learned to use a computing machine will be apt to agree. However, the subject of graphic computation is in itself a very interesting one, and there is also the chance that certain problems in geometry and mechanics which are too complicated for arithmetic analysis may yield to some such methods as we have here. It is important that every method be as fully developed as possible whether it may compare favorably or unfavorably with other methods. This has been done for the method of graphic computation in this book with scholarly thoroughness.

D. N. Lehmer

From Determinant to Tensor. By W. F. Sheppard. New York, Oxford University Press, American Branch, 1923. 127 pp.

This is an excellent little book the aim of which is to familiarize the student with tensors and to give an idea of their applications. It is, perhaps, not sufficiently realized that tensors are not merely a part of "relativity", the fact being that they permeate almost all mathematics.

Professor Sheppard gives illustrations of the applicability of tensor algebra to statistical theory and this chapter of his book is the most novel. The fact that tensor theory would be useful in this theory is à priori evident from the prominent position given in statistics to a quadratic form.

What caught the reviewer's attention first, on reading the book, was that Professor Sheppard had a happy thought when he tried to introduce the beginner to tensors by starting with determinants. The connection between alternating tensors, which are those that occur naturally in the study of geometrical figures and their attached integrals, and determinants is a deep-lying one and it is in some respects a good plan to reverse the process and obtain the principal results on determinants from very elementary theorems of tensor algebra. A very good account of determinants and their uses is given in the first few chapters of the book.

We wish to recommend this book heartily, but we must refer briefly to a point in it which we regard as unfortunate. Reduced to ordinary vector symbolism, the writer says that we can write the scalar product idea in the form $m/\overline{A} = \overline{B}$ this being understood as equivalent to $m = (\overline{A}\cdot\overline{B})$ where m is a scalar quantity and $\overline{A}$ and $\overline{B}$ are two vectors. If we also have $n/\overline{C} = \overline{B}$ the author equates $m/\overline{A}$ to $n/\overline{C}$ and says that $m/\overline{A} = n/\overline{C}$ is a way of stating that m is the same linear function of $\overline{A}$ as n is of $\overline{C}$ (p. 75). But $\overline{B}$ is not determined by $m/\overline{A}$, as we can write $m/\overline{A} = \overline{B} + \lambda\overline{X}$ where $\overline{X}$ is any vector perpendicular to $\overline{A}$. We do not, therefore, follow the author in his discussion of the "direct proportion of simple sets" (vectors) but this disagreement is after all a small one, being largely a question of terminology.

The beginner will find the book a valuable introduction and the expert an interesting review with a refreshing novelty of presentation.

F. D. MURNAGHAN

The Mathematical Groundwork of Economics: An Introductory Treatise. By A. L. Bowley. Oxford, Oxford University Press, 1924. vi + 98 pp. $2.35.

Professor Bowley's prefatory statement of the purpose and scope of his book is an adequate description of his accomplishment as well as of his aim. He says "Though the simple application of mathematics made by competent writers and lecturers can be appreciated by any intelligent readers and students, the more complicated analyses are only within the power of those who have mathematical aptitude, and it is for them that this book is arranged. . . . . . I have attempted to reduce to a uniform notation, and to present as a properly related whole, the main part of the mathematical methods used by Cournot, Jevons, Pareto, Edgeworth, Marshall, Pigou, and Johnson, so far as these are applied to the fundamental equations of exchange and to the elementary study of taxation. . . . I have not intended to advance any new theorems in economics, nor do I claim any originality in mathematical results."

The author points out that the science of wealth and welfare has two aspects, one subjective, the other objective. On the subjective side, satisfactions and desires are not arithmetically measurable, but, on the objective side, goods and services are measurable in terms of quantity and price. "At first sight," he continues, "it might appear that mathematical reasoning was confined to the objective aspect, but this is not the case." Mathematical methods may be applied to the subjective as well as the objective aspect of economics, for we can detect "equality and inequality, relationship, continuity, variation, and other properties" of satisfactions and desires. The mathematical treatment of economic concepts, therefore, embraces both incommensurable and measurable entities.

The chief contribution of the author in this book is to make available a uniform, orderly, and consistent treatment of the concepts of economics which lend themselves to mathematical analysis. The book will prove to be a valuable reference work for the economist with mathematical training. But it will have still greater value and influence if its applications of mathematics to economics are used as illustrative material by teachers of mathematics in their courses in analytic geometry and calculus.

W. M. PERSONS

NOTES

Mr. G. Y. Rainich, of Johns Hopkins University, and Professor J. F. Ritt, of Columbia University, have been appointed associate editors of the TRANSACTIONS OF THIS SOCIETY.

The following university courses for the academic year 1925-1926 are announced·

COLUMBIA UNIVERSITY.—By Professor T. S. Fiske: Fundamental concepts of mathematics; Differential equations.—By Professor F. N. Cole: Theory of groups.—By Professor C. J. Keyser: Modern theories in geometry; Introduction to mathematical philosophy (first semester).—By Professor D. E. Smith: History of mathematics, first and second courses (first semester); History of mathematics, advanced course (first semester); Seminar in the history of mathematics (first semester).—By Professor Edward Kasner: Seminar in differential geometry.—By Professor W. B. Fite: Differential equations.—By Professor J. F. Ritt: Elliptic functions (first semester); Advanced course in the theory of functions of a complex variable (second semester).—By Professor G. A. Pfeiffer: Analysis situs (second semester).—By Dr. M. H. Stone: Fourier series and related topics.

HARVARD UNIVERSITY.—By Professor W. F. Osgood: Advanced calculus; Functions of real variables; Linear differential equations of the second order, complex variables.—By Professor J. L. Coolidge: Subject matter of elementary mathematics; Probability; Algebraic plane curves.—By Professor E. V. Huntington: Fundamental concepts of mathematics.—By Professor O. D. Kellogg: Elementary theory of differential equations; Theory of potential functions; Dynamics (second course).—By Professor G. D. Birkhoff: Space, time, and relativity.—By Professor W. C. Graustein: Introduction to modern geometry; Projective geometry; Non-euclidean geometry.—By Dr. H. W. Brinkmann: Theory of functions; Partial differential equations of mathematical physics.—By Professor E. T. Bell (University of Washington): Theory of numbers.—By Dr. L. M. Graves (National Research Fellow): Calculus of variations.—By Mr. B. O. Koopman. Analytical theory of heat, problems in elastic vibrations, Modern methods in dynamics. Professor Bell and Dr. Brinkmann will conduct a fortnightly seminar in analysis and the theory of numbers. Courses of research are also offered by Professor Osgood in analysis, by Professor Coolidge in geometry, by Professor Kellogg in potential theory, by Professor Birkhoff in the theory of differential equations, by Professor Graustein in geometry, and by Dr. Brinkmann in the theory of groups.

From forty to fifty fellowships for advanced study abroad will be awarded annually by the John Simon Guggenheim Memorial Foundation, beginning with the academic year 1926–27. These fellowships are open to men and women; research or other productive work may be pursued in any academic subject or in the arts. The annual stipend is about $2500. Applications should be sent to the office of the Foundation, 2300 Pershing Square Building, New York City.

Applications for American Field Service Fellowships for study in French universities for the academic year 1926–27 should be sent to Stephen P. Duggan, 522 Fifth avenue, New York City, not later than December 15, 1925. Mr. W. E. Byrne, of Rensselaer Polytechnic Institute, has been appointed fellow in mathematics for 1925–26.

The Belgian Academy announces the following subjects for prizes in mathematics for the year 1926: *A contribution to the infinitesimal geometry of curved surfaces*, *A contribution to the theory of the absorption of light in interstellar space.*

In 1914 a committee was formed under the auspices of the Paris Academy of Sciences, with Professor Paul Appell as chairman, to found a prize and medal in honor of Henri Poincaré, for the encouragement of research in the fields of mathematical analysis, celestial mechanics, mathematical physics, and the philosophy of sciences. The work of this committee, which was interrupted by the war, has now been taken up again, and it is hoped to begin the annual awards shortly.

Cambridge University has awarded a Smith's prize to Mr. T. G. Room, of St. John's College, for an essay entitled *Varieties generated by collinear stars in hyperspace.* The Adams prize for an essay on *The physical state of matter at high temperatures* (see this BULLETIN, vol. 29, p. 235) has been awarded to Mr. R. H. Fowler, of Trinity College. The subject announced for the Adams prize for 1925–26 is *The constitution of the interior of the earth and the propagation of waves through the interior and over the surface of the earth.*

The Barnard medal has been awarded by Columbia University to Professor Neils Bohr, of the University of Copenhagen, in recognition of his work on atomic structure.

On the occasion of his recent visit to Buenos Aires, Professor Albert Einstein was elected an honorary member of the Argentine Academy of Science.

Professor Willy Wien, of Wurzburg, has been elected an honorary fellow of the Physical Society of London.

Oxford University has conferred an honorary doctorate on Professor Paul Appell.

Professors Paul Appell and Emile Picard have been elected honorary members of the Russian Academy of Sciences.

Professor L. Lecornu has been promoted to be a commander of the Légion d'Honneur.

Eugène Fichot, director of the French naval hydrographic service, and known for his work in tidal theory, has been elected a member of the Paris Academy of Sciences in the section of geography and navigation.

The following mathematicians have been recommended by the president and council of the Royal Society of London for election as fellows: R. H. Fowler, Harold Jeffreys, Joseph Proudman.

The University of Edinburgh has conferred an honorary degree on Professor A. S. Eddington.

Dr. J. H. Jeans has been elected president of the Royal Astronomical Society.

Professor Richard Gans, of the University of La Plata, has been appointed professor of physics at the University of Konigsberg.

Professor Hans Rademacher, of the University of Hamburg, has been appointed professor of mathematics at the University of Breslau.

Professor Lothar Schrutka, of the Brünn Technical School, has been appointed professor of mathematics at the Technical School at Vienna.

Professor G. Togliatti, of Turin, has been appointed professor of applied mathematics at the University of Zurich.

At the University of Paris, Professor Paul Montel has been transferred to the professorship of rational mechanics; Professor Gaston Julia succeeds him in the chair of general mathematics. J. Chazy has been appointed chargé de cours.

A readership in mathematical physics has been created recently at the University of Bristol, from the Wills bequest; Dr. J. E. Jones has been appointed the first reader.

Mr. W. L. Fellar, of the University of Edinburgh, has been appointed to a tutorial fellowship at Hertford College, Oxford.

The following 34 doctorates with mathematics or mathematical physics as major subject were conferred by American universities during 1924; the university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case.

R. W. Babcock, Wisconsin, October, *On thermal convection.*

Herman Betz, Yale, June, *Surface transformations applied to dynamical systems with two degrees of freedom.*

A. D. Campbell, Cornell, September, philosophy, *Linear systems of conics in the Galois field.*

Elizabeth Carlson, Minnesota, June, physics, *On the convergence of certain methods of closest approximation.*

G. H. Collignon, Johns Hopkins, June, geophysics, *Problems of flow in connection with mapping of spherical polygons.*

Julia T. Colpitts, Cornell, June, philosophy, *On a certain class of entire functions.*

Julia Dale, Cornell, June, physics, *Some properties of the exponential mean.*

Marguerite D. Darkow, Chicago, August, *Arithmetics of certain algebras of generalized quaternions.*

W. W. Elliott, Cornell, June, physics, *Generalized Green's functions for compatible differential systems.*

Bess M. Eversull, Cincinnati, June, physics, *The summability of the triple Fourier's series at points of discontinuity of the function developed.*

F. J. Gerst, Johns Hopkins, June, geophysics, *Image points and Riemann's theorem.*

Cornelius Gouwens, Chicago, August, *Invariants of the linear group modulo* $p_1^{\lambda_1} \cdots p_n^{\lambda_n}$.

L. M. Graves, Chicago, August, *The derivative as independent function in the calculus of variations.*

C. F. Gummer, Chicago, August, *The relative distribution of the real roots of a system of polynomials.*

J. W. Hedley, Chicago, August, *Ruled surfaces whose flecnode curves belong to linear complexes.*

P. E. Hemke, Johns Hopkins, June, physics, *A transformation involving* ζ*-functions with an aerodynamical application.*

J. L. Holley, Harvard, *Linear spaces and their fixed points.*

Harold Hotelling, Princeton, June, *Three-dimensional manifolds of states of motion.*

Jewell C. Hughes, Chicago, August, *A problem in the calculus of variations in which one end-point is variable on a one-parameter family of curves.*

Mildred Hunt, Chicago, June, *The arithmetics of certain linear algebras.*

M. H. Ingraham, Chicago, June, *A general theory of linear sets.*

C. M. Jensen, Minnesota, June, physics, *Some problems in the approximate representation of a function by a Sturm-Liouville interpolation formula.*

C. G. Latimer, Chicago, August, *Arithmetic of generalized quaternions.*

Harry Levy, Princeton, June, *Tensors determined by a hypersurface in a Riemann space.*

J. B. Linker, Johns Hopkins, June, geophysics, *Equations of motion.*

L. H. McFarlan, Missouri, June, physics, *Transformation of the Euler equations in the calculus of variations.*

Aristotle Michal, Rice Institute, June, (a) *Integro-differential expressions invariant under Volterra group of transformations*; (b) *Functions of curves invariant under point transformations of the plane.*

J. A. Nyswander, Chicago, June, *A direct method of obtaining the solutions of systems of linear differential equations having constant coefficients.*

R. G. Putnam, Chicago, June, *On solutions of special, linear, third-order differential systems.*

B. P. Reinsch, Illinois, June, mechanics, *Expansion problems in connection with the hypergeometric differential equation.*

D. S. Saund, California, May, analytic mechanics, *The periodic solutions of the equation of the elliptic cylinder.*

J. H. Taylor, Chicago, June, *A generalisation of Levi-Civita's parallelism and the Frenet formulas.*

Marian M. Torrey, Cornell, June, physics, *On monoidal space transformations in which the monoids have a fixed tangent cone.*

D. V. Widder, Harvard, *Theorems of mean value and trigonometric interpolation.*

The following degree, conferred in 1923, should have been included in our list for that year (this BULLETIN, vol. 30, pp. 474–75):

Frederick Wood, Wisconsin, June, *Group-velocity and the propagation of disturbances in dispersive media.*

Professor E. A. Bailey, of La Grange College, has been appointed dean.

Dr. Henry Blumberg, of the University of Illinois, has been appointed professor of mathematics at Ohio State University.

At Brown University, Assistant Professor R. W. Burgess has resigned and will continue the statistical work with the Western Electric Company which he has been doing during the year 1924–25 while on leave of absence. Dr. C. R. Adams has been promoted to an assistant professorship.

At Cornell University, Assistant Professors W. B. Carver and D. C. Gillespie have been promoted to full professorships.

Assistant Professor C. A. Garabedian, of Northwestern University, has been appointed associate professor of mathematics in the University of Cincinnati.

At Harvard University, Professor J. L. Walsh has been granted leave of absence for the academic year 1925–26, and Professor G. D. Birkhoff for the second semester. Professor E. T. Bell has been appointed visiting lecturer for the first half-year.

Professor J. N. Michie, of the University of Texas, has been appointed head of the department of mathematics at Texas Technology College.

Associate Professor W. L. Miser, of the Armour Institute of Technology, has been appointed professor of mathematics at Vanderbilt University.

Professor D. E. Smith, of the faculty of pure science and Teachers College, Columbia University, retires at the end of the first semester of 1925-26. Dr. W. D. Reeve has been appointed associate professor of mathematics at Teachers College.

Professor F. B. Wiley, of Denison University, served as acting head of the department of mathematics at Roberts College, Constantinople, for the academic year 1924-25.

Assistant Professor L. T. Wilson, of the United States Naval Academy, has been promoted to an associate professorship.

Professor Alexander Ziwet, of the University of Michigan, has retired from active teaching.

The following appointments to instructorships are announced.

Cornell University, Messrs. R. P. Agnew, H. I. Lane, J. Wolfender, and Miss F. Farnum.

Harvard University, Dr. H. W. Brinkmann, Mr. H. L. Garabedian, Dr. L. M. Graves (first semester), Mr. B. O. Koopman (Peirce instructor), Mr. H. B. Hammatt, Mr. W. A. Jenkins, Mr. Malcolm MacClaren, Mr. Morris Marden, Mr. F. W. Perkins, Mr. H. P. Stabler, Mr. D. E. Whitford.

The death is announced of Professor C. Guichard, of the University of Paris

Professor P. A. Nekrassoff, of the University of Moscow, died December 20, 1924, at the age of seventy-three.

Mr. W. W. Rouse Ball, fellow and formerly tutor of Trinity College, Cambridge, died April 4, 1925, at the age of seventy-four.

Dr. Oliver Heaviside, known for his work in electromagnetism, died February 3, 1925, in his seventy-fifth year.

Professor P. A. Lambert, of Lehigh University, died February 15, 1925, at the age of sixty-three. Professor Lambert had been a member of the American Mathematical Society since 1893.

Dr. D. J. McAdam, professor emeritus of mathematics at Washington and Jefferson College, died February 15, 1925, at the age of eighty-two.

Shirely E. Roberts, formerly professor of mathematics at the University of the Philippines, died March 9, 1925, at the age of forty-seven.

Professor C. S. Sperry, of the University of Colorado, died in July, 1924. Professor Sperry had been a member of the American Mathematical Society since 1921.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS.

BERNOULLI (J.). Die Differentialrechnung von Johann Bernoulli aus dem Jahre 1691/92. Nach der in der Basler Universitätsbibliothek befindlichen Handschrift übersetzt, mit einem Vorwort und Anmerkungen versehen von P. Schafheitlin. (Ostwalds Klassiker der exakten Wissenschaften, Nr. 211.) Leipzig, Akademische Verlagsgesellschaft, 1924. 56 pp.

BOREL (E.). Principes d'algèbre et d'analyse. Paris, Albin Michel, 1924. 8+310 pp.

GRELLING (K.). Mengenlehre. (Mathematisch-physikalische Bibliothek, Nr. 58.) Leipzig, Teubner, 1924. 49 pp.

GRISAR (C. M.). Über eine Verallgemeinerung des Tauberschen Satzes und seine Ausdehnung auf n-fache Reihen. (Diss.) München, 1924.

INSOLERA (F.). Complementi di matematiche generali. Torino, Lattes, 1924. 12+256 pp.

NIKOLAJEW (B.). Bedingungen der Möglichkeit der Gleichungen $x^n+y^n=z^n$ in ganzen Zahlen. Leningrad, 1924.

PASCH (M.). Mathematik und Logik. Vier Abhandlungen. 2te Auflage. Leipzig, Engelmann, 1924. 46 pp.

PETERS (L.). Vektoranalysis. (Mathematisch-physikalische Bibliothek, Nr. 57.) Leipzig, Teubner, 1924.

PÓLYA (G.) und SZEGO (G.). Aufgaben und Lehrsätze aus der Analysis. Band 1: Reihen, Integralrechnung, Funktionentheorie. (Die Grundlehren der mathematischen Wissenschaften, Band 19.) Berlin, Springer, 1925. 16+338 pp.

SCHAFHEITLIN (J.). See BERNOULLI (J.).

SCHWATT (I. J.). An introduction to the operations with series. Philadelphia, University of Pennsylvania Press, 1924. 10+287 pp.

SCHWERDT (H.). Lehrbuch der Nomographie. Berlin, Springer, 1924. 8+267 pp.

SZEGO (G.). See PÓLYA (G.).

THOMAS (R. G.). Essentials of applied calculus. New York, Van Nostrand, 1924.

THOMPSON (A. J.). Logarithmetica Britannica; being a standard table of logarithms to twenty decimal places. Part 9 (the first to be published): Numbers 90,000 to 100,000. London, Cambridge University Press, 1924. 4to. 18+100 pp.

WIELEITNER (H.). Die Geburt der modernen Mathematik. I: Analytische Geometrie. Karlsruhe, Braun, 1924 (?). 61 pp.

PART II. APPLIED MATHEMATICS

ANNUAIRE pour l'an 1925 publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1925. 16mo. 686 pp.

AUERBACH (F.). Die Methoden der theoretischen Physik. Leipzig, Akademische Verlagsgesellschaft, 1925. 10+436 pp.

BACH (C.) und BAUMANN (R.). Elastizität und Festigkeit. 9te vermehrte Auflage. Berlin, Springer, 1924. 687 pp.

BARKHAUSEN (H.). Elektronenrohren. Band 1: Elektronentheoretische Grundlagen. Verstarker. 2te umgearbeitete Auflage. Leipzig, Hirzel, 1924. 6+140 pp.

BARRIOL (A.). Théorie et pratique des opérations financières. 3e édition. Paris, Doin, 1925. 400 pp.

BAUMANN (R.). See BACH (C.).

BENEDICKS (C.). Space and time. An experimental physicist's conception of these ideas and of their alteration. London, Methuen, 1924. 14+98 pp.

BERGET (A.). See CHAPPUIS (J.).

BINGEN (J. N.) et CROMBEZ (R.). Unités de mesure scientifiques et industrielles. Bruxelles, 1923. 47 pp.

BLEICH (F.). Theorie und Berechnung der eisernen Brücken. Berlin, Springer, 1924. 593 pp.

BOEHM (F.). Versicherungs-Mathematik. I: Elemente der Versicherungsrechnung. (Sammlung Göschen.) Berlin, de Gruyter, 1925. 144 pp.

BOQUET (F.). Histoire de l'astronomie. Paris, Payot, 1925. 509 pp.

BORN (M.). Vorlesungen uber Atommechanik. Herausgegeben unter Mitwirkung von F. Hund. Band 1. (Struktur der Materie in Einzeldarstellungen, Band 2.) Berlin, Springer, 1925. 367 pp.

BOUASSE (H.). Houle, rides, seiches et marées. Paris, Delagrave, 1924. 540 pp.

——. Oscillations électriques. Paris, Delagrave, 1925. 375 pp.

BREISIG (F.). Theoretische Telegraphie. Eine Anwendung der Maxwellschen Elektrodynamique auf Vorgänge in Leitungen und Schaltungen. Braunschweig, Vieweg, 1924. 14+548 pp.

BRUCKNER (A.). Grundzüge der Brillenlehre. Band 1. Berlin, Springer, 1924. 167 pp.

CABREIRA (A.). Méthode pour obtenir les coordonnées géographiques et astrobales mécanique et solaire. Lisbonne, 1924. 18 pp.

CHAPPUIS (J.) et BERGET (A.). Leçons de physique générale. 2e édition entièrement refondue. Tome 1 et tome 3. Paris, Gauthier-Villars, 1924. 12+692+7+502 pp.

CLERC (A.). See JEANS (J. H.).

CROMBEZ (R.). See BINGEN (J. N.).

DANNEMANN (F.). Die Naturwissenschaften in ihrer Entwicklung und in ihrem Zusammenhange. 2te Auflage. Band IV: Das Emporblühen der modernen Naturwissenschaften seit der Entdeckung des Energieprinzips. Leipzig, Engelmann, 1923. 12 + 630 pp.

DEBYE (P.). See MARX (E.).

DRIESCH (H.). Relativitatstheorie und Philosophie. Karlsruhe, G. Braun, 1924. 52 pp.

EDDINGTON (A. S.). Vues générales sur la théorie de la relativité. Traduction par T. Greenwood. Préface de P. Painlevé. Paris, Gauthier-Villars, 1924. 24 + 103 pp.

EINSTEIN (A.). Quatre conférences sur la théorie de la relativité. Traduction de M. Solovine. Paris, Gauthier-Villars, 1925. 96 pp.

—— Sur l'électrodynamique des corps en mouvement. Traduction de M. Solovine. Paris, Gauthier-Villars, 1925. 56 pp.

FORCHHEIMER (P.). Hydraulik. 2te Auflage. Leipzig, Teubner, 1924. 10 + 566 pp.

GREENWOOD (T.). See EDDINGTON (A. S.).

GRIFFITH (R. O.). See LEWIS (W. C. M.).

HAAS (A.). Einführung in die theoretische Physik. 3te und 4te vollig umgearbeitete Auflage. Band 2. Berlin, de Gruyter, 1924. 379 pp.

VON HELMHOLTZ (H.). Treatise on physiological optics. Translated from the third German edition by J. P. C. Southall. Volumes 1–2. Ithaca, Optical Society of America, 1924. 9 + 480 pp.

HERZ (N.). Allgemeine Theorie zentrierter Linsensysteme. Leipzig und Wien, Deuticke, 1924. 4 + 68 pp.

HUND (F.). See BORN (M.).

HUNTER (H.). The chemical significance of optical dispersion. (Diss., University of London.) London, the Author, 1924. 117 pp.

INTERNATIONAAL CONGRES voor Technische Mechanica, Delft, 22.–28. April 1924. Uittreksels der Voodrachten. 131 pp.

JACQUEMIN (R.). See NICEFORO (A.).

JEANS (J. H.). Théorie dynamique des gaz. Traduit sur la 3e édition anglaise par A. Clerc. Paris, Blanchard, 1925. 510 pp.

JOOS (G.). See MARX (E.).

KLEINERT (H.). Die Prüfungsmöglichkeiten der Einsteinschen Relativitätstheorie. Allgemein verständliche Darstellung. Bern und Leipzig, Bircher, 1924. 63 pp.

LAMB (H.). Statics including hydrostatics and the elements of the theory of elasticity. 2d edition. Cambridge, University Press, 1924.

LANG (E.). See SPILLMAN (W. J.).

VON LAUE (M.). See MARX (E.).

LECHER (E.). See RIECKE (E.).

LEREDU (R.). L'équivoque d'Einstein. Paris, Les Presses Universitaires de France, 1925. 66 pp.

LORENTZ (H. A.). See MARX (E.).

LEWIS (W. C. M.). A system of physical chemistry. Volume 3: Quantum theory. With certain appendices by James Rice, A. M'Keown, and R. O. Griffith. London, Longmans, 1924. 10 + 407 pp.

LORENZ (H.). Lehrbuch der technischen Physik. Band 1, Teil 1: Mechanik ebener Gebilde. 2te, vollstandig neubearbeitete Auflage der Technischen Mechanik starrer Systeme. Berlin, Springer, 1924. 288 pp.

LUDWIG (W.). Lehrbuch der darstellenden Geometrie. Teil 3· Das rechtwinklige Zweitafelsystem (Krumme, Flachen, Axonometrie, Perspektive). Berlin, Springer, 1924. 174 pp.

M'KEOWN (A.). See LEWIS (W. C. M.).

MANES (A.). Versicherungswesen. 4te neudurchgesehene Auflage. Band 1: Allgemeine Versicherungslehre. Band 2: Besondere Versicherungslehre. Leipzig, Teubner, 1924. 14 + 231 + 14 + 357 pp.

MARX (E.), herausgegeben von. Handbuch der Radiologie. Band 6: Die Theorien der Radiologie. Bearbeitet von M. von Laue, P. Zeeman, H. A. Lorentz, A. Sommerfeld, G. Wentzel, G. Joos, E. Riecke, L. Vegard, P. Debye. Leipzig, Akademische Verlagsgesellschaft, 1925. 11 + 806 pp.

MASCART (J.). Le plus vieil instrument européen d'astronomie nautique (IIIe siècle). Lyon, 1923. 16 pp.

MARTIN (L. C.). Optical measuring instruments: their construction, theory and use. London, Blackie, 1924. 9 + 270 pp.

MAURAIN (C.). Physique du globe. Paris, Colin, 1923. 204 pp.

MENZEL (H.). Die Theorie der Verbrennung, die stochiometrischen und thermochemischen Grundlagen der Verbrennungs- und Vergasungs-Vorgange. Dresden, Steinkopff, 1924. 8 + 120 pp.

MILLIKAN (R. A.). The electron. 2d edition. Chicago, University of Chicago Press, 1924. 14 + 293 pp.

MILLOT (S.). Théorie nouvelle de la probabilité des causes. Paris, Gauthier-Villars, 1925.

MILLS (F. C.). Statistical methods applied to economics and business. New York, Holt, 1924. 16 + 604 pp. $3.50

MOLESWORTH (F. C.). The military uses of astronomy. London, Longmans, 1924. 12 + 112 pp.

NICEFORO (A.). La méthode statistique et ses applications aux sciences naturelles, aux sciences sociales et à l'art. Traduit de l'italien par R. Jacquimin. Paris, Marcel Girard, 1925. 8 + 652 pp.

NOYES (A.). See WASHBURN (E. W.).

PAINLEVÉ (P.). See EDDINGTON (A. S.).

PETZOLDT (J.). Die Stellung der Relativitatstheorie in der geistigen Entwicklung der Menschheit. 2te verbesserte und vermehrte Auflage. Leipzig, Barth, 1923. 4 + 98 pp.

RICE (J.). See LEWIS (W. C. M.).

RIECKE (E.). Lehrbuch der Physik. 7te Auflage, herausgegeben von E. Lecher. Band 1. Berlin, de Gruyter, 1923. 16 + 656 pp.

—— See MARX (E.).

ROUSSILHE (H.). Cours d'astronomie appliquée et géodésie professé à l'Ecole spéciale des travaux publics, du bâtiment et de l'industrie. Paris, Léon Eyrolles, 1924.

SCHOUTEN (J. A.). Raum, Zeit und Relativitätsprinzip. Leipzig. Teubner, 1924. 40 pp.

SCHOY (K.). Die Gnomik der Araber. (Die Geschichte der Zeitmessung, Band 1.) Berlin, de Gruyter, 1923.

SILBERSTEIN (L.). The theory of relativity. 2d edition, enlarged. London, Macmillan, 1924. 10 + 563 pp.

SOLOVINE (M.). See EINSTEIN (A.).

SOMMERFELD (A.). See MARX (E.).

SOUTHALL (J. P. C.). See VON HELMHOLTZ (H.).

SPILLMAN (W. J.) and LANG (E.). The law of diminishing returns. Yonkers, World Book Company, 1924. 12 + 178 pp.

TAKABEYA (E.). Zur Berechnung des beiderseits eingemauerten Tragers unter besonderer Berücksichtigung der Längskraft. Berlin, Springer, 1924. 4 + 52 pp.

THOMSON (G. H.). How to calculate correlations. A non-mathematical book of instructions. London, Harrap, 1925. 24 pp.

URBAIN (G.). Les notions fondamentales d'élément chimique et l'atome. Paris, Gauthier-Villars, 1925. 2 + 172 pp.

VARCOLLIER (H.). La relativité dégagée d'hypothèses métaphysiques. Paris, Gauthier-Villars, 1925. 19 + 544 pp.

VEGARD (L.). See MARX (E.).

WASHBURN (E. W.). Principes de chimie du point de vue de l'atomistique et de la thermodynamique modernes. Traduit par H. Weiss et A. Noyes. Paris, Payot, 1925. 16 + 574 pp.

WEIGHARDT (E.). Mathematische Geographie und Astronomie. München, Oldenbourg, 1924. 9 + 127 pp.

WEINBACH (M. F.). Principles of transmission in telephony. New York, Macmillan, 1924. 14 + 303 pp.

WEISS (H.). See WASHBURN (E. W.).

WENTZEL (G.). See MARX (E.).

ZEEMAN (P.). See MARX (E.).

ZIMMERMANN (H.). Zur Relativitätslehre. Gedanken eines Technikers. Berlin, Wilhelm Ernst, 1924. 52 pp.

THE MAY MEETING IN NEW YORK

The two hundred forty-second regular meeting of the Society was held at Columbia University, on Saturday, May 2, 1925, extending through the usual morning and afternoon sessions. The attendance included the following sixty-three members.

Alexander, R. L. Anderson, C. R. Ballantine, J. P. Ballantine, Boyajian, Brant, R. W. Burgess, Carson, Alonzo Church, Dostal, Dresden, Eversull, Feldstein, Fenn, Fine, Fiske, Fite, D. A. Flanders, Fort, R. M. Foster, Philip Franklin, Frink, Gafafer, Garretson, Gehman, Gill, Glenn, W. C. Graustein, Gray, Gronwall, C. C. Grove, Hazlett, Hille, Himwich, Hoyt, Joffe, Kerékjártó, Kline, Lefschetz, Harry Levy, MacColl, S. P. Mead, Meder, Metz, Molina, Paxton, Pell, R. G. Putnam, Ritt, R. B. Robbins, Saurel, Schelkunoff, Seely, J. H. Taylor, J. M. Thomas, Tracey, Veblen, Wedderburn, Whited, Widder, Widmark, Wiener, Zobel

There was no meeting of the Council or of the Trustees. At the morning session Professor Arnold Dresden, Assistant Secretary of the Society, gave a report on the business transacted by the Council at the April meeting in Chicago and by mail.

Vice-President Wedderburn presided, relieved by Ex-Presidents Fine and Veblen and Professor Alexander. At the beginning of the afternoon session addresses were presented, at the request of the program committee, by Mr. J. R. Carson and Dr. T. H. Gronwall on *The Heaviside operational calculus and its applications to electric circuit theory.* A number of engineers and physicists were present during this session by invitation, in addition to members of the Society.

Titles and abstracts of the papers presented at this meeting follow below. The papers of Dr. Castellani and Professor Hollcroft were read by title.

1. Professor Norbert Wiener: *The operational calculus.*

The author employs the method of separating the Fourier integral of a function into low and high frequency ranges to justify the asymptotic expansions derived by Oliver Heaviside by operational methods for the solution of the differential equations of the electric circuit.

2. Professor Norbert Wiener and Dr. Philip Franklin: *Analytic approximations to topological transformations.*

The leading theorem of this paper asserts the existence of an analytic transformation approximating to any degree of exactness any given one to one continuous transformation of the spherical surface into itself. The analytic transformation is obtained after an approximation with continuous partial derivatives is set up. This last construction depends on a dissection of the sphere and its transform into convex polygons which are nearly transforms of one another. All the vertices are triple, and we use affine transformations at these vertices. We then interpolate our transformation appropriately on the sides and in the interiors.

3. Dr. Philip Franklin: *Functions with assigned derivatives.*

In this paper it is shown that, given an infinite number of isolated points in the complex plane, with one branch cut drawn to infinity from each point, a function exists in the cut plane which is analytic at all points except those of the given set, and at these points possesses derivatives of all orders whose values may be arbitrarily assigned. The method of proof is an extension of that used by J. F. Ritt for a simpler, but related, problem. The function is also shown to exist when the further condition is imposed on it that, in a finite region not including any points on the branch cuts, it approximates uniformly to a given degree of exactness a given analytic function.

4. Professor Olive C. Hazlett: *On the types of division algebras.*

In this paper the author extends somewhat the results of a paper by Dickson in the TRANSACTIONS OF THIS SOCIETY for 1924 and one by Wedderburn in the TRANSACTIONS for 1921. It is proved that any quadratic division algebra of the type discussed by Cezioni in the RENDICONTI DI PALERMO for 1923 is a direct product of Dickson algebras.

5. Professor T. R. Hollcroft: *On conditions for self-dual curves.*

In accordance with the Lefschetz postulate of singularities for a given order $n \geqq 14$, self-dual plane curves exist only for a genus within and including the limits 0 and $2n-7$. This theorem results from Plücker's equations and a theorem of Klein's: For an algebraic curve whose equation has real coefficients, a self-dual curve, and a self-dual curve only, may have all of its point and line singularities (1) real with real tangents and points, (2) imaginary. Six sets of equalities among the singularities of a curve are found all of which are necessary for the curve to be self-dual. Of these, three sets only are always sufficient. The remaining three are sufficient except for certain definite curve types, dual in pairs. For space self-dual curves, in general the plane projection curve is not self-dual, and conversely. Such curves whose plane projection curves are also self-dual exist only for $n \geqq 8$ and for a very limited genus. There are eight equalities among the characteristics of space curves, four of which are both necessary and sufficient for the curve to be self-dual, and four necessary, but sufficient only with the exception of certain curve types.

6. Professor O. E. Glenn: *The invariant system of two associated bilinear connexes.*

Systems of invariants of connexes were studied by Clebsch and Gordan, who published, jointly, a paper on the theory in MATHEMATISCHE ANNALEN, vol. 1. The subject of complete systems remained stationary until 1916, when the present writer improved the algorism for simultaneous connexes (TRANSACTIONS OF THIS SOCIETY, vol. 17), and determined a complete system for a conic p_x^2 and a connex $a_x \alpha_u$. The present paper gives the determination of a fundamental system of two bilinear connexes $a_x \alpha_u$, $p_x \theta_u$. Subject to some reduction methods yet to be tried, the system contains, approximately, 130 invariant formations.

7. Dr. Maria Castellani: *Algebraic surfaces with reducible bitangent and osculating hyperplanar sections.*

The object of this note is to study, first, the algebraic surfaces in hyperspace which are cut by any bitangent hyperplane in reducible curves and, second, algebraic surfaces which are cut by any osculating hyperplane in reducible curves. Among the results is the following theorem: The algebraic irreducible surfaces $S_r (r > 5)$, not cones, which any bitangent hyperplane cuts in reducible curves are the ruled surfaces and the surfaces with hyperplanar sections of genus $p = 1$.

8. Professor J. H. M. Wedderburn: *The elementary divisors of a real symmetric matrix.*

This paper gives a short proof of the well known theorem that a real symmetric matrix has real roots and simple elementary divisors. The proof is based on the reduced characteristic equation introduced by Frobenius and on the fact that $a^2 + b^2 = 0$ implies $a = b = 0$ when a and b are real symmetric matrices.

9. Professor J. H. M. Wedderburn: *The absolute value of the product of two matrices.*

This paper appears in the July number of this BULLETIN.

10. Dr. T. H. Gronwall: *The algebraic structure of the formulas in plane trigonometry.*

The set of formulas connecting the sides and angles of a plane triangle is investigated from the point of view of Hilbert's theory of algebraic forms (MATHEMATISCHE ANNALEN, vol. 36). A linear basis of the set of formulas is constructed, the syzygies of various orders are set up, and finally the group of birational transformations leaving the formulas invariant is considered.

11. Dr. T. H. Gronwall: *The behavior at infinity of the gamma and associated functions.*

Regions are constructed in which the functions $\Gamma(z)$, $P(z)$, and $Q(z)$ tend uniformly to a limit as $|z| \to \infty$ (the limit being either zero or infinity) and asymptotic formulas are given for the roots of the equations $\Gamma(z) = a$, $P(z) = a$, $Q(z) = a$.

12. Professor Norbert Wiener: *A canonical form for biunivocal continuous sense-preserving transformations of a sphere.*

The author proves a theorem suggested by Professor Alexander, to the effect that it is possible to interpolate a continuous sequence of continuous biunivocal transformations between two biunivocal continuous sense-preserving transformations of a sphere into itself. In the course of this argument, these transformations are reduced to a canonical form.

13. Dr. Harry Levy: *Ricci's canonical congruences.*

Ricci has associated with a given congruence of curves in a Riemann space $n-1$ others which he calls the congruences *canonical* with respect to the given one. In this paper we obtain a geometrical interpretation of the canonical congruences dependent upon the parallelism of Levi-Civita; we derive the conditions that congruences be mutually canonical, and from them we deduce several theorems of geometric interest.

14. Mr. Orrin Frink: *Operations of boolean algebras.*

In this paper the problem of finding all operations of boolean algebras which have all the formal properties of a given boolean operation is solved by means of a theory of transformation. The author also finds all boolean operations which are unique, i. e., which share all their formal properties with no one other operation. In terms of such a unique triadic operation taken as a single fundamental operation a postulate system for boolean algebras is given. All boolean operations with idemfacient and nilfacient elements are determined. A method is given of deriving from a noninvariant relation between n elements an invariant relation between $n+1$ elements, with similar properties.

15. Mr. Alonzo Church: *Alternatives to Zermelo's assumption.* Preliminary report.

If we reject the axiom of choice as a principle of logic applicable to all classes, the following possibilities for the second ordinal class appear: (A) There exists an assignment of a unique fundamental series to every ordinal of the second kind in the second ordinal class. (B) There exists no assignment of a unique fundamental series to every ordinal of the second kind in the second ordinal class; but, given any ordinal, α, of the second ordinal class, there exists an assignment of a unique fundamental series to every ordinal of the second kind less than α. (C) There is an ordinal, α, of the second ordinal class, such that there exists no assignment of a unique fundamental series to every ordinal of the second kind less than α. These three possibilities may be regarded as postulates, determining three distinct kinds of second ordinal classes, and the consequences of each may be investigated on this basis, with an open mind about the possibility of obtaining a contradiction from one or more of them. Any one of these postulates is found to be an effective instrument in obtaining theorems. On grounds of elegance, the theory resulting from (B) seems preferable, although it is in contradiction to the axiom of choice.

16. Professor Oswald Veblen and Dr. J. M. Thomas: *Projective normal coordinates for the geometry of paths.*

This paper introduces into the geometry of paths a system of coordinates which have the following properties: (1) they are independent of the particular affine connection associated with the paths and are therefore projective in character; (2) in terms of them the equations of the paths through the origin are linear in a specially chosen projective parameter; (3) they undergo a linear fractional transformation when the general coordinate system suffers an arbitrary analytic transformation. A projective normal tensor Q^i_{jkl} is defined in terms of them, and the Weyl projective curvature tensor is found to be $Q^i_{jkl}-Q^i_{jlk}$. An outline of this paper appeared in the PROCEEDINGS OF THE NATIONAL ACADEMY for April, 1925.

17. Dr. J. M. Thomas: *Conformal correspondence of Riemann spaces.*

In this paper is pointed out the existence of a set of quantities which are formed from the Christoffel symbols of the second kind and have the same value at corresponding points of two Riemann spaces mapped conformally on each other so that corresponding points have the same coordinates in both spaces. These quantities are useful in obtaining conformal invariants. The conformal curvature tensor of Weyl is obtained by expressing integrability conditions of their law of transformation under a change of coordinate system.

18. Dr. J. M. Thomas: *Asymmetric displacement of a vector.*

The changes in the components of a vector ξ^i by displacement from a point x^i to a nearby point x^i+dx^i of an n-dimensional manifold are assumed to be $-H^i_{jk}\xi^j dx^k$, where the quantities H have the same law of transformation as the Christoffel symbols of the second kind for a Riemann space, but are otherwise absolutely general. When the H's are changed, it is found that the directions of all displaced vectors are preserved if, and only if, the changes in the symmetric and skew symmetric parts of H have the forms $\delta_j^i \varphi_k + \delta_k^i \varphi_j$ and $\delta_j^i \varphi_k - \delta_k^i \varphi_j$ respectively, φ_j denoting an arbitrary vector. The most general change of affine connection which preserves geodesics (cf. Weyl, GÖTTINGER NACHRICHTEN, 1921, p. 99) is brought about, therefore, by such a change in the H's. Tensors independent of φ_j are found, and a process for forming tensors of the same nature but of higher rank is given. A necessary and sufficient condition for the H's to assume the special form considered by Friedmann and Schouten (MATHEMATISCHE ZEITSCHRIFT, vol. 21, p. 18) is the existence of a coordinate system in which the original and displaced vectors have proportional components. The corresponding H's can be made symmetric by a change preserving displaced directions.

19. Dr. J. H. Taylor: *On the inverse problem of the calculus of variations.*

In this paper application of existing theory is made to the problem of determining when a given system of differential equations are such that they define the extremals for a definite integral, whose integrand is expressed in parametric form and which is homogeneous of degree one in the first derivatives. It is shown that if the "paths" of the geometry of paths minimize a definite integral of the above type its integrand is necessarily the square root of a quadratic form, that is, the paths then admit of being the geodesics of a Riemann space.

20. Dr. J. P. Ballantine: *On a certain functional equation.*

In taking a mean between two numbers x_1 and x_2, one has an equation of the form $p_1 f(x_1) + p_2 f(x_2) = (p_1 + p_2) f(x_3)$. The mean based on the function f is called the f-mean. The problem is to find all functions f whose f-mean has the property that if x_1 and x_2 are each multiplied by the same number a, then the mean, x_3, is also multiplied by that same number. Additive and multiplicative constants in f have no effect on the mean, and, disregarding these, it is shown that x^n and $\log x$ are the only such functions. If a instead of being multiplied by x_1, x_2, and x_3 is added to them, the only functions are x and k^x, where k is an arbitrary constant.

ARNOLD DRESDEN,
Assistant Secretary.

THE FORTY-SIXTH REGULAR MEETING OF THE SAN FRANCISCO SECTION

The forty-sixth regular meeting of the San Francisco Section of the Society was held at the University of Oregon on Friday, June 19, 1925. Professor E. R. Hedrick presided and Professor W. E. Milne acted as Secretary. The total attendance was eighteen, including the following fourteen members of the Society:

Bernstein, A. F. Carpenter, De Cou, Dederick, Eells, E. R. Hedrick, Kent, Lehmer, W. E. Milne, Neikirk, Moritz, Robertson, Smail, Winger.

It was decided to hold the next Northwest meeting of the Section at the University of Washington on a date to be determined later.

The visiting members and their families were generously entertained by the University of Oregon members on the day of the meeting and on the day following.

Titles and abstracts of papers read at the meeting follow. The papers of Bell, Cajori, Cramlet, and Stager were read by title.

1. Professor E. T. Bell: *On the representations of integers as sums or differences of two positive integer cubes.*

This paper appeared in the July number of this BULLETIN.

2. Professor E. T. Bell: *Generalizations of the eight-square and similar identities.*

A generalization for the eight-square identity similar to that of Lagrange for the four-square is given; and it is shown that the corresponding norm theorems for algebraic numbers can be similarly generalized. This paper will appear in the ANNALS OF MATHEMATICS.

3. Professor B. A. Bernstein: *Sets of postulates for the logic of propositions.*

Defining, with Schroder, the logic of propositions as a two-element logic of classes, the author obtains four extremely simple sets of postulates for this logic, the number of the postulates being respectively 5, 4, 3, 1. The complete existential theory is established for each set.

4. Professor Florian Cajori: *Newton's indebtedness to Descartes.*

Ordinarily, we think only of the long struggle for supremacy between the Newtonian and Cartesian explanations of the mechanics of the solar system, and we overlook Newton's indebtedness to Descartes, in such cases as the following: (1) Descartes, rather than Euclid, initiated the young Newton into geometry, (2) Descartes' algebra, not Oughtred's or Harriot's, familiarized Newton with the modern exponential notation for positive integral exponents which he later generalized by the introduction of negative and fractional exponents, (3) Newton studied Descartes' rule of signs for equations, and gave it a precise statement, (4) he acknowledged the Cartesian optical theory to be a "good step," (5) he referred appreciatively to Descartes' theory of vision, (6) he used Descartes' (and Snell's) law of refraction in the explanation of total reflection, (7) according to Conduit, it was Descartes' physical theory of color that started Newton's experiments with prisms.

5. Professor A. F. Carpenter: *Self-correspondence of the complex developables of a ruled surface under a point-line transformation of space.*

By means of a transformation of the points of space into the lines of a linear congruence, the developables whose cuspidal edges are the two branches of the complex curve of a ruled surface are shown to be self-corresponding in such a way that to the points of any curve C lying upon one of these developables correspond the lines of that developable which are intersected by C.

6. Professor D. N. Lehmer: *Note on the construction of tables of linear forms belonging to a given quadratic residue.*

This paper appears in full in the present number of this BULLETIN.

7. Mr. C. M. Cramlet: *Some determinant theorems as tensor equations.*

The h^2 differential equations $a_{ir}(\partial\varphi/\partial a_{ij}) = \varphi\cdot\delta_r^j$ have $\varphi = Aa$ as a general solution, where a is the determinant of the a_{ij}'s and A is a constant of integration. The most general operations with determinants which merely introduce a factor are found from the condition that these equations be invariant. The determination of the constant of integration under these transformations gives the multiplication theorems.

Using a tensor defined by Murnaghan in a note in the AMERICAN MATHEMATICAL MONTHLY, vol. 32, a list of general determinant theorems is given in tensor notation. It is shown that a determinant of a tensor of second rank is a tensor with h^{2n} components of $+a$'s, $-a$'s, and 0's. Knowing this, the relative tensor, discussed by Veblen and Thomas, TRANSACTIONS OF THIS SOCIETY, vol. 26, ceases to be an extension. Similar investigations will be made with ϱ-way determinants.

8. Professor R. E. Moritz: *A new generalization of Wilson's theorem.*

In volume 31 of the QUARTERLY JOURNAL OF MATHEMATICS, Glaisher proved that the sum of the products of the first $n-1$ integers taken $p-1$ together, p being any prime not greater than n, is congruent to $-r$, mod p, where r is the integral part of the quotient obtained on dividing n by p. If $n=p$ we have Wilson's theorem. The author shows that this theorem may be extended, with certain restrictions, to any n successive integers.

9. Professor R. E. Moritz: *A theorem in number congruences.*

The author proves that if n is any prime number greater than 3, the combinations of all numbers of the form $kn-1$ taken $n-1$ together leave the same remainder when divided by n^3, that is to say

$$C_{n-1}^{kn-1}-C_{n-1}^{ln-1} \equiv 0, \quad (\text{mod } n^3).$$

When $l=1$, we have

$$C_{n-1}^{kn-1} \equiv 1, \quad (\text{mod } n^3).$$

The special case of our corollary for $k=2$ has been previously proved by Wolstenholme, QUARTERLY JOURNAL OF MATHEMATICS, volume 5.

10. Dr. H. P. Robertson: *On certain solutions of Einstein's cosmological equations.* Second paper.

The author here considers manifolds whose line elements are of the form

$$ds^2 = f(x,y,z,t)\{dx^2+dy^2+dz^2\}-g(x,y,z,t)dt^2,$$

i. e., orthogonal four-spaces containing a family of three-spaces, $t=$ constant. The cosmological equations for the case of a dynamical field of this type are immediately reduced to six second order equations in one dependent and four independent variables. These equations contain six arbitrary functions (arising through the integration of third order equations) which are determined by the conditions of integrability. All manifolds of this type are found; interpreted as relativistic space-times they represent all dynamical worlds which admit of orthogonal coordinates in which the velocity of light is independent of direction at a point. The solutions yield geometric rather than physical interpretations.

11. Dr. H. W. Stager: *A factor table of the second and succeeding ten millions.*

In this paper the author describes the method to be employed in constructing a factor table to the ultimate limit of 100,000,000. Following out the general method used in constructing his *Sylow Factor Table,* the table will first be extended to 50,000,000. The use of an automatic device and a special form of rubber type in making the entries assures almost absolute accuracy. These tables will be of the same general form as Lehmer's well known *Factor Table,* but will only show the least divisor of numbers not divisible by 2, 3, 5, 7, or 11. The omission of numbers divisible by 11 will reduce the size of the volume for each 10,000,000 approximately one-eleventh, without materially increasing the labor of finding the factors of a number.

12. Professor R. M. Winger: *The ternary Hesse group and its invariants.*

The Hesse collineation group has received attention at the hands of several writers, including Jordan, Witting, Muth, Maschke, Newson, Burnside, Steinitz, and Blichfeldt. The present author makes a systematic analysis of the group on the basis of its generating transformations and the theory of collineations, discussing in particular those properties of the invariant configuration and curves which follow most directly from the properties of the group.

B. A. BERNSTEIN,
Secretary of the Section.

ON THE NUMBER OF ELEMENTS OF A GROUP WHICH HAVE A POWER IN A GIVEN CONJUGATE SET*

BY LOUIS WEISNER

1. *Introduction.* A fundamental theorem on abstract groups is Frobenius' theorem: The number of elements in a group of order g whose nth powers belong to a given conjugate set is zero or a multiple of the greatest common divisor of g and n. In this paper, I will prove the following theorems, which are also concerned with the number of elements having a power in a given conjugate set.

THEOREM 1. *The number of elements of a group whose nth powers are in a given conjugate set is either zero, or a multiple of the number of elements in the conjugate set.*

THEOREM 2. *In a group of order g, the number of elements which have a power in a given conjugate set of elements of order n is a multiple of the greatest divisor of g that is prime to n.*

An interesting deduction from Theorem 2 is the following theorem.

THEOREM 3. *In a group of order g, the number of elements whose orders are multiples of n is either zero, or a multiple of the greatest divisor of g that is prime to n.*

2. *Proof of Theorem* 1. Let $t_1, t_2, \cdots, t_x$ be the elements of a group G which satisfy the equation $t^n = s_1$, and let the conjugates of s_1 under G be $s_1, s_2, \cdots, s_m$. There exist elements $u_1, u_2, \cdots, u_m$ in G such that

$$u_i^{-1} s u_i = s_i, \qquad (i = 1, 2, \cdots, m).$$

Since $t_a^n = s_1$,

$$(u_i^{-1} t_a u_i)^n = u_i^{-1} s u = s_i.$$

---

* Presented to the Society, February 28, 1925.

Hence there are exactly x elements of G whose nth powers are s_i. If $u_i^{-1}t_a u_i = u_k^{-1}t_b u_k$, then, raising both members to the nth power, we have

$$u_i^{-1}su_i = u_k^{-1}su_k,$$

or $s_i = s_k$, whence $i = k$. It follows that $t_a = t_b$, whence $a = b$.

The distinct elements of G whose nth powers are conjugate to s_1 are therefore

$$u_i^{-1}t_a u_i, \qquad \begin{pmatrix} a = 1, 2, \cdots, x \\ i = 1, 2, \cdots, m \end{pmatrix},$$

and their number is mx.

3. *Proof of Theorem* 2. Suppose, first, that the conjugate set consists of only one element s, which is therefore invariant under the group G. Let k be the greatest divisor of g that is prime to n; and let $t^a = s$.

CASE 1: *a prime to n*. If $aa' \equiv 1 \pmod{n}$, then $t = s^{a'}$. An element u of G of order m prime to n, being commutative with s, is commutative with t. Hence

$$(tu)^{xm} = s^{a'xm}u^{xm} = s,$$

where $xa'm \equiv 1 \pmod{n}$. *Hence s is a power of tu.*

By Frobenius' theorem, G contains λk (λ integral) elements whose orders divide k. Denote these elements by $u_1, \cdots, u_{\lambda k}$. We have just proved that the only elements of G satisfying the equation $t^a = s$, where a assumes all values prime n, are

$$s^{c'_\mu}u_z, \qquad \begin{pmatrix} \mu = 1, \cdots, \varphi(n) \\ z = 1, \cdots, k \end{pmatrix}, \tag{1}$$

where $c_1, \cdots, c_{\varphi(n)}$ are the integers not greater than n and prime to n. These elements are distinct;[*] hence their number is a multiple of k.

CASE 2: *a not prime to n*. Let $a = n_1 b$, where b is the greatest divisor of a that is prime to n. We may write $t = t_1 t_2$,[†] where t_1 and t_2 are powers of t, and the order of t_2 is the greatest divisor of the order of t that is prime to n, while the order of t_1 is a multiple n_2 of n which is not divisible by a number prime to n (except unity). If the order of t_2 is c, then the order of t is n_2c. Since $s = t^{n_1b} = t_1^{n_1b}t_2^{n_1b}$, we may write

---

[*] W. Burnside, *Theory of Groups*, 1911, § 16.

[†] Burnside, loc. cit.

$$s^n = t_1^{nn_1b}t_2^{nn_1b} = 1, \tag{2}$$

whence

$$t_1^{nn_1b} = t_2^{nn_1b} = 1.$$

Hence c is a divisor of nn_1b and therefore of b. It follows from (2) that $s = t_1^{n_1b}$. If u, of order m prime to n, is commutative with t, and if $m'm \equiv 1 \pmod{n_2}$, then

$$(t_1u)^{m'mn_1b} = t_1^{m'mn_1b}u^{m'mn_1b} = t_1^{n_1b} = s. \tag{3}$$

Hence s is a power of t_1u.

Let n_3k_1 be the order of the normaliser N of t in G and let $g/n_3k_1 = n_4k_2$, where k_1 and k_2 are the greatest divisors of n_3k_1 and n_4k_2 respectively that are prime to n and hence to n_2. The number of elements of N whose orders are prime to n is of the form ak_1; denote these elements by

$$u_1, \cdots, u_{ak_1}. \tag{4}$$

It follows from (3) that s is a power of

$$t_1u_1, \cdots, t_1u_{ak_1}. \tag{5}$$

It is noteworthy that t_2 is in (4) and hence $t = t_1t_2$ is in (5). Let $t_1' = w^{-1}t_1w$ be a conjugate of t_1. Since s is invariant under G,

$$t_1'^{n_1b} = w^{-1}t_1^{n_1b}w = w^{-1}sw = s.$$

Now there are exactly ak_1 elements in G whose orders are prime to n and which are commutative with t_1'. Denoting these by $u_1', \cdots, u_{ak_1}'$, it follows that s is a power of

$$t_1'u_1', \cdots, t_1'u_{ak_1}'. \tag{6}$$

Moreover, no element in (6) is equal to an element in (5).* There being n_4k_2 conjugates of t_1, we obtain $an_4k_1k_2$ elements of which s is a power. Observing that $k = k_1k_2$ is the greatest divisor of g prime to n, it follows that t_1 and its conjugates give rise in the manner described above to a multiple of k elements of which s is a power. These elements are evidently distinct from those obtained under Case 1.

---

* Burnside, loc. cit.

If s is a power of τ and τ is not one of the elements already obtained, let $\tau = \tau_1\tau_2$, where the order of τ_2 is the greatest divisor of the order of τ that is prime to n. Then τ_1 and its conjugates give rise to a multiple of k elements of which s is a power. Let $\tau_1 v$ be one of these elements, and, if possible, let it be equal to an element in (5), say $\tau_1 v = t_1 u = v$. Since the order of u and the order of v are both equal to the greatest divisor of the order of w that is prime to n, we must have $\tau_1 = t_1$, $v = u$. This is not the case, and hence τ_1 and its conjugates give rise to a multiple of k new elements of which s is a power.

The theorem now follows under the assumption that s is invariant under G.

Suppose next that the conjugate set consists of powers of s, so that (s) is invariant under G. Denote the conjugates of s by

$$s^{c_1}, \cdots, s^{c_r}, \qquad (c_1 = 1), \ (r > 1). \tag{7}$$

Let H, of order h, be the normaliser of s in G; and let k be the greatest divisor of h that is prime to n. The number of elements in H of which s is a power is a multiple of k, which we denote by λk. Since $c_1, \cdots, c_r$ are prime to n, $s^{c_1}, \cdots, s^{c_r}$ are powers of these same λk elements. Hence H contains exactly λk elements which have a power in (7). If t has a power in (7), so has t^{c_i} $(i = 1, 2, \cdots, r)$. Hence the number of elements of G which have powers in (7) is a multiple of r. The order of G is $g = rh$; for G/H is simply isomorphic with the group obtained by establishing an isomorphism of s with $s^{c_1}, \cdots, s^{c_r}$, and is of order r. The number of elements of G which have powers in (7) is a multiple of r and a multiple of k and is therefore a multiple of the greatest divisor of g that is prime to n.

Finally, suppose the conjugate set of elements does not consist of the powers of one of them. The elements in the conjugate set may be separated into subsets

$$\begin{array}{lll}
(8_1) & s_1^{c_1}, \cdots, s_1^{c_r}, & (c_1 = 1), \\
(8_2) & s_2^{c_1}, \cdots, s_2^{c_r}, & (r \geqq 1), \\
 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
(8_m) & s_m^{c_1}, \cdots, s_m^{c_r}, & (m > 1),
\end{array}$$

such that any two elements in the same subset are powers of each other, whereas no element is a power of an element in another subset.

Let H_i, of order h, be the normaliser of (s_i), $(i = 1, \cdots, m)$. An element of G which has a power in (8_i) is commutative with the elements of (8_i) and hence is in H_i. Therefore H_i contains all the elements of G which have powers in (8_i). Since the elements of (8_i) form a complete set of conjugates under H_i, the number of elements of G which have powers in (8_i) is λk, where λ is an integer, and k is the greatest divisor of h that is prime to n. If t has a power in (8_i), t cannot have a power in (8_j), $(i \neq j)$. For if $t^x = s_i^c$ and $t^y = s_j^{c'}$, then t^x and t^y are of the same order n, so that each is a power of the other, whence $i = j$. It follows that the number of elements of G which have a power in (8) is $\lambda kg/h$, and this number is evidently a multiple of the greatest divisor of g that is prime to n.

4. *Proof of Theorem* 3. If the group G contains an element of order n, separate the elements of order n into complete sets of conjugates, which we denote by $C_1, \cdots, C_x$. Of these, we select a subset $C_1, \cdots, C_y$, such that no element in C_i has a power in C_j $(i \neq j)$, $(i, j = 1, \cdots, y)$. Every element of G whose order is a multiple of n has a power in one and only one of the sets $C_1, \cdots, C_y$; and the order of every element which has a power in one of these sets is a multiple of n. Since the number of elements of G which have a power in C_i is a multiple of the greatest divisor of g that is prime to n, it follows that the number of elements of G whose orders are multiples of n is a multiple of the greatest divisor of g that is prime to n.

UNIVERSITY OF ROCHESTER

NOTE ON THE CONSTRUCTION OF TABLES OF LINEAR FORMS*

BY D. N. LEHMER

Certain theorems concerning the linear forms that go with a given quadratic residue seem not to have been noticed, or if noticed, not to have been appreciated by those who have constructed tables of divisors of quadratic forms. In connection with work on factor stencils, the author has found it necessary to extend these tables of linear forms, and the construction and checking has been much facilitated and rendered more accurate by the simple observation that if a and b are two entries in such a table so also is the product ab. In particular, the powers of any entry are also in the list of entries. This indicates a very rapid method of computation that is well adapted to a computing machine. If R is of the form $4n+1$, then in the table for R, if an entry a appears so also will $-a$. This furnishes an important check on the accuracy of the work. The numerous errors in Legendre's tables could not have escaped detection if this check had been employed.

The table for $-R$ may be easily written down without computation from the table for $+R$ and conversely, when it is observed that entries of the form $4n+1$ appear in the same place in both tables while entries of the form $4n-1$ which appear in one table must not appear in the other.

Availing himself of these theorems and checks, the author has already extended the tables of forms as far as $+300$ and -300. The tables for $+58$ and -58 are subjoined to indicate the form used and to illustrate the method of construction and the checks. Thus the first table shows that the forms for $+58$ are $232n+1$, 3, 7, 9, 11, 19, 21, 23, 25, 27, 33, $\cdots$, these entries being indicated by 1's. Numbers not prime to the modulus 58 are indicated by circles. Since (58/3) = 1, we can start with the entry 3 in this table and enter the powers of 3, casting out multiples of 232 as they arise. This process gives all the entries for this table. The entries may appear in several cycles, of course, instead of one as in this case.

---

* Presented to the Society, San Francisco Section, June 19, 1925.

The table for $-R$ is now built from this by setting over into it the entries of the form $4n+1$ as they stand, such as 1, 9, 21, 25, 33, 37, $\cdots$, and then filling in the blanks in the first table which correspond to numbers of the form $4n-1$, such as 15, 31, 35, 39, 47, $\cdots$, avoiding the numbers marked with circles. It is then carefully verified that the table for $+58$ reads backwards and forwards the same, while in the table for -58 the blanks at one end of the table match with the 1's at the other. The theorems upon which these checks are based are easily deduced from Jacobi's extension of Legendre's Law of Reciprocity.

$R = +58$

	$232n+$				
	1	3	5	7	9
0	1	1		1	1
1	1				1
2	1	1	1	1	○
3		1		1	
4		1			1
5				1	
6	1	1	1		1
7	1		1	1	
8	1		1	○	
9					1
10	1	1			
1	1				
2	1				1
3	1	1			
4			○	1	
5	1		1	1	
6	1	1		1	1
7	1		1		
8		1			1
9			1		1
20		○	1	1	1
1	1	1			
2	1	1	1		1
3	1				

$R = -58$

	$232n+$				
	1	3	5	7	9
0	1				1
1			1		
2	1		1		○
3	1	1	1	1	1
4				1	1
5	1		1	1	1
6	1		1	1	1
7				1	1
8	1	1	1	○	
9	1		1		
10	1			1	
1			1		1
2	1	1		1	1
3		1	1		1
4		1	○		
5				1	1
6	1				1
7					1
8				1	1
9	1				
20		○	1		1
1		1	1		1
2	1		1	1	1
3					

THE UNIVERSITY OF CALIFORNIA

CRITERIA THAT ANY NUMBER OF REAL POINTS IN n-SPACE SHALL LIE IN AN $(n—k)$-SPACE

BY H. S. UHLER

The object of the present paper is to establish an algebraic identity from which may be deduced necessary and sufficient conditions that any large number of real points in n-dimensional linear space shall lie in a linear $(n—k)$-space.

Let the following matrix, in which the number of columns is m and the number of rows is $n+1$ $[m \geqq (n+1)]$, be compounded with its conjugate:

$$\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ x_{1,1} & x_{2,1} & \cdots & x_{m,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{m,2} \\ \cdot & \cdot & \cdots & \cdot \\ x_{1,n} & x_{2,n} & \cdots & x_{m,n}. \end{array}$$

The determinant of the resulting symmetric square array is

$$\begin{vmatrix} m & \Sigma x_{i,1} & \Sigma x_{i,2} & \cdots & \Sigma x_{i,n} \\ \Sigma x_{i,1} & \Sigma x_{i,1}x_{i,1} & \Sigma x_{i,1}x_{i,2} & \cdots & \Sigma x_{i,1}x_{i,n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \Sigma x_{i,n} & \Sigma x_{i,n}x_{i,1} & \Sigma x_{i,n}x_{i,2} & \cdots & \Sigma x_{i,n}x_{i,n} \end{vmatrix} \equiv \Delta;$$

$$(i = 1, 2, 3, \cdots, m).$$

Multiply all of the rows of Δ except the top row by m, compensate by prefixing m^{-n}, and remove the factor m now common to the constituents of the first column to get

$$\Delta = m^{1-n}\begin{vmatrix} 1 & \Sigma x_{i,1} & \Sigma x_{i,2} & \cdots & \Sigma x_{i,n} \\ \Sigma x_{i,1} & m\Sigma x_{i,1}x_{i,1} & m\Sigma x_{i,1}x_{i,2} & \cdots & m\Sigma x_{i,1}x_{i,n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \Sigma x_{i,n} & m\Sigma x_{i,n}x_{i,1} & m\Sigma x_{i,n}x_{i,2} & \cdots & m\Sigma x_{i,n}x_{i,n} \end{vmatrix};$$

$$(i = 1, 2, 3, \cdots, m).$$

Next subtract $\sum_{i=1}^{i=m} x_{i,k}$ times the first column from the $(k+1)$th column, $(k=1, 2, 3, \cdots, n)$, in order to reduce to zero all the constituents of the top row, except the leading constituent, and to find $\Delta = U_n/m^{n-1}$, where

$$U_n \equiv \begin{vmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdot & \cdot & \cdot & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_{2,2} & \cdot & \cdot & \cdot & \sigma_{2,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_{n,1} & \sigma_{n,2} & \cdot & \cdot & \cdot & \sigma_{n,n} \end{vmatrix}$$

and

$$\sigma_{p,q} \equiv m \sum_{i=1}^{i=m} x_{i,p}x_{i,q} - \left(\sum_{i=1}^{i=m} x_{i,p}\right)\left(\sum_{i=1}^{i=m} x_{i,q}\right)$$

$$= \sum_{i=j+1}^{i=m} \sum_{j=1}^{j=m-1} [(x_{i,p}-x_{j,p})(x_{i,q}-x_{j,q})] = \sigma_{q,p}.$$

Now the determinant Δ produced by compounding the matrices specified above is known to equal the sum of the squares of all the ν determinants of order $n+1$ that can be formed from the columns of the original matrix, where

$$\nu \equiv \binom{m}{n+1}.$$

Let any one of these determinants be denoted by D_r; then the required algebraic identity is

$$(1) \qquad U_n = m^{n-1} \sum_{r=1}^{r=\nu} (D_r^2).$$

Thus far no special meaning has been assigned to the x's; they may represent complex quantities, etc.

To obtain the criteria contemplated advantage will be taken of the fact that D_r is squared in identity (1) so that if the x's are real numbers D_r^2 will be incapable of becoming negative. Accordingly let the rectangular coordinates of a system of real points in n-dimensional flat space be

$$(x_{i,1}, x_{i,2}, \cdots, x_{i,n});\ i = 1, 2, 3, \cdots, m;\ m \geqq (n+1).$$

Also let S_t symbolize a linear space of t dimensions, a t-flat.

Now the vanishing of $\sum(D_r^2)$ is a necessary and sufficient condition that the m given real points shall lie in the same S_{n-1}, hence, by formula (1), *a necessary and sufficient condition that any number* $m\,[\geqq(n+1)]$ *of real points in* S_n *shall lie in the same* S_{n-1} *is the vanishing of* U_n.

When $m=n+1$, $\nu=1$ so that there is only one D_r in $\sum(D_r^2)$. This D_r represents $n!$ times the content of the hyper-figure or simplex having the $n+1$ given points as vertices.* Hence, for $m>(n+1)$, $\sum(D_r^2)$ is proportional to the sum of the squares of the contents of all the simplexes that can be formed from the m points taken $n+1$ at a time as vertices of each geometric figure. Accordingly the above italicized statement may also be interpreted as meaning that the contents of all the simplexes involved vanish.

Keeping $m=n+1$, and giving n successively the values $1, 2, 3, 4, \cdots, n$, we may derive from the identity (1) the following expressions for the respective magnitudes of the length of a segment in S_1, the area of a triangle in S_2, the volume of a tetrahedron in S_3, the hyper-volume of a pentahedroid in $S_4, \cdots$, the content of a simplex in S_n:

$$\sigma_{1,1}^{1/2}, \qquad \frac{|\sigma_{1,1},\ \sigma_{2,2}|^{1/2}}{2\sqrt{3}}, \qquad \frac{|\sigma_{1,1},\ \sigma_{2,2},\ \sigma_{3,3}|^{1/2}}{24},$$

$$\frac{|\sigma_{1,1},\ \sigma_{2,2},\ \sigma_{3,3},\ \sigma_{4,4}|^{1/2}}{120\sqrt{5}}, \cdots, \frac{|\sigma_{1,1},\ \sigma_{2,2},\ \cdots,\ \sigma_{n,n}|^{1/2}}{n!\,(n+1)^{(n-1)/2}}.$$

The extension of the above italicized statement from S_{n-1} to S_{n-k} is an immediate consequence of the well known properties of orthogonal projections of linear spaces. The fundamental idea is that identity (1) holds for a smaller number of coordinates than n and hence it may be applied to the orthogonal projections of the m given points upon *all* of the

$$\binom{n}{n-k+1}$$

---

* P. H. Schoute, *Mehrdimensionale Geometrie*, Part 2, §§ 36, 37.

coördinate-S_{n-k+1}'s. In other words the original matrix is to be replaced by

$$\binom{n}{n-k+1}$$

matrices having the same top row of m 1's while the remaining rows are composed of $n-k+1$ of the original rows of x's. There will now be

$$\binom{n}{n-k+1}$$

new systems of points,—one in each coördinate-S_{n-k+1},—to all of which the above italicized test must be applied. The orders of the U_n's and D_r's of formula (1) will be $n-k+1$ and $n-k+2$ respectively. Without further comment it should be perfectly clear that *necessary and sufficient conditions that any number of real points in n-dimensional flat space shall lie in an* $(n-k)$*-dimensional flat space are that all the*

$$\binom{n}{n-k+1}$$

determinants U *of order* $n-k+1$ *in the* σ*'s shall vanish while one, at least, of the determinants* U *of order* $n-k$ *shall be finite.*

The last sentence may be stated in terms of the rank of the U of order n.* Incidentally the writer has found it possible to express the general criteria analytically in terms of only two determinants involving polynomial constituents composed of the σ's.

YALE UNIVERSITY

---

* G. Kowalewski, *Determinantentheorie*, § 52.

CHARACTERISTIC PARAMETER VALUES FOR AN INTEGRAL EQUATION*

BY W. A. HURWITZ

1. *Introduction*; *Hermitian Kernel.* It is well known, though seldom explicitly mentioned, that the properties of real symmetric kernels hold, with only trivial alterations in the usual proofs, for complex Hermitian kernels. It is the purpose of this note to show that kernels of certain other kinds have analogous properties, with the role of the real axis as bearer of the characteristic parameter values taken by any other straight line in the plane.

All functions of x, y, s which appear will be understood to be complex functions defined for real values of each of the variables in the closed interval (a, b); all integrals will be taken between the limits a, b, which will not be written. We call λ a *characteristic parameter value* (hereafter abbreviated cpv) for the kernel $K(x, y)$ if there exist non-trivial solutions of the equations

$$u(x) = \lambda \int K(x, s) u(s) ds, \tag{1}$$

$$v(y) = \lambda \int v(s) K(s, y) ds. \tag{2}$$

The cpv's and the corresponding solutions of (1) (normalized and orthogonalized), arranged in the usual order, will sometimes be denoted by λ_p, $\varphi_p(x)$ $[p = 1, 2, \cdots]$. A cpv is a pole (for at least some values of x, y) of the resolvent function $K(x, y; \lambda)$, which is elsewhere analytic in λ. The resolvent satisfies the equation

$$K(x, y; \lambda) - K(x, y; \mu) = (\lambda - \mu) \int K(x, s; \lambda) K(s, y, \mu) ds \tag{3}$$

whenever neither λ nor μ is a cpv. Since $K(x, y; 0) = K(x, y)$, (3) reduces for $\lambda = 0$ and for $\mu = 0$ to the simpler resolvent formulas used in the process of solving the Fredholm equation.

---

* Presented to the Society, December 30, 1924.

A continuous[*] kernel $K(x, y)$ is *closed* if the equation $\int K(x, s)u(s)ds = 0$ is satisfied by no continuous function except $u(x) = 0$.

The properties of Hermitian kernels with which we are particularly concerned may be stated as follows:

If $K(x, y)$ *is continuous and not identically zero, and*[†] $\bar{K}(y, x) = K(x, y)$, *then* (I) *every cpv is real*; (II) *every pole of the resolvent is of the first order*; (III) *there is at least one cpv*; *if* K *is closed, the set of cpv's is infinite*; (IV) *if for a cpv* λ, $\varphi(x)$ *is a solution of* (1), *then* $\bar{\varphi}(y)$ *is a solution of* (2); (V) *the series*[‡] $\sum \varphi_p(x)\bar{\varphi}_p(y)/\lambda_p^m$ *converges uniformly in* x, y *and represents* $K_m(x, y)$ *for* $m \geqq 2$; *it represents* $K(x, y)$ *for* $m = 1$ *provided it converges uniformly*; (VI) *any function* $f(x)$ *expressible in the form* $\int K(x, s)u(s)ds$, *where* $u(x)$ *is continuous, can be expanded in the uniformly convergent series* $f(x) = \sum \varphi_p(x) \int \bar{\varphi}_p(s)f(s)ds$.

2. *Straight Line through Origin.* Suppose that instead of the Hermitian condition we have $\bar{K}(y, x) = \zeta K(x, y)$, where K is not identically zero; then

$$\bar{K}(x, y) = \zeta K(y, x) = \zeta\bar{\zeta}\bar{K}(x, y),$$

so that $|\zeta| = 1$. Denote by $\sqrt{\zeta}$ either square root of ζ, and let $L(x, y) = \sqrt{\zeta}\, K(x, y)$. If λ, μ denote corresponding cpv's for K, L, then $\lambda = \mu\sqrt{\zeta}$. Since

$$\bar{L}(y, x) = \frac{1}{\sqrt{\zeta}}\bar{K}(y, x) = \sqrt{\zeta}\, K(x, y) = L(x, y),$$

L is Hermitian and μ is real. Thus λ lies on a line through the origin and the point $\sqrt{\zeta}$. The other properties are readily verified, and we have the following result.

---

[*] The assumption of continuity is made for simplicity; the usual extensions, with the use of Lebesgue integrals, are possible.

[†] A bar over a symbol for a *variable* denotes the conjugate of that variable; a bar over a symbol for a *function* denotes the conjugate of the whole expression represented by the function; for instance, $\bar{K}(x, y; \lambda)$ is an abbreviation for $\overline{K(x, y; \lambda)}$.

[‡] If the number of cpv's is finite, the series terminates, and the "convergence" is uniform in x, y for any value of m.

If $K(x, y)$ is continuous and not identically zero, and $\bar{K}(y, x) = \zeta K(x, y)$, then $|\zeta| = 1$; (I) *every cpv is on the line through the origin and* $\sqrt{\zeta}$; (II–VI) *the other properties stated in* § 1 *hold without change.*

A case of especial interest is that of the skew-Hermitian kernel: $\bar{K}(y, x) = -K(x, y)$; here $\zeta = -1$ and the cpv's are pure imaginary. Current treatises* prove the results of § 1 only for real symmetric kernels, and are therefore able to deduce the corresponding facts for real skew-symmetric kernels only by repeating the proofs throughout; whereas the procedure here given makes the second theory essentially a mere special case of the first.

3. *Special Straight Line not through Origin.* Now suppose that K is not identically zero and satisfies one of the relations†

$$(4) \qquad K(x, y) + \bar{K}(y, x) = 2\int K(x, s)\bar{K}(y, s)\,ds,$$

$$(5) \qquad K(x, y) + \bar{K}(y, x) = 2\int \bar{K}(s, x)K(s, y)\,ds;$$

for definiteness, say (4). By (3), if $K(x, y; 2)$ exists,

$$K(x, y; 2) = K(x, y) + 2\int K(x, s)K(s, y; 2)\,ds;$$

and, conversely, this relation is satisfied only by $K(x, y; 2)$; hence (4) is equivalent to

$$(6) \qquad K(x, y; 2) = -\bar{K}(y, x),$$

and (5) holds also. Since K has at most a countably infinite set of cpv's, there is on the line $R(\lambda) = 1$ a point $\lambda = \lambda_0$ which is not a cpv; thus

$$(7) \quad K(x, y; \lambda_0) = K(x, y) + \lambda_0\int K(x, s)K(s, y; \lambda_0)\,ds$$

---

* Lalesco, *Introduction à la Théorie des Équations Intégrales*, pp. 64, 73; Vivanti, *Elementi della Teoria delle Equazione Integrali Lineari*, pp. 207, 241.

† I omit the simple heuristic steps by which one is led to these equations as the natural generalization of the Hermitian condition.

and

$$(8)\quad K(x,y;\lambda_0) = K(x,y;2) + (\lambda_0 - 2)\int K(x,s;\lambda_0)K(s,y;2)\,ds.$$

From (8) we deduce, by interchanging x, y and taking conjugates,

$$(9)\quad \bar{K}(y,x;\lambda_0) = \bar{K}(y,x;2) + (\bar{\lambda}_0 - 2)\int \bar{K}(s,x;2)\bar{K}(y,s;\lambda_0)\,ds,$$

or, by (6) and the relation $\lambda_0 + \bar{\lambda}_0 = 2$,

$$(10)\quad \bar{K}(y,x;\lambda_0) = -K(x,y) + \lambda_0\int K(x,s)\bar{K}(y,s;\lambda_0)\,ds.$$

Call $M(x,y) = K(x,y;\lambda_0) + \bar{K}(y,x;\lambda_0)$. By (7) and (10),

$$M(x,y) = \lambda_0\int K(x,s)M(s,y)\,ds,$$

so that $M(x,y) = 0$ and

$$(11)\quad \bar{K}(y,x;\lambda_0) = -K(x,y;\lambda_0).$$

If we write $N(x,y) = -K(x,y;\lambda_0)$, and call $N(x,y;\mu)$ the resolvent to $N(x,y)$, then $N(x,y;\lambda - \lambda_0) = K(x,y;\lambda)$; also

$$(12)\quad K(x,y) = N(x,y) - \lambda_0\int K(x,s)N(s,y)\,ds,$$
$$= N(x,y) - \lambda_0\int N(x,s)K(s,y)\,ds.$$

By (11), $\bar{N}(y,x) = -N(x,y)$; and by § 2, with $\zeta = -1$, any cpv μ for N is a pure imaginary: $R(\mu) = 0$. Thus a cpv λ for K must satisfy $R(\lambda - \lambda_0) = 0$ or $R(\lambda) = 1$.

This condition replaces (I) of § 1. The other conditions may also be carried over; only (V), (VI), and the second part of (III) require any special consideration, which we now proceed to give.

In view of (12), it is readily seen that the equations

$$(13)\quad f(x) = \int K(x,s)u(s)\,ds, \qquad f(x) = \int N(x,s)v(s)\,ds$$

are equivalent if $v(x) = u(x) - \lambda_0 f(x)$; hence the hypothesis concerning f in (V), involving K, may be stated equivalently involving in the same way N, for which by § 2 the conclusion follows. Also by putting $f(x) = 0$ in (13), we see that the hypothesis of the second part of (II) may be put in terms of N instead of K.

This leaves only (VI) to be proved. By (VI) of § 2 stated for N, with $\zeta = -1$, $m = 2$, we have

$$N_2(x, y) = \sum \frac{\varphi_p(x)\overline{\varphi}_p(y)}{(\lambda_p - \lambda_0)^2}, \tag{14}$$

the series converging uniformly in x, y. From (12),

$$K_2(x, y) = N_2(x, y) - 2\lambda_0 \int K(x, s) N_2(s, y)\, ds + \lambda_0^2 \int K_2(x, s) N_2(s, y)\, ds,$$

by means of which (14) yields

$$K_2(x, y) = \sum \frac{\varphi_p(x)\overline{\varphi}_p(y)}{\lambda_p^2}, \tag{15}$$

the series converging uniformly. Direct computation from (15) of the higher kernels gives the corresponding series for greater values of m. For $m = 1$, suppose the series converges uniformly, and write

$$A(x, y) = \frac{\varphi_p(x)\overline{\varphi}_p(y)}{\lambda_p}.$$

By use of the relation

$$\varphi_p(x) + \lambda_0 \int N(x, s)\varphi_p(s)\, ds = \frac{\lambda_p}{\lambda_p - \lambda_0}\varphi_p(x),$$

which is an immediate consequence of (12), we find

$$A(x, y) + \lambda_0 \int N(x, s) A(s, y)\, ds = \sum \frac{\varphi_p(x)\overline{\varphi}_p(y)}{\lambda_p - \lambda_0}.$$

This series also converges uniformly and hence represents $N(x, y)$. Thus

$$A(x, y) + \lambda_0 \int N(x, s) A(s, y)\, ds = N(x, y)$$

and $A(x, y) = K(x, y)$.

All the expected results have now been shown to hold in the present case; I defer their formal statement until a further extension is made.

4. *Straight Line not through Origin.* More generally, let

$$\bar{\alpha}K(x, y) + \alpha\bar{K}(y, x) = 2\int K(x, s)\bar{K}(y, s)ds$$

or else

$$\bar{\alpha}K(x, y) + \alpha\bar{K}(y, x) = 2\int \bar{K}(s, x)K(s, y)ds,$$

where $\alpha \neq 0$. Writing $K(x, y) = \alpha L(x, y)$, we see that L satisfies the condition stated for K in (4) or (5). Hence a cpv μ for L satisfies $R(\mu) = 1$, and a cpv λ for K satisfies $R(\alpha\lambda) = 1$.

The verification of all the other results of § 3 is easy. We have the following theorem.

If $K(x, y)$ is continuous and not identically zero, and satisfies one of the conditions

$$\bar{\alpha}K(x, y) + \alpha\bar{K}(y, x) = 2\int K(x, s)\bar{K}(y, s)ds,$$

$$\bar{\alpha}K(x, y) + \alpha\bar{K}(y, x) = 2\int \bar{K}(s, x)K(s, y)ds,$$

where $\alpha \neq 0$, it satisfies the other also; (I) *every cpv λ is on the straight line $R(\alpha\lambda) = 1$*; (II–VI) *the other properties stated in* § 1 *hold without change.*

5. *Conclusion.* We have given conditions on the kernel which will cause the cpv's to lie on any chosen line through the origin in § 2, and on any line not through the origin in § 4. The results of § 2 are not contained in those of § 4; but if we put $\alpha = ri/\sqrt{\zeta}$, where $|\zeta| = 1$, r is real and positive, we may view them as limiting cases for $r \rightarrow \infty$. A slight change of notation would of course make possible the inclusion of all the results in a single theorem.

Cornell University

ON THE POLYNOMIAL OF THE BEST APPROXIMATION TO A GIVEN CONTINUOUS FUNCTION*

BY J SHOHAT (JACQES CHOKHATE)

1. *A Theorem on Minimizing Polynomials.* Let $f(x)$ and $p(x)$ be defined on a finite interval (a, b); $f(x)$ is bounded and integrable, $p(x)$ is integrable and not negative.

THEOREM I.† *If there exist two numbers α, β such that $a \leqq \alpha < \beta \leqq b$, and such that*

$$\int_c^d p(x)\,dx > 0$$

whenever $\alpha \leqq c < d \leqq \beta$, then there exists one and only one polynomial of degree $\leqq n$ minimizing the integral

$$I_{nk} = \int_a^b p(x)\,|f(x) - U_{nk}(x)|^k\,dx,$$

where

$$U_{nk}(x) = \sum_{i=0}^{n} u_{ik}x^i,$$

provided that $k > 1$. If $k = 1$, the proof of existence applies without change; and the approximating polynomial is unique, if $f(x)$ is continuous on (a, b), and if

$$\int_c^d p(x)\,dx > 0$$

whenever $a \leqq c < d \leqq b$.

The proof may be organized as follows.

---

[*] Presented to the Society, December 26, 1924. The author wishes to acknowledge with appreciation many helpful suggestions made by Professor D. Jackson in connection with this paper.

[†] Cf. G. Pólya, *Sur un algorithme . . .*, COMPTES RENDUS, vol. 157 (1913), pp. 840–843; D. Jackson, *On functions of closest approximation*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 117–128, *Note on a class of polynomials of approximation*, ibid., vol. 22 (1921), pp. 320–326, *A generalized problem in weighted approximation*, ibid., vol. 26 (1924), pp. 133–154, *Note on the convergence of weighted trigonometric series*, this BULLETIN, vol. 29 (1923), pp. 259–263.

2. *Existence of a Minimizing Polynomial.* The value of n being regarded as fixed, take

$$\delta = \frac{\beta - \alpha}{8n^2}. \tag{1}$$

For $y \pm \delta$ belonging to (α, β), we have

$$\int_y^{y+\delta} p(x)\,dx, \quad \int_{y-\delta}^{y} p(x)\,dx \geqq h(\delta) = h_n, \tag{2}$$

since these integrals are continuous and positive functions of y; h_n depends on n only. For an arbitrarily large $A > 0$ we can find, according to Kirchberger,* certain K_i such that *any one* of the inequalities $|u_{ik}| > K_i$ $(i = 0, 1, 2, \cdots, n)$ implies

$$\begin{aligned} \max|U_{nk}(x)| \text{ in } (\alpha, \beta) = |U_{nk}(\xi)| &= M > 2\Phi + 2\sqrt[k]{A/h_n}, \\ \Phi = \max|f(x)| \text{ in } (a, b), \qquad & \alpha \leqq \xi \leqq \beta. \end{aligned} \tag{3}$$

Using Markoff's theorem† and (1), we find, for $|x - \xi| \leqq \delta$,

$$|U_{nk}(x) - U_{nk}(\xi)| \leqq |x - \xi| \cdot \frac{2n^2 M}{\beta - \alpha} \leqq \frac{M}{4},$$

$$|U_n(x)| > \frac{M}{2} > \Phi + \sqrt[k]{A/h_n},$$

$$|f(x) - U_{nk}(x)| > \sqrt[k]{A/h_n}, \tag{4}$$

whence it follows that $I_{nk} > A$, since at least one of the intervals $(\xi - \delta, \xi)$, $(\xi, \xi + \delta)$ belongs to (α, β). Therefore we must take $|u_{ik}| \leqq K_i$ $(i = 0, 1, 2, \cdots, n)$, which proves the existence of a minimizing polynomial. We use the notation

$$M_{nk} = \min I_{nk} = \int_a^b p(x)\,|f(x) - P_{nk}(x)|^k\,dx. \tag{5}$$

---

[*] *Ueber Tchebychefsche Annäherungsmethoden*, Dissertation, Gottingen, 1902.

[†] Cf., e. g., M. Riesz, *Eine trigonometrische Interpolationsformel, . . .*, JAHRESBERICHT DER DEUTSCHEN - MATHEMATIKER - VEREINIGUNG, vol. 23 (1914), pp. 354-368; pp. 359-360.

3. *Uniqueness of the Solution.* Case I: $k > 1$. It can be easily shown that

$$|\tfrac{1}{2}(x+y)|^k < \tfrac{1}{2}(|x|^k + |y|^k) \tag{6}$$

if $k > 1$ and $x \neq y$. Assuming the existence of two non-identical solutions $\varphi_1(x) = f(x) - P'_{nk}(x)$, $\varphi_2(x) = f(x) - P''_{nk}(x)$, we get, using in (6) the function $\varphi_3(x) = \tfrac{1}{2}[\varphi_1(x) + \varphi_2(x)]$,

$$\int_a^b p(x)|\varphi_3(x)|^k dx < \tfrac{1}{2}M_{nk} + \tfrac{1}{2}M_{nk} = M_{nk}, \tag{7}$$

which is impossible.

Case II: $k = 1$. Proceeding as above, we get the impossible inequality (7), unless $\varphi_1(x)\varphi_2(x) \geqq 0$ for $a \leqq x \leqq b$. Evidently every function of the type

$$\varphi_3(x) = h\varphi_1(x) + l\varphi_2(x) = \varphi_1(x) + l\psi(x), \tag{8}$$

where $\psi(x) = \varphi_2(x) - \varphi_1(x)$, $h, l > 0, h + l = 1$, is also a solution. Inasmuch as $\varphi_1(x)$ and $\varphi_2(x)$ can never have opposite signs, all roots of $\varphi_3(x)$ are the roots common to $\varphi_1(x)$ and $\varphi_2(x)$, or, what is the same, to $\varphi_1(x)$ and $\psi(x)$. Since $\psi(x)$ is a polynomial of degree $\leqq n$, we conclude:

If there exist two solutions $\varphi_1(x), \varphi_2(x)$, there exists necessarily a third one $\varphi_3(x)$, which has not more than n roots in (a, b). We shall prove that the last conclusion leads to a contradiction.[*]

Let the zeros of $\varphi_3(x)$ in (a, b) be

$$x_1 < x_2 < \cdots < x_m \qquad (m \leqq n). \tag{9}$$

We have necessarily $m > 0$; otherwise the function $\varphi(x) = \varphi_3(x) + \eta$ with a properly chosen η gives $I_{nk} < M_{nk}$. Consider now two groups of the roots (9):

$$z_1, z_2, \cdots, z_{m'}; \quad y_1, y_2, \cdots, y_{m''} \qquad (m' + m'' = m),$$

where the z's are the roots, if any, at which $\varphi_3(x)$ changes sign, and the y's are those at which it does not. Form the polynomial

---

[*] In the case that $f(x) \equiv x^{n+1}$, it was established in my Thesis that the number of roots is $n+1$.

$$(10)\qquad \prod(x) = \pm\prod_{i=1}^{m'}(x-z_i) \quad \left(\prod(x)=1 \text{ for } m'=0\right),$$

where the sign $\pm$ is chosen so that $\prod(x)\varphi_3(x)\geqq 0$ in (a, b). Draw the curve $y=|\prod(x)|$ and the line $y=\varepsilon$ with $\varepsilon>0$ sufficiently small. Project the points of intersection on the x-axis. Thus we get a set of points Z, made up of m' intervals

$$(11)\qquad (z_i-\delta_i', z_i+\delta_i'') \quad (i=1, 2, \cdots, m'),$$

and a complementary set $K=b-a-Z$, such that

$$(12)\qquad \varepsilon = \max\left|\prod(x)\right| \text{ on } Z = \min\left|\prod(x)\right| \text{ on } K.$$

It is evident that for ε sufficiently small all maxima points of the curve and all y's are outside Z, and

$$(13)\qquad \int_K p(x)\,dx > 3\int_Z p(x)\,dx.$$

The numbers ε and δ_i', δ_i'' being fixed, we take $\delta>0$ so small that the set of points

$$(14)\qquad Y:\quad y_j-\delta\leqq x\leqq y_j+\delta \quad (j=1, 2, \cdots, m'')$$

is outside Z, and

$$(15)\qquad \begin{aligned} &\int_Y p(x)\,dx < \frac{\varepsilon}{2M}\left(\int_K p(x)\,dx-\int_Z p(x)\,dx\right),\\ &\int_Y p(x)\,dx < \int_Z p(x)\,dx, \end{aligned}$$

where M is the maximum of $|\prod(x)|$ in (a, b). Since $\varphi_3(x)$ is continuous and has no zeros in

$$(16)\qquad K' = b-a-Z-Y = K-Y,$$

we have

$$(17)\qquad |\varphi_3(x)|\geqq h'>0 \quad \text{on } K'.$$

We form now the function

$$(18)\qquad \varphi(x) = \varphi_3(x)-\eta\prod(x), \qquad 0<\eta<h'/M,$$

and observe that

$$(19)\qquad \begin{cases} |\varphi(x)| = |\varphi_3(x)| - \eta \left|\prod(x)\right| & \text{on } K', \\ |\varphi(x)| \leqq |\varphi_3(x)| + \eta \left|\prod(x)\right| & \text{on } Z+Y, \end{cases}$$

whence, by combination with (12), (13), and (15), we get the desired contradiction:

$$\int_a^b p(x)|\varphi(x)|\,dx < \int_a^b p(x)|\varphi_3(x)|\,dx = M_{n1}.$$

The existence proof gives incidentally the following corollary.

COROLLARY. *To an arbitrarily large* $A > 0$ *there correspond certain* K_i $(i = 0, 1, 2, \cdots, n)$ *such that any one of the inequalities* $|u_i| > K_i$ *implies that*

$$\int_a^b p(x)\left|f(x) - \sum_{i=0}^{n} u_i x^i\right|^k dx > A,$$

where $P(x)$ *and* $f(x)$ *have the properties specified, and* $k \geqq 1$.

4. *Tchebychef Approximating Polynomials.* THEOREM II. *Let* $f(x)$ *be continuous in* (a, b), $p(x) \geqq 0$, *and*

$$\int_\alpha^\beta p(x)\,dx > 0$$

whenever $a \leqq \alpha < \beta \leqq b$. *Then, for* $k \to \infty$, *the minimizing polynomial* $P_{nk}(x)$ *tends uniformly to the polynomial* $T_n(f)$, *of degree* $\leqq n$, *which gives the closest approximation to* $f(x)$ *in the interval* (a, b), *in the sense of Tchebychef.*

The proof is very similar to that given by G. Pólya, loc. cit. We assume, without loss of generality,

$$(20)\qquad \int_a^b p(x)\,dx = 1,$$

replacing, if necessary, $p(x)$ by $cp(x)$, $c > 0$ being properly chosen. We have evidently, taking $k > 1$,

$$(21)\qquad \begin{aligned} M_{nk} &= \min I_{nk} < E_n^k, \\ E_n &= \max |f(x) - T_n(f)| \text{ for } a \leqq x \leqq b. \end{aligned}$$

If $a < \alpha < \beta < b$, and $|f(x) - P_{nk}(x)|$ attains its minimum in (α, β) at $x = z$, then we get, using (21),

$$\begin{gathered} |P_{nk}(z)| < \frac{E_n}{\sqrt[k]{\int_a^b p(x)\,dx}} + \Phi < \frac{E_n}{\int_a^b p(x)\,dx} + \Phi, \\ \Phi = \max |f(x)| \text{ in } (a, b). \end{gathered} \tag{22}$$

We form now $n+1$ partial intervals (α_i, β_i) all belonging to (a, b), and separated by segments of a certain length δ, and we use (22). *Since the right-hand member in* (22) *does not depend on* k, we can follow the reasoning of G. Pólya and prove (a) the coefficients in $P_{nk}(x)$, considered as functions of k, are bounded, so that the sequence (P_{nk}) admits one or several limiting polynomials; (b) all these limiting polynomials coincide with the polynomial of best approximation $T_n(f)$, which is known to be unique.

COROLLARY. *The polynomial minimizing the integral*

$$\int_a^b p(x)\,|x^n + \cdots|^k dx$$

tends for $k \to \infty$ *uniformly to*

$$\frac{1}{B} \cos(n \arccos y), \qquad y = \frac{2x - a - b}{b - a},$$

for every $p(x)$ *satisfying the conditions of Theorem* II, *where* $B (= 2^{2n-1}/(b-a)^n)$ *is the coefficient of* x^n *in* $\cos(n \arccos y)$.

In fact, $x^n - (1/B)\cos(n \arccos y)$ is the polynomial of degree $\leq n-1$ giving the best approximation to x^n in (a, b).

The results given above hold if we replace polynomials by trigonometric sums, provided $p(x)$ and $f(x)$ are periodic functions.

THE UNIVERSITY OF CHICAGO

RESOLVENT SEXTICS OF QUINTIC EQUATIONS

BY L. E. DICKSON

1. *Introduction.* The object of this paper is to give simple derivations of the classic resolvents which have been obtained heretofore by elaborate computations.

Jacobi[*] established the form of a remarkable resolvent, but neither found the values of the coefficients nor gave the simple details (§ 2 below) which lead directly to that form.

Cayley[†] was not aware of Jacobi's work when he fully computed the same resolvent. Noting that its roots are functions of the differences of the roots x_i of the quintic, he first computed at length the resolvent sextic under the restriction that $x_5 = 0$. Then the coefficients were "completed by the introduction of the terms involving the constant coefficient of the quintic." No details were given of the latter long computation, which may perhaps be best made by utilizing the fact that the coefficients are seminvariants. The simple new method employed here (§ 3) makes initial use of the latter fact as well as of a lemma which reduces the search for the needed seminvariants of the quintic to a mere inspection of the invariants of a quartic.

From the Jacobi-Cayley resolvent (which is a simple transform of the old Malfatti resolvent) it is an immediate step (§ 5) to the noteworthy covariant resolvent discovered by Perrin,[‡] and independently by McClintock,[§] each time as the final step of a long computation.

---

[*] JOURNAL FUR MATHEMATIK, vol. 13 (1835), pp. 340–52; WERKE, vol 3, 1884, pp. 269–84.

[†] PHILOSOPHICAL TRANSACTIONS, London, vol. 151 (1861), pp. 263–76; COLLECTED MATHEMATICAL PAPERS, vol. 4, pp. 309–24.

[‡] COMPTES RENDUS DU DEUXIÈME CONGRÈS INTERNATIONAL DES MATHÉMATICIENS, Paris, 1902, pp. 199–223. Announced in BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 11 (1882–83), pp. 64–65.

[§] AMERICAN JOURNAL, vol. 8 (1886), pp. 45–84; vol. 20 (1898), pp. 157–192.

2. *The Symmetric Functions of* $z_1, \cdots, z_6$. Writing ij for x_ix_j, we consider the function

$$12345 = 12+23+34+45+51.$$

It is unaltered by the substitutions $a=(12345)$, $b=(25)(34)$. Since $b^{-1}ab = a^{-1}$, a and b generate a group of ten even substitutions. Since 12345 is therefore unaltered by these ten, it takes at most 60/10 distinct values under the group G of all 60 even substitutions. It actually takes the six distinct values given by the first (positive) parts of

$$z_1 = 12345-13524, \quad z_2 = 12453-14325,$$
$$z_3 = 12534-15423, \quad z_4 = 15243-12354,$$
$$z_5 = 14235-12543, \quad z_6 = 13254-12435.$$

In fact, (345), (354), (253), (243), (23)(45) replace z_1 by $z_2, \cdots, z_6$. Hence $z_1, \cdots, z_6$ are merely permuted by each of the 60 even substitutions. Next, every odd substitution O replaces each z_i by the negative of some z. For, (2354) replaces z_1 by $-z_1$. Let E be one of the even substitutions which replaces z_i by z_1 and write E_1 for the even substitution $(2354)^{-1}E^{-1}$. Then $O=E(2354)E_1$ replaces z_i by the function by which E_1 replaces $-z_1$ and that function is the negative of some z.

Hence any homogeneous symmetric function of $z_1, \cdots, z_6$ of even degree is symmetric in $x_1, \cdots, x_5$. But if it is of odd degree in the z's, it merely changes sign when any two x's are interchanged and hence is divisible by the product of the ten differences of the x's, the quotient being symmetric in the x's.

3. *The Jacobi-Cayley Resolvent.* The discriminant Δ of

$$f(x,y) = a_0x^5+5a_1x^4y+10a_2x^3y^2+10a_3x^2y^3+5a_4xy^4+a_5y^5$$

is defined to be the polynomial such that $5^5a_0^{-8}\Delta$ is equal to the product of the squares of the ten differences of the roots x_i of $f(x,1)=0$.

In the sextic having the roots $z_1, \cdots, z_6$, the coefficients of z^5 and z^3 are zero by § 2, being of odd degrees 1 and 3 in the z's, so that their degrees in the x's are less than the degree 10 of the product Π of the differences of the x's. The coefficient of z is the product of a numerical constant by Π or by $a_0^{-4}\sqrt{\Delta}$. It is convenient to multiply the sextic by a_0^6. We get

$$a_0^6 z^6 + a_0^4 u_1 z^4 + a_0^2 u_2 z^2 + u_3 = \nu a_0^2 \sqrt{\Delta} z,$$

where ν is a numerical constant, while $a_0^{-2i} u_i$ is the sum of the products of the z's taken $2i$ at a time and hence is of total degree $4i$ in the x's and of degree $2i$ in any one root x. By § 2, it is symmetric in the x's. It is expressible as a polynomial in the differences of the x's, since

$$z_1 = (1-5)(2-5) + (2-5)(3-5) + (3-5)(4-5)$$
$$-(2-5)(4-5)-(4-5)(1-5)-(1-5)(3-5).$$

It follows* that u_i is a seminvariant of f of degree $2i$ and weight $4i$. By a seminvariant S of f is meant a homogeneous isobaric polynomial in $a_0, \cdots, a_5$ for which $\Omega S = 0$, i. e., is annihilated by the operator

$$\Omega = a_0 \frac{\partial}{\partial a_1} + 2 a_1 \frac{\partial}{\partial a_2} + 3 a_2 \frac{\partial}{\partial a_3} + 4 a_3 \frac{\partial}{\partial a_4} + 5 a_4 \frac{\partial}{\partial a_5}.$$

Since u_1 is of degree 2 and weight 4, it lacks a_5 and is the product of I by a numerical constant, as shown by the following lemma.

LEMMA. *If a seminvariant S of the quintic $f(x, y)$ lacks a_5, it is a seminvariant of the quartic*

$$q = a_0 x^4 + 4 a_1 x^3 y + 6 a_2 x^2 y^2 + 4 a_3 x y^3 + a_4 y^4.$$

If the weight of S is double its degree, it is an invariant of q and hence is a polynomial in

$$I = a_0 a_4 - 4 a_1 a_3 + 3 a_2^2,$$
$$J = a_0 a_2 a_4 - a_0 a_3^2 + 2 a_1 a_2 a_3 - a_1^2 a_4 - a_2^3.$$

For, S is homogeneous and isobaric and is annihilated by the operator derived from Ω by suppressing the final derivative. Hence S is a seminvariant of q. A seminvariant of degree d and weight w of a binary form of order p is the leader of a unique covariant of order $pd-2w$ (*Invariants,* p. 43 and Ex. 1, p. 40). It is therefore an invariant if $p=4$, $w=2d$.

---

* Dickson, *Algebraic Invariants*, New York, 1914, p. 53.

A known seminvariant of the same degree 4 and weight 8 as u_2 is

$$T = a_0^2a_3a_5-3a_0a_1a_2a_5-5a_0a_1a_3a_4+10a_0a_2^2a_4-4a_0a_2a_3^2$$
$$+2a_1^3a_5-5a_1^2a_2a_4+14a_1^2a_3^2-16a_1a_2^2a_3+6a_4^2$$

(it suffices to verify that $\Omega T\equiv 0$). We delete the term $a_0^2a_3a_5$ from u_2 by subtracting a multiple of T. In the resulting seminvariant v, the only terms involving a_5 are those in $a_5(\alpha a_0a_1a_2+\beta a_1^3)$, since the terms in parenthesis, together with the deleted term $a_0^2a_3$, are the only possible terms of degree 3 and weight 3. By inspection, Ωv is the sum of

$$\alpha a_0^2a_2a_5+(2\alpha+3\beta)a_0a_1^2a_5$$

and terms free of a_5. Hence $\alpha=\beta=0$. Thus v lacks a_5 and by the Lemma is an invariant of q and hence is a product of I^2 by a constant. Thus u_2 is a linear combination of I^2 and T.

To determine u_3 we shall employ the seminvariant P of the same degree 6 and same weight 12 (cf. § 6).

$$P = a_0^2a_2a_5^2-2a_0^2a_3a_4a_5+a_0^2a_4^3-a_0a_1^2a_5^2-4a_0a_1a_2a_4a_5$$
$$+8a_0a_1a_3^2a_5-2a_0a_1a_3a_4^2-2a_0a_2^2a_3a_5+14a_0a_2^2a_4^2$$
$$-22a_0a_2a_3^2a_4+9a_0a_3^4+6a_1^3a_4a_5-12a_1^2a_2a_3a_5-15a_1^2a_2a_4^2$$
$$+10a_1^2a_3^2a_4+6a_1a_2^3a_5+30a_1a_2^2a_3a_4-20a_1a_2a_3^3$$
$$-15a_2^4a_4+10a_2^3a_3^2.$$

We use a_0P and TI to delete the terms $a_0^3a_2a_5^2$ and $a_0^3a_3a_4a_5$ from u_3. In the resulting seminvariant S, the terms having the factor a_5 are those in

$$\alpha a_0^2a_1^2a_5^2+\beta a_0^2a_1a_2a_4a_5+\gamma a_0^2a_1a_3^2a_5+a a_0^2a_2^2a_3a_5$$
$$+b a_0a_1^3a_4a_5+c a_0a_1^2a_2a_3a_5+d a_0a_1a_2^3a_5+e a_1^4a_3a_5+g a_1^3a_2^2a_5.$$

These terms alone furnish the part of ΩS involving a_5:

$$2\alpha\, a_0^3 a_1 a_5^2 + \beta\, a_0^3 a_2 a_4 a_5 + \gamma\, a_0^3 a_3^2 a_5 + (3b + 2\beta + 10\alpha)\, a_0^2 a_1^2 a_4 a_5$$
$$+ (2c + 4a + 4\beta + 6\gamma)\, a_0^2 a_1 a_2 a_3 a_5 + (d + 3a)\, a_0^2 a_2^3 a_5$$
$$+ (2c + 4b + 4e)\, a_0 a_1^3 a_3 a_5 + (3c + 3g + 6d)\, a_0 a_1^2 a_2^2 a_5$$
$$+ (4g + 3e)\, a_1^4 a_2 a_5.$$

The conditions that this be zero identically require that $\alpha, \beta, \gamma, a, b, c, d, e, g$ shall all vanish. Since S therefore lacks a_5, the lemma shows that it is a sum of terms $I^r J^s$ whose degree is $2r + 3s = 6$, whence either $s = 0,\ r = 3$ or $s = 2,\ r = 0$. Thus u_3 is a linear combination of $I^3, J^2, a_0 P, TI$. Hence the sextic is of the form

$$a_0^6 z^6 + a_0^4 \alpha I z^4 + a_0^2(\beta I^2 + \gamma T) z^2 + \delta I^3 + \varepsilon J^2 + \lambda a_0 P + \mu TI$$
$$= a_0^2 \nu \sqrt{\Delta} z,$$

where the Greek letters are numerical constants. To find their values, we employ the special quintic having $x_3 = -x_1$, $x_4 = -x_2$, $x_5 = 0$:

$$x(x^2 - x_1^2)(x^2 - x_2^2) = x^5 + 10 a_2 x^3 + 5 a_4 x = 0,$$

where $10 a_2 = -x_1^2 - x_2^2$, $5 a_4 = x_1^2 x_2^2$. Then

$$z_1 = z_3 = (x_1 + x_2)^2, \qquad z_2 = z_4 = -(x_1 - x_2)^2,$$
$$z_5 = x_1^2 - x_2^2 - 4x_1 x_2, \qquad z_6 = x_2^2 - x_1^2 - 4x_1 x_2.$$

Using temporarily the abbreviations

$$s = x_1^2 + x_2^2, \qquad p = x_1^2 x_2^2, \qquad t = x_1 x_2,$$

we see that z_1 and z_2 are the roots of $z^2 - 4tz + 4p - s^2 = 0$, while z_5 and z_6 are the roots of $z^2 + 8tz + 20p - s^2 = 0$. Hence the sextic is

$$z^6 - (20p + 3s^2) z^4 + (240p^2 - 8ps^2 + 3s^4) z^2 + (4p - s^2)^2 (20p - s^2)$$
$$= 128 tp (4p - s^2) z.$$

Replacing s, p, t by their values in terms of a_2, a_4, we get

$$z^6 - 100(a_4 + 3a_2^2) z^4 + 2000(3a_4^2 - 2a_4 a_2^2 + 15 a_2^4)$$
$$+ 40000(a_4^3 - 11 a_4^2 a_2^2 + 35 a_4 a_2^4 - 25 a_2^6)$$
$$= 12800 a_4 (a_4 - 5a_2^2) \sqrt{5a_4} z.$$

But for $a_0 = 1$, $a_1 = a_3 = a_5 = 0$, we have

$$I = a_4 + 3a_2^2, \qquad J = a_2a_4 - a_2^3, \qquad T = 10a_2^2a_4 + 6a_2^4,$$
$$\Delta = 16^2a_4^3(a_4 - 5a_2^2)^2, \qquad P = a_4^3 + 14a_2^2a_4^2 - 15a_2^4a_4.$$

We see at once that $\alpha = -100$, $\beta = 6000$, $\gamma = -4000$, $\nu = 800\sqrt{5}$. By the terms free of z,

$$\delta + \lambda = 40000, \quad 9\delta + \epsilon + 14\lambda + 10\mu = -11\cdot 40000,$$
$$27\delta - 2\epsilon - 15\lambda + 36\mu = 35\cdot 40000,$$
$$27\delta + \epsilon + 18\mu = -25\cdot 40000.$$

Hence $\lambda = 40000$, $\epsilon = -25\lambda$, $\mu = \delta = 0$. The resolvent sextic is therefore

$$a_0^6z^6 - 100a_0^4Iz^4 + 2000a_0^2(3I^2 - 2T)z^2 - 800\sqrt{5}\,a_0^2\sqrt{\Delta}\,z$$
$$+ 40000(a_0P - 25J^2) = 0.$$

4. *Canonizant C.* The covariant of f,

$$C = Jx^3 + J_1x^2y + J_2xy^2 + J_3y^3,$$

having the leader J, may readily be found by means of the annihilator (*Invariants*, p. 39)

$$O = 5a_1\frac{\partial}{\partial a_0} + 4a_2\frac{\partial}{\partial a_1} + 3a_3\frac{\partial}{\partial a_2} + 2a_4\frac{\partial}{\partial a_3} + a_5\frac{\partial}{\partial a_4}.$$

We get

$$OJ = J_1 = \begin{vmatrix} a_0 & a_1 & a_3 \\ a_1 & a_2 & a_4 \\ a_2 & a_3 & a_5 \end{vmatrix}, \qquad \tfrac{1}{2}OJ_1 = J_2 = \begin{vmatrix} a_0 & a_2 & a_3 \\ a_1 & a_3 & a_4 \\ a_2 & a_4 & a_5 \end{vmatrix};$$

$$\tfrac{1}{3}OJ_2 = J_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \qquad J = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}.$$

Hence

$$C = \begin{vmatrix} a_0x + a_1y & a_1x + a_2y & a_2x + a_3y \\ a_1x + a_2y & a_2x + a_3y & a_3x + a_4y \\ a_2x + a_3y & a_3x + a_4y & a_4x + a_5y \end{vmatrix}.$$

The name canonizant is given to C since its three linear factors u, v, w have the property* that $f = u^5 + v^5 + w^5$. For a direct derivation of C, see § 6. An elementary verification that C is a covariant may be made by manipulating a single determinant. Under the transformation $x = X + \varepsilon Y$, $y = Y$, f becomes $F = A_0 X^5 + 5A_1 X^4 Y + \cdots$, where†

$$\begin{aligned}
A_0 &= a_0,\\
A_1 &= a_1 + \varepsilon a_0,\\
A_2 &= a_2 + 2a_1\varepsilon + a_0\varepsilon^2,\\
A_3 &= a_3 + 3a_2\varepsilon + 3a_1\varepsilon^2 + a_0\varepsilon^3,\\
A_4 &= a_4 + 4a_3\varepsilon + 6a_2\varepsilon^2 + 4a_1\varepsilon^3 + a_0\varepsilon^4,\\
A_5 &= a_5 + 5a_4\varepsilon + 10a_3\varepsilon^2 + 10a_2\varepsilon^3 + 5a_1\varepsilon^4 + a_0\varepsilon^5.
\end{aligned}$$

To the elements of the second row of C add the products of those of the first row by ε. To the elements of the third row add the products of those of the first row by ε^2 and those of the original second row by 2ε. Replace x and y by their values. To the new determinant apply the corresponding operations on columns instead of rows. We get a determinant of type C written in capital letters. Finally, the interchange of x with y and hence of a_0 with a_5, a_1 with a_4, a_2 with a_3, replaces C by a determinant which reduces to C by writing its rows in reverse order and then the columns in reverse order.

5. *Covariant Resolvent.* We employ the linear covariant $L = Px + Qy$, where P was defined in § 3 and Q is derived from P by the substitution $(a_0 a_5)(a_1 a_4)(a_2 a_3)$ induced by the interchange of x and y. The constant term $40000(a_0 P - 25J^2)$ of the resolvent in § 3 is therefore the leader of the covariant $K = 40000(fL - 25C^2)$ of order 6. Equated to zero, it gives the covariant resolvent of Perrin, which was rediscovered by McClintock and called the central resolvent.

---

* Salmon, *Modern Higher Algebra*, 4th ed., p. 153; German translation by Fiedler, p. 199.

† We may write symbolically a^i for a_i, $f = (x + a_1 y)^5$. Then $F = (X + \beta Y)^5$, $\beta = a_1 + \varepsilon$. After expansion, the terms free of a_1 are to be multiplied by a_0. From β^i we get A_i.

For its use as a resolvent of the quintic $f(x, 1) = 0$, it is essential to know the expressions for its roots in terms of $x_1, \cdots, x_5$. To find these expressions, we shall give a process indicated without proof by McClintock, but requiring correction by inserting factors a_0. Write

$$\Phi_1 = a_0 z_1 = a_0 \sum x_1 (x_2 - x_3),$$

where (and below) each of the five terms of $\sum$ are derived from the preceding term by the substitution (12345). Replacing each x_i by its reciprocal, we get

$$a_0 \sum \frac{(x_3 - x_2)\, x_4 x_5}{x_1 x_2 x_3 x_4 x_5} = -\frac{a_0}{a_5} \Psi_1, \quad \Psi_1 = a_0 \sum (x_3 - x_2)\, x_4 x_5.$$

Similarly, from Ψ_1 we get $-\Phi_1 a_0 / a_5$. We shall prove that

$$G = \prod_{i=1}^{6} (\Phi_i x - \Psi_i y)$$

is a covariant of f, where $\Phi_i = a_0 z_i$, and Ψ_i is derived from Φ_i by replacing each root x by its reciprocal. Since the leader of G is $\Phi_1 \cdots \Phi_6$, which is equal to the constant term of the resolvent in § 3, and hence to the leader of K, it will follow that $G = K$.

Apply transformation $x = Y$, $y = X$ to $f = a_0 \prod (x - x_1 y)$. We get $F = a_5 \prod (X - x_1^{-1} Y)$. The function Φ_1 for F is

$$a_5 \sum x_1^{-1} (x_2^{-1} - x_3^{-1}) = \frac{a_5}{a_0} \left(-\frac{a_0}{a_5} \Psi_1 \right) = -\Psi_1.$$

The function Ψ_1 for F is the product of a_5 / a_0 by the function $-\Phi_1 a_0 / a_5$ obtained above from Ψ_1 by replacing each x_i by its reciprocal. Hence G for F is

$$\prod (-\Psi_1 X + \Phi_1 Y) = \prod (\Phi_1 x - \Psi_1 y) = G.$$

Next, apply transformation $x = X + tY$, $y = Y$ to f. We get $F = a_0 \prod (X - X_1 Y)$, where $X_1 = x_1 - t$. The function Φ_1 for F is the seminvariant Φ_1 itself. The function Ψ_1 for F is

$$a_0 \sum (x_3 - x_2)(x_4 - t)(x_5 - t) = \Psi_1 - t\Phi_1 \equiv M_1.$$

Hence G for F is

$$\prod(\Phi_1 X - M_1 Y) = \prod[\Phi_1 x - (t\Phi_1 + M_1)y] = G_i.$$

Thus *the covariant resolvent* $K(x, 1) = 0$ *has the roots* Ψ_i/Φ_i.

A like process enables us to write down at once the linear factors of a covariant of order n whose leader is a seminvariant which is the product of n rational functions of the x's.

6. *Another Derivation of C and L.* If in a covariant φ of f we replace $x^r y^s$ by $(-1)^s \partial^{r+s}/(\partial y^r\, \partial x^s)$, i. e. replace the products of powers of x and y by symbolic products of powers of $\partial/\partial y$ and $-\partial/\partial x$, and apply the resulting operator to another covariant ψ of f, we obtain a covariant $[\varphi, \psi]$ of f (*Invariants*, top p. 61).

The quintic f has the covariant*

$$i = Ix^2 + I_1 xy + I_2 y^2; \quad I_1 \doteq OI = a_0 a_5 - 3a_1 a_4 + 2a_2 a_3,$$
$$I_2 = \tfrac{1}{2} O I_1 = a_1 a_5 - 4a_2 a_4 + 3a_3^2.$$

Then

$$-\tfrac{1}{30}[i, f] = C, \quad -\tfrac{1}{2}[i, C] = L = Px + Qy.$$

THE UNIVERSITY OF CHICAGO

---

[*] It is the invariant I of the fourth polar of f.

AN ACKNOWLEDGEMENT

BY J. H. M. WEDDERBURN

Professor G. Scorza has kindly called my attention to the fact that the result of my note entitled *A theorem on simple algebras* (this BULLETIN, vol. 31, pp. 11–13) was given by him in his book, *Corpi Numerici e Algebre* (1921), pp. 346–352. I regret that I was unaware of this at the time the paper was published, and I take this means of acknowledging Professor Scorza's priority.

IMPROPER DOUBLE INTEGRALS

BY C. A. SHOOK

1. *Introduction.* It can be shown by examples that there exist integrals of the type

$$\int_a^c dy \int_b^d f(x, y)\, dx$$

where a, b, c and d are constants, in which an alteration of value occurs when the order of integration is reversed. Jordan[*] has given the following example. Let

$$f(x, y) = \frac{\partial^2 V}{\partial x\, \partial y},$$

where $V = \arctan(y/x)$; and let $a = b = 0$, and $c = d = 1$. I adduce one more example in which

$$f(x, y) = 2\frac{x - y}{(x + y)^3},$$

and in which the limits are the same as in the previous example. In this example, the value obtained is -1 or $+1$ according as we integrate first with respect to x or with respect to y.

Conditions sufficient to change the order of integration have been investigated by de la Vallée Poussin,[†] who showed that the whole question rests on the *uniform convergence* of the improper integrals involved. The question has also been studied by Brunel,[‡] Jordan,[*] Stolz,[§] and Osgood.[||]

---

[*] *Cours d'Analyse*, vol. 2, pp. 66 et seq.

[†] *Sur la convergence des intégrales définies*, JOURNAL DE MATHÉMATIQUES, (4), vol. 8 (1892), p. 421.

[‡] ENCYKLOPADIE DER MATHEMATISCHEN WISSENSCHAFTEN, II, 1, A. 1, p. 135.

[§] *Differential- und Integralrechnung*, vol. 3, Chapter 16.

[||] ANNALS OF MATHEMATICS, (2), vol. 3, p. 129.

The following theorem is due to Osgood:

Let $f(x, y)$ *be continuous in the region* $a \leqq x$, $b \leqq y$, *and let*

(1) $\int_b^\infty f(x, y)\, dy$ *converge uniformly in an arbitrary finite interval,* $a \leqq x \leqq G$;

(2) $\int_a^\infty f(x, y)\, dx$ *converge uniformly in an arbitrary finite interval,* $b \leqq y \leqq H$,

(3) $\int_a^\infty dx \int_b^y f(x, y)\, dy$ *converge uniformly in the infinite interval,* $b \leqq y \leqq \infty$.

Then $\int_a^\infty dx \int_b^\infty f(x, y)\, dy$ *and* $\int_b^\infty dy \int_a^\infty f(x, y)\, dx$ *both converge and are equal.*

The purpose of this paper is to show that if we introduce the hypothesis that $f(x, y)$ does not change sign in the region, then (3) above can be replaced by the condition that

$$(3') \qquad \int_a^\infty dx \int_b^\infty f(x, y)\, dy \text{ converge.}$$

Next, the theorem as thus modified will be extended to include the case in which the integrand does not remain finite at either or both lower limits. Finally the extended theorem will be applied to an important iterated integral arising in the theory of the gamma function.

2. *Definitions and Lemma.* Let $f(x, y)$ be continuous, $c \leqq x \leqq d$, $a < y$. Suppose that $f(x, y)$ does not remain finite as y tends to a. Finally, let $b > a$. Then the integral, $\int_b^\infty f(x, y)\, dy$, (or the integral $\int_a^b f(x, y)\, dx$), is said to converge uniformly in the interval $c \leqq x \leqq d$, if, corresponding to an arbitrarily small positive quantity ϵ, there exists a positive quantity Υ, (or a positive quantity δ) *independent of* x, such that for every value of x in its interval,

$$\left| \int_{y_1}^{y_2} f(x, y)\, dy \right| < \epsilon, \text{ provided } y_1, y_2 \geqq \Upsilon, \text{ (or } a \leqq y_1 . y_2 \leqq a + \delta).$$

To establish uniform convergence we have recourse to the Vallée-Poussin* μ-test. This may be stated as follows.

---

* Loc cit

If there exists a function $\mu(y)$ *such that*

(A) $|f(x, y)| \leqq \mu(y)$ *for all values of* x *and* y *in the region considered;*

(B) $\int_b^\infty \mu(y)\,dy \left(or \int_a^b \mu(y)\,dy\right)$ *converges;*

then $\int_b^\infty f(x, y)\,dy \left(or \int_a^b f(x, y)\,dy\right)$ *converges uniformly,* $c \leqq x \leqq d$.

We can now state and prove the following lemma.

LEMMA. *If*

(a) $f(x, y)$ *is continuous,* $a \leqq x \leqq b, \quad c \leqq y$;

(b) $\int_c^\infty f(x, y)\,dy$ *converges uniformly,* $a \leqq x \leqq b$;

then $\int_a^b dx \int_c^\infty f(x, y)\,dy = \int_c^\infty dy \int_a^b f(x,y)\,dx$.

Since the integral $\int_c^\infty f(x, y)\,dy$ converges uniformly, $a \leqq x \leqq b$, it represents a continuous function of x in that interval, so that the integral

$$\int_a^b dx \int_c^\infty f(x,y)\,dy$$

has a definite meaning. Let n be any number greater than c and write

$$\int_a^b dx \int_c^\infty f(x,y)dy - \int_a^b dx \int_c^n f(x,y)\,dy = \int_a^b dx \int_n^\infty f(x,y)\,dy.$$

Also, the absolute value of the left hand member is equal to the absolute value of the right hand member. But, by (*b*), we have

$$\left|\int_n^\infty f(x,y)\,dy\right| < \epsilon,\ n > m, \qquad m \text{ independent of } x,$$

where ϵ is an arbitrarily small positive number and m is a positive number corresponding to ϵ. Also

$$\int_a^b dx \int_c^n f(x,y)\,dy = \int_c^n dy \int_a^b f(x,y)\,dx,$$

since both are equal to a certain definite volume. Hence

$$\left|\int_a^b dx \int_c^\infty f(x,y)\,dy - \int_c^n dy \int_a^b f(x,y)\,dx\right| < \varepsilon(b-a), \quad n > m,$$

and this proves the lemma.

3. *The Modified Theorem.* In proving the modified theorem, use is made of an important theorem of Osgood* which may be stated as follows:

Let $F(x,y)$ be a function of the two independent variables x and y, satisfying the following conditions:

(1) $\lim\limits_{x\to a} F(x,y)$ *exists, say equals* $f(y)$;

(2) $\lim\limits_{y\to b} F(x,y)$ *exists, say equals* $g(x)$;

(3) $F(x,y)$ *converges uniformly when x tends to a, that is,* $|F(x',y)-F(x'',y)|<\varepsilon$, $a \leqq x', x'' \leqq a+\delta$, δ *independent of y.*

Then we conclude

(4) $\lim\limits_{y\to b} f(y)$ *exists, say equals* A;

(5) $\lim\limits_{x\to a} g(x)$ *exists, say equals* B;

(6) $A = B$.

The theorem, with the proper modification of the third condition, holds when a and b are infinite.

Suppose now that $f(x,y)$ remains always positive, and let

$$F(x,y) = \int_a^x dx \int_b^y f(x,y)\,dy = \int_b^y dy \int_a^x f(x,y)\,dx.$$

We will show that $F(x,y)$ satisfies the conditions of the theorem just stated, taking the a and b of the theorem infinite. We have

$$\lim_{x\to\infty} \int_a^x dx \int_b^y f(x,y)\,dy = \int_a^\infty dx \int_b^y f(x,y)\,dy$$

$$= \int_b^y dy \int_a^\infty f(x,y)\,dx, \text{ by the Lemma;}$$

$$\lim_{y\to\infty} \int_b^y dy \int_a^x f(x,y)\,dx = \int_b^\infty dy \int_a^x f(x,y)\,dx$$

$$= \int_a^x dx \int_b^\infty f(x,y)\,dy, \text{ by the Lemma.}$$

---

* *Funktionentheorie*, vol. 1, p. 519. This theorem is fundamental in the consideration of double limits, on which see Osgood, this BULLETIN, vol. 3, p. 59; also *Funktionentheorie*, vol. 1, p. 66.

We now have to show that $F(x,y)$ converges uniformly when x increases without limit, that is, that

$$\left|\int_a^{x'} dx \int_b^y f(x,y)\,dy - \int_a^{x''} dx \int_b^y f(x,y)\,dy\right| < \epsilon$$

provided x' and x'' are both greater than a certain suitably chosen number. Combining integrals, we have

$$\left|\int_{x'}^{x''} dx \int_b^y f(x,y)\,dy\right| < \epsilon.$$

This is precisely the condition that the integral

$$\int_a^\infty dx \int_b^y f(x,y)\,dy$$

should converge uniformly. To apply the μ-test, take

$$\mu(x) = \int_b^\infty f(x,y)\,dy,$$

and notice that the conditions of the test are satisfied. Hence

$$\lim_{y\to\infty} \int_b^y dy \int_a^\infty f(x,y)\,dx = \lim_{x\to\infty} \int_a^x dx \int_b^\infty f(x,y)\,dy$$

or

$$\int_b^\infty dy \int_a^\infty f(x,y)\,dx = \int_a^\infty dx \int_b^\infty f(x,y)\,dy$$

which proves the modified theorem.

4. *The Extended Theorem.* We may now state a more extended theorem.

THEOREM A. *Let $f(x, y)$ be a function of x and y satisfying the conditions*

(1) *$f(x, y)$ is continuous, $a < x$, $b < y$, and may increase without limit as x tends to a or as y tends to b;*

(2) *$f(x, y) \geqq 0$ in the region considered;*

(3) *$\int_a^c f(x, y)\,dx$ converges uniformly, $b < t \leqq y$, $c > a$;*

(4) *$\int_c^\infty f(x, y)\,dx$ converges uniformly, $b < t \leqq y$, $c > a$;*

(5) *$\int_b^d f(x, y)\,dy$ converges uniformly, $a < s \leqq x$, $d > b$;*

(6) *$\int_d^\infty f(x, y)\,dy$ converges uniformly, $a < s \leqq x$, $d > b$;*

(7) $\int_a^\infty dx \int_b^\infty f(x, y)\, dy$ *converges.*

Then the order of integration can be reversed, that is,

$$\int_a^\infty dx \int_b^\infty f(x, y)\, dy = \int_b^\infty dy \int_a^\infty f(x, y)\, dx.$$

To prove this theorem, we write, omitting the integrand for brevity,

$$\int_a^\infty \int_b^\infty = \int_a^c \int_b^d + \int_a^c \int_d^\infty + \int_c^\infty \int_b^d + \int_c^\infty \int_d^\infty .$$

The last of the four integrals of the right hand member falls directly under the modified theorem, stated in § 1 and proved in § 3. The other three integrals can be dealt with in a precisely similar way by making obvious changes. The hypotheses under Theorem A suffice for these proofs. But when the order of integration is changed in each of the four integrals, their sum becomes

$$\int_b^\infty \int_a^\infty$$

and this proves the theorem.

5. *Application of Theorem* A *to a Particular Integral.* In obtaining the relation between beta and gamma functions,*

$$B(m, n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$$

it is necessary to change the order of integration, of the integral

$$\int_0^\infty dy \int_0^\infty x^{m-1} y^{m+n-1} e^{-y(1+x)}\, dx$$

where m and n are any positive numbers. This change of order can be justified by Theorem A. To apply the theorem take $c = d = 1$. Conditions 1, 2 and 7 are obviously satisfied. To show that conditions 3 to 6 inclusive are satisfied we have to exhibit a set of μ-functions covering the various cases. Such a set follows:

---

* Byerly, *Integral Calculus*, p. 113.

	$0<t\leqq y\leqq 1$	$1\leqq y$
$\int_0^1 f(x.y)\,dx$	$\mu(x) = x^{m-1}e^{-tx}G$, $1<m+n$; $= x^{m-1}e^{-tx}H$, $0<m+n<1$	$\mu(x) = x^{m-1}e^{-x}G$
$\int_1^\infty f(x,y)\,dx$	$\mu(x) = x^{m-1}e^{-tx}G$, $1<m+n$; $= x^{m-1}e^{-tx}H$, $0<m+n<1$	$\mu(x) = x^{m-1}e^{-x}G$
	$0<s\leqq x\leqq 1$	$1\leqq x$
$\int_0^1 f(x,y)\,dy$	$\mu(x) = y^{n}e^{-y}M$, $1<m$; $= y^{m+n-1}e^{-y}N$, $0<m\leqq 1$	$\mu(y) = y^{n}e^{-y}M$, $1<m$ $= y^{m+n-1}e^{-y}$, $0<m\leqq 1$
$\int_1^\infty f(x,y)\,dy$	$\mu(x) = y^{m+n-1}e^{-y}M$, $1<m$: $= y^{m+n-1}e^{-y}N$, $0<m\leqq 1$	$\mu(y) = y^{m+n-1}e^{-y}M$

$$G = \left(\frac{|m+n-1|}{e}\right)^{|n-1|m+}, \qquad H = e^{-t}t^{m+n-1};$$

$$M = \left(\frac{|m-1|}{e}\right)^{|m-1|}, \qquad N = s^{m-1};$$

$f(x, y)$ denotes the integrand of the integral under consideration.

Yale University

THE FUNDAMENTAL REGION FOR A FUCHSIAN GROUP*

BY L. R. FORD

1. *Introduction.* The present paper is an attempt to lay the groundwork of the theory of Fuchsian groups by basing the treatment on concepts of a very simple sort. The fundamental region to which we are led is not new. It is given by the Fricke-Klein method† under certain circumstances and is identical with that given by Hutchinson‡ in an important paper. However, we make use neither of non-euclidean geometry nor of quadratic forms, and we are able to derive the major results of the theory of Fuchsian groups in an unexpectedly simple manner.

2. *The Group.* Given a group of linear transformations with an invariant circle or straight line K', the interior of K' (or the half-plane on one side of K') being transformed into itself by each transformation of the group. We shall assume that there exists a point A, not on K', such that there are no points congruent to A in a sufficiently small neighborhood of A.

Let G be a linear transformation carrying K' into the unit circle K with center at the origin and carrying A to the origin. Let S be any transformation of the group; then the set of transformations

$$T = GSG^{-1}$$

is a group with K as principal circle.§ Configurations which are congruent by transformations of the new group are carried by G^{-1} into configurations which are congruent by transformations of the original group. It will suffice, then, to find a fundamental region for the new group.

---

* Presented to the Society, September 11, 1925.

† Fricke-Klein, *Vorlesungen über die Theorie der automorphen Funktionen*, vol. I, Chap. II.

‡ J. I. Hutchinson, *A method for constructing the fundamental region of a discontinuous group of linear transformations*, TRANSACTIONS OF THIS SOCIETY, vol. 8 (1907), pp. 261–269.

§ We use this order to mean the transformation G^{-1}, followed by S, followed by G. That is, writing $z' = S(z)$, $z' = G(z)$, etc., as the combining transformations, the new transformation is $z' = T(z) = G\{S[G^{-1}(z)]\}$.

The condition that

$$z' = T(z) = \frac{az+b}{cz+d}$$

leave the unit circle unchanged is that $b = \bar{c}$, $d = \bar{a}$, where bars indicate conjugate imaginaries. Then

$$T = \frac{az+\bar{c}}{cz+\bar{a}}.$$

Since the origin, 0, must transform into an interior point, and $T(0) = \bar{c}/\bar{a}$, we must have $|c| < |a|$. Hence the determinant $a\bar{a} - c\bar{c}$, which is real, must be positive. We shall insert such a positive factor in numerator and denominator that

$$a\bar{a} - c\bar{c} = 1.$$

There is no point congruent to 0 in a suitably small neighborhood of 0. In particular, 0 is not a fixed point for any transformation; hence $c \neq 0$, unless T be the identical transformation.

3. *Two Locus Problems.* Excluding the identical transformation, we shall solve the following two locus problems for the transformation T.

I. *Find the locus of a point in the neighborhood of which lengths and areas are unchanged in magnitude.*

Infinitesimal lengths are multiplied by $|T'(z)|$ and infinitesimal areas are multiplied by $|T'(z)|^2$. Since

$$T'(z) = \frac{1}{(cz+\bar{a})^2},$$

we have as the required locus the circle

$$C: \qquad |cz+\bar{a}| = 1,$$

or

$$|z+\bar{a}/c| = 1/|c|.$$

This circle, which will be called the C-circle of the transformation T, has its center at the point $-\bar{a}/c$; its radius is $1/|c|$.

Writing the relation $a\bar{a}-c\bar{c}=1$ in the form

$$1+\frac{1}{|c|^2}=\left|\frac{\bar{a}}{c}\right|^2,$$

we see that the sum of the squares of the radii of K and C is equal to the square of the distance between their centers. Hence C is orthogonal to K. It follows that 0 is outside C.

If z is within C then

$$|z+\bar{a}/c|<1/|c|, \qquad |cz+\bar{a}|<1, \qquad |T'(z)|>1,$$

and lengths and areas in the neighborhood of z are increased in magnitude when transformed by T. Similarly, if z is outside C lengths and areas are decreased in magnitude when transformed by T.

II. *Find the locus of a point whose distance from 0 is unchanged.*

The required locus is

$$|z|=\left|\frac{az+\bar{c}}{cz+\bar{a}}\right|$$

or

$$|cz^2+\bar{a}z|=|az+\bar{c}|,$$

or

$$(cz^2+\bar{a}z)(\bar{c}\bar{z}^2+a\bar{z})=(az+\bar{c})(\bar{a}\bar{z}+c).$$

On expanding and making use of the relation $c\bar{c}=a\bar{a}-1$ this factors into

(a) $$[z\bar{z}-1]\,[(cz+\bar{a})(\bar{c}\bar{z}+a)-1]=0,$$

whence

$$|z|=1, \quad \text{or} \quad |cz+\bar{a}|=1.$$

The complete locus, then, consists of the two circles C and K.

If z is inside both C and K or outside both, the first member of (a) is greater than 0, and we find, on retracing our steps, that

$$|z|>\left|\frac{az+\bar{c}}{cz+\bar{a}}\right|.$$

The transform of z is nearer 0 than z is. Similarly, if z is inside one circle and outside the other the transform is farther from 0 than z is.

4. *Geometric Interpretation of T.* The inverse of T,

$$T^{-1} = \frac{-\bar{a}z + \bar{c}}{cz - a},$$

has the C-circle

$$C': \quad |z - a/c| = 1/|c|.$$

It is clear that T carries C into C'. For, T carries C into a circle C_0 without alteration of lengths; then T^{-1} transforms C_0 without alteration of lengths; whence C_0 coincides with C'.

The region exterior to both C and C' is transformed by T into the interior of C' and by T^{-1} into the interior of C. Let z be a point of the region, and let z be carried by T and T^{-1} into z' and z'' respectively. In both cases there is diminution of lengths and areas near z since z is exterior to both C-circles. Then the inverses, T^{-1} and T, carry z' and z'' respectively back to z with increase of lengths and areas. Hence z' is in C' and z'' is in C.

Let A, B and A', B' be the intersections of C and C' with K, the points being so designated that motion around C from A to B inside K is counter-clockwise about C, and motion from B' to A' around C' in K is counter-clockwise about C'. Now, on applying T, the interior arc AB of C is carried without alteration of length into either $A'B'$ or $B'A'$. The latter is impossible, for it is equivalent to a suitable rotation with 0 as fixed point, which is contrary to hypothesis. Hence A is transformed into A' and B into B'.

We can now give simple geometric interpretations of T. Let L be the perpendicular bisector of the line joining the centers of C and C'. L passes through 0. (In the special case that C and C' coincide let L be the line joining 0 to the center of C.) Either of the following pairs of inversions transforms the points of C exactly as T does and hence is identical with it:

(1) A reflection in L followed by an inversion in C';
(2) An inversion in C followed by a reflection in L.

We can show from these inversions that T is hyperbolic, elliptic, or parabolic according as C and C' are exterior to one another, intersect, or are tangent. The fixed points of the transformation are easily found geometrically.

5. *The Arrangement of the C-Circles.* We shall show first that there is an upper bound of the radii of the C-circles defined by the transformations of the group. There exists, by hypothesis, a circle of radius $\varepsilon < 1$ with 0 as center having no point congruent to 0 in its interior. We have then for any transformation of the group

$$|T(0)| = \left|\frac{\bar c}{\bar a}\right| \geqq \varepsilon, \text{ whence } \left|-\frac{\bar a}{c}\right| \leqq \frac{1}{\varepsilon};$$

that is, the distance from 0 to the center of C is not greater than $1/\varepsilon$. Since 0 is outside C it follows that the radius of C is less than $1/\varepsilon$.

Let

$$T_1 = \frac{\alpha z + \bar\gamma}{\gamma z + \bar\alpha}$$

be a second transformation of the group. Designate by C_1, C_1' the C-circles of T_1 and T_1^{-1}. Thus C_1 is $|z + \bar\alpha/\gamma| = 1/|\gamma|$. Let us now make the transformation

$$TT_1^{-1} = \frac{(-a\bar\alpha + \bar c\gamma)z + a\bar\gamma - \bar c\alpha}{(-c\bar\alpha + \bar a\gamma)z + c\bar\gamma - \bar a\alpha},$$

which, since $T_1 \neq T$, is not the identical transformation. Designating the radii of C, C_1, and the C-circle of TT_1^{-1} by r, r_1, r_2, respectively, we have

$$r_2 = \frac{1}{|-c\bar\alpha + \bar a\gamma|} = \frac{1}{|c\gamma| \cdot \left|-\frac{\bar\alpha}{\gamma} - \left(-\frac{\bar a}{c}\right)\right|} = \frac{rr_1}{d},$$

where d is the distance between the centers of C and C_1. Since $r_2 < 1/\varepsilon$ we have

$$d = \frac{rr_1}{r_2} > \varepsilon rr_1.$$

Consider now all C-circles whose radii equal or exceed some positive number k. The distance between the centers of any two of the circles satisfies the inequality

$$d > \epsilon k^2.$$

Since the centers lie in the finite region bounded by K and by the circle of radius $1/\epsilon$ with center at 0 it follows that the number of such circles is finite.

It follows from the result just found that any closed region lying entirely within K, for example a circle with 0 as center and radius less than 1, is exterior to all but a finite number of C-circles.

Another consequence is that the transformations of the group are denumerable, since the radii of the corresponding C-circles are denumerable.

6. *The Fundamental region.* Let R be the region within K which lies outside all C-circles formed for the transformations of the group. The region is connected, since any of its points can be joined to 0 by a straight line segment which does not cross the boundary. We shall show that R is a fundamental region for the group; that is, (1) that no two interior points of R are congruent, and (2) that no region adjacent to R and lying in K can be added to R without the inclusion of points congruent to points of R.

The proof of the first property is immediate. The transform of any interior point of R by any transformation of the group, the identical transformation excepted, lies within some C-circle and hence is exterior to R.

To establish the second property we shall show first that if P, a point of C, the C-circle of some transformation T of the group, lies on the boundary of R, then P', the transform of P by T, also lies on the boundary of R. P' lies on C', the C-circle of T^{-1}.

Suppose P' does not lie on the boundary of R. Then P' lies within the C-circle of some transformation T_1 of the group. Consider the transformation T_1T. By the transformation T lengths in the neighborhood of P are carried without alteration of magnitude into the neighborhood of P'. By T_1 lengths in the neighborhood of P' are magnified. Hence T_1T magnifies lengths in the neighborhood of P; consequently P is within the C-circle of T_1T. This is contrary to hypothesis; hence P' is on the boundary of R.

It follows from the preceding that if an arc ab of C forms part of the boundary of R the congruent arc $a'b'$ of C' is a part of the boundary. The transformation T^{-1} carries R into a region abutting R along ab. Any region adjacent to R abuts along some C-circle and contains points congruent by a suitable transformation to points of R. The second property is thus established.

The following properties of the region R are consequences of the preceding analysis:

1. R is bounded by arcs of circles orthogonal to K. The number of bounding arcs in a circle $|z| = r < 1$ is finite.

2. The bounding arcs are arranged in congruent pairs. Two congruent arcs of the boundary are equal in length, and congruent points thereof are equidistant from the center of K.

3. The vertices of a cycle (congruent vertices) lie on a circle concentric with K, since all are equidistant from the center of K. If the vertices of a cycle lie within K their number is finite.

4. R is the fundamental region of maximum area. For, a different fundamental region must contain points congruent to all points of R, and a shift of any part of R to a congruent position effects a diminution of area.

We shall now prove that R *and the regions congruent to it fill up without overlapping the whole interior of* K.

Suppose R_i and R_j, the transforms of R by T_i and T_j, overlap. Let z_1, z_2, z_3 be three points common to R_i and R_j. These are the transforms by T_i of three points z_1', z_2', z_3' of R, and the transforms by T_j of three points z_1'', z_2'', z_3'' of R. If $z_1' = z_1''$, $z_2' = z_2''$, $z_3' = z_3''$ then T_i and T_j are the same transformation, since they transform three points in the same way, and R_i and R_j coincide. Otherwise the points of one pair, say z_1' and z_1'', are unequal. Being both congruent to z_1 they are congruent, which is impossible.

Let K_r be the circle $|z|=r<1$. Let $a_1b_1, a_1'b_1', \cdots, a_nb_n, a_n'b_n'$ be the sides of R lying wholly or in part within K_r; and let T_i, $i=1,2,\cdots,n$, be the transformation carrying a_ib_i into $a_i'b_i'$. The transforms of all remaining sides of R are exterior to K_r, since the distance from 0 of any point on such a side is not decreased by any transformation of the group.

By applying $T_1, \cdots, T_n$ and their inverses we get regions congruent to R abutting on R along the sides $a_1b_1, \cdots, a_n'b_n'$. The sides of the new regions which lie in K_r are congruent to the sides just mentioned. By combinations of $T_1, \cdots, T_n$ and their inverses we can adjoin further regions along sides of these new regions, provided the sides lie in K_r; and the process can be continued as long as there are any free sides in K_r.

This process will end in a finite number of steps; for, each transformation carries R into the interior of a particular C-circle, and there is but a finite number of C-circles intersecting K_r. Hence K_r is covered by a finite number of regions. Since r may be chosen as near 1 as we like, it follows that the whole interior of K is covered.

We note from the preceding that all transformations of the group are formed by combinations of the transformations connecting congruent sides of R. These are therefore called generating transformations of the group. A further interesting fact is that a transformation which carries R into a region lying wholly or in part in a circle K_r concentric with K is a combination of those generating transformations only whose C-circles intersect K_r.

7. *An Important Special Case.* The sides of R may be finite or infinite in number. There are certain groups in which a fundamental region not extending to the principal circle is known to exist; for example, some of the groups

arising in connection with the uniformization problem.[*] For this case we have the following proposition.

If there exists a fundamental region F lying within K_r $(|z| = r < 1)$, *then R lies within* K_r, *and the number of sides and of generating transformations is finite.*

R and a finite number of its transforms, $R_1, \cdots, R_m$, will cover F completely. Carry the portion of F lying in each R_i into R by means of the transformation which carries R_i into R. The totality of these transforms of parts of F, together with the portion of F originally in R, fill up R completely. If this were not so we could construct a region D adjacent to one of these transformed regions and containing no points congruent to points of F. On carrying D back to the boundary of F we should have a region abutting on F and containing no points congruent to points of F, which is contrary to hypothesis.

Finally R is in K_r, for on transforming the parts of F into R the distance of no point from 0 is increased.

Since only a finite number of C-circles intersect K_r, it follows that R has a finite number of sides and that the number of generating transformations is finite.

THE RICE INSTITUTE

---

[*] See Osgood, *Lehrbuch der Funktionentheorie*, vol. I, 2d ed., pp 721-25.

SOME PHASES OF DESCRIPTIVE GEOMETRY*

BY W. H. ROEVER

The purpose of this paper is to recall those phases of descriptive geometry which are involved in the construction of adequate pictures of space objects. These remarks are particularly addressed those who attempt to use pictures and thus admit a need for them. To quote from Klein, "Is it not as worthy an object of mathematics to be able to draw correctly as to be able correctly to calculate?"

The need for descriptive geometry was probably first felt by the artist and the architect. The former was interested in conveying by means of his drawings a clear notion of the spatial form of the object represented. The latter, on the other hand, made use of his process of drawing, not only to instruct the builder as to the form and size of the objects which he represented by his pictures, but, in addition, to solve by means of plane constructions the problems of space which were encountered by the mason and carpenter. Thus the art of stereotomy was developed in the Middle Ages. In separating the geometric constructions of this art from their application, Frézier, in 1738, laid the cornerstone of modern descriptive geometry. It was Gaspard Monge (1746 – 1818) however, who developed this new constructive geometry of space so as to elevate it to the dignity of a pure science to which he gave the name descriptive geometry.

At this point let us observe that the process of drawing can be performed only upon a surface, and, in particular, upon a plane surface. It thus becomes possible to execute graphically, i. e., with pencil, ruler and compasses (instruments of the geometer), the following fundamental operations of plane geometry (postulates of construction):

---

* An address delivered before the Southwestern Section of this Society at Ames, Iowa, November 29, 1924, by invitation of the program committee.

I
1. To determine the (straight) line connecting two points.
2. To find the point of intersection of two lines.
3. To construct a circle of given center and radius.

The corresponding fundamental operations of space are·

II
1. To find the line connecting two points of space.
2. To find the line of intersection of two planes.
3. To find the plane determined by a point and a line.
4. To find the point in which a line pierces a plane.
5. To construct a sphere of given center and radius.

There is, however, no process in space which is analogous to that of drawing in the plane. One might thus be tempted to say that the problems of space are incapable of graphical solution. In the art of stereotomy, solutions of problems of type II were virtually reduced to those of problems of type I, thus opening the way for the graphical solution of space problems.*

The principal purposes of descriptive geometry thus appear to be the following:

1. Representation of the objects of space by means of figures which lie in a plane (or upon a surface).

2. Solution of the problems of space by means of constructions which can be executed in the plane.

The criterion of what shall be meant by a good plane representative of a space object has, more or less unconsciously, been taken to be *that the plane representative when properly placed, shall produce upon the retinal surface of the eye an image which differs but little from that produced by the object itself.* Since the optical properties of the eye may be compared with those of the camera obscura, it is easy to see† that *the above criterion may be satisfied*, at least approximately, *by a plane picture which*

---

* See Gino Loria, *Vorlesungen über Darstellende Geometrie*, Leipzig. Teubner, 1907.

† See A. Schoenflies, *Einfuhrung in die Hauptgesetze der Zeichnerischen Darstellungsmethoden*, Leipzig, Teubner, 1908.

is a central (or, by accommodation, also a parallel) *projection.* A photograph is such a plane picture.

In order that the solution of a space problem may be made possible by a construction in the plane, it is essential that there exist an *unambiguous correspondence between space and the plane.* To make this clear, we will observe that a single projection does not satisfy this requirement. For, while to each point of space P there corresponds a definite projection P' on the picture plane, it is *not* true, conversely, that to each point of the projection there is a definite point of space. Hence, in general, a single projection does not suffice if no information concerning the object projected is known. The reason that a single projection, like a photograph, may convey an adequate notion of an object (like a building) is because we have some information about the object, such, for instance, as the perpendicularity of some of its edges. If, however, such a relation concerning the object to be represented is not known, or not true, one projection of it is not sufficient to convey to the mind an adequate notion of its form. The projection of a cube might, for instance, also be the projection of any hexahedron of which the vertices are on the lines connecting the vertices of the cube with the center of projection. That two projections do, in general, suffice to set up an unambiguous correspondence between space and the plane, is illustrated by the stereoscope, which enables the user to combine the images of two pictures taken from different points of view and thus obtain the impression of solidity or relief.

Let us now illustrate several of the principal methods of descriptive geometry, showing how, in each of these methods, use is made either of *two projections* or of *one projection and information about the object projected.* For this purpose let us use, as an object to be represented, a solid of revolution obtained by revolving around one of its sides $O''\ Z''$ the part $O''\ D''\ E''\ F''\ Z''$ of a rectangle which remains after a corner has been removed by a circular cut $E''\ F''$ (see Fig. II). The solid of revolution thus obtained is bounded by two planes (generated by $O''D''$ and $Z''F''$), by a circular cylinder (generated by $D''E''$) and by a torus (generated by the circle $E''F''$). Let us further suppose that points have been taken on the circle $E''F''$ in such a way that their (orthographic) projections on the line $O''Z''$ are equally spaced. In the revolution, these points generate circles of latitude of which the planes are perpendicular to the axis $O''Z''$ and are equally spaced.

Let us now choose several planes on which to project our solid. As a first plane of projection we will take the plane base π_1 (generated by $O''D''$) and as a second plane of projection, a plane π_2 parallel to the axis of revolution $O''Z''$. We will think of π_2 as coincident with the plane of our drawing, and of π_1 as intersecting π_2 in the ground line g_{12}. Then we will think of π_1 as rotated around g_{12} until it becomes coincident with π_2. The plane of the paper (π_2) is then a Mongean drawing plane. In it Fig. I is the orthographic projection of our solid on π_1 and Fig. II is that on π_2. Together these figures constitute a *Mongean representation* of our solid. It is easily seen that one of these projections alone is not sufficient, but that the two are sufficient. Thus we see that associated with a point of space P there are two points P', P''. in the drawing plane which lie on the same perpendicular to the ground line, and, conversely, to such a pair of points in the drawing plane there corresponds a point in space (namely, the point with which was associated this pair). From this it follows that associated with a line of space p there is a pair of lines p', p'' of the drawing plane; and conversely, in general, to a pair of lines in the drawing plane, there corresponds a line in space (namely, the line with which was associated this pair). A line of space p instead of being represented by its projections (p', p'') may also be represented by the points (P_1, P_2) (traces) in which it pierces the planes of projection π_1, and π_2 (see drawing). It is the extension of the latter method which is used to represent a plane. Thus a plane μ is represented by the lines (m_1, m_2), (traces)

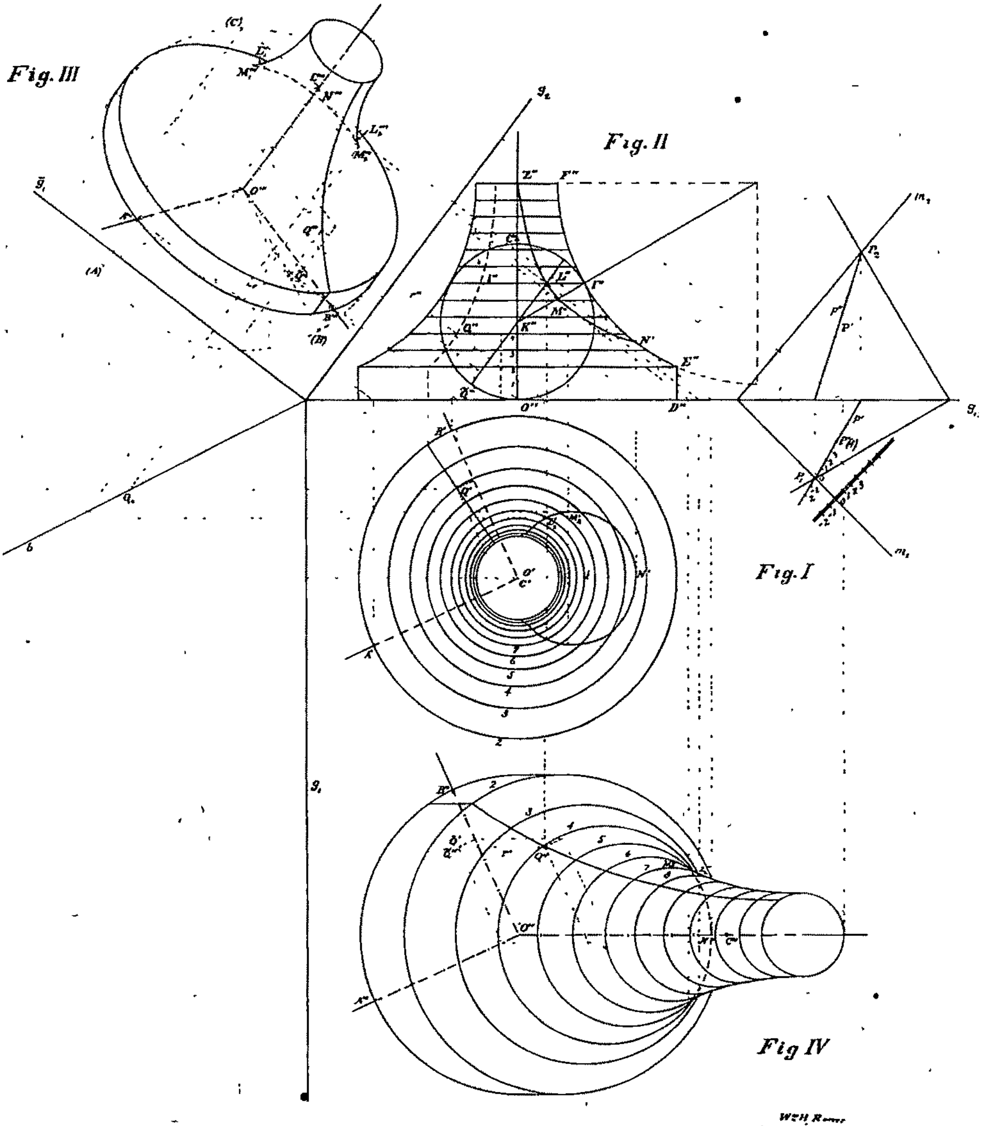


Fig. III
Fig. II
Fig. I
Fig IV
g_1
g_2
m_1
m_2
Z''
F'''
C''
L''
I''
M''
K''
N'
E''
D''
O''
Q''
P''
P'
P_1
P_2
O'
C'
N'
M_1'
A'
B'
A''
B''
O'''
N'''
Q'''
F'''
(A)
(B)
(C)
1
2
3
4
5
6
7
8
Wm H. Roever

in which it cuts the planes π_1 and π_2 (see drawing). Since these traces necessarily intersect on the ground line g_{12}, it follows that a general plane is adequately represented by two lines of the drawing plane which intersect on the ground line. It is thus evident that the Mongean method of representing space (by means of its elements, points, lines, planes) in the (drawing) plane, possesses the property of unambiguous correspondence.

A circle of latitude, such as that generated by the point I'', is represented by the circle i' in Fig. I and the straight line i'' in Fig. II. The corresponding representatives of other circles of latitude are also shown in these figures. These circles might be regarded as contour (level) lines if we now think of our solid as a mountain peak. If we attach to the circles in Fig. I numbers $(\cdots, 2, 3, 4, \cdots)$ representing the distances of these contour lines above the (datum) plane π_1, Fig. I becomes a *topographic representation* of our solid. In this method, a point is adequately represented by its (orthographic) projection and a number indicating its altitude. A (straight) line is represented by its projection and a scale, the intervals of which are the projections of the intervals into which the line in space is divided by equally spaced level planes; and finally, a plane is represented by its line of greatest slope. Thus, in the drawing, the point P' with the accompanying number (4) represents the point P (represented in the Mongean method by the pair of points P', P''), the line p' with the accompanying scale $(\cdots, -2, -1, 0, 1, 2, \cdots)$ represents the line p (represented in the Mongean method by the pair of lines p', p''), and the double line m' with the accompanying scale $(\cdots, -2, -1, 0, 1, 2, \cdots)$ represents the plane μ (represented in the Mongean method by the traces m_1, m_2). This method of representation also possesses the property of unambiguous correspondence.

Let us now find the (orthographic) projection (Fig. III of our solid on a plane $\pi \equiv [g_1, g_2]$ which is inclined to π_1, but perpendicular to π_2. The horizontal trace g_1 of

π will be perpendicular to g_{12}. We will then think of the plane π as rotated around its vertical trace g_2 (just as π_1 was rotated around g_{12}) until it comes into coincidence with the plane of the drawing π_2. By this rotation the trace g_1 assumes the new position $\bar{g}_1$, perpendicular to g_2 at the intersection of g_2 with g_{12}. The two figures, Fig. II and Fig. III, together constitute another Mongean representation of our solid, with g_2 as the corresponding ground line.

The points of Fig. III may be obtained from those of Figs. I and II by a knowledge of the following facts:

If Q denote a general point of space and Q', Q'', Q''' its projections on π_1, π_2, π, respectively, then just as Q' and Q'' lie on the same perpendicular to the ground line g_{12} so Q'' and Q''' lie on the same perpendicular to the ground line g_2. Furthermore, since the points Q' and Q''' are at equal distances from g_{12} and g_2 respectively, a line through Q' parallel to g_{12} meets a line trough Q''' parallel to g_2 on the bisector b of the angle formed by g_1 and $\bar{g}_1$ (see the drawing). Thus it is easy to obtain point by point the projection on π of the circular edges of our solid (generated by D'', E'' and F'').

In order to obtain the other bounding curve of Fig. III, let us observe that if we regard the rays which project our solid on the plane π as rays of light, this bounding curve is the projection on π of the *line of shade,* i. e., the curve which separates the illuminated portion of our solid from the unilluminated portion. Points of such a line of shade may be determined for a surface of revolution by the following well known construction. A surface of revolution may be regarded as the envelope of a one-parameter family of spheres of which the centers lie on the axis of revolution. Each sphere of this family has for its line of shade a great circle whose plane is perpendicular to the direction of the illuminating rays. On the other hand, each such sphere has contact with the envelope along a common circle of latitude of these two surfaces. The points in which these two circles intersect are points of the line of shade of the envelope. To find the center of the sphere having contact with our surface along the circle of latitude generated by the point I'', we erect at I'' in the drawing plane a normal to the meridian curve $E''\ I''\ F''$. This cuts the axis $O''\ Z''$ in the required center K''. Through K'' we then draw a line parallel to g_2, since the rays of light are perpendicular to the plane π. This is the projection on π_2 of the line of shade of the sphere of center K, and cuts the corresponding projection i'' of the circle of contact in the point L''. This point is the vertical projection of two points of our line of shade. The horizontal projections of these points are found by drawing through L'' a line perpendicular to g_{12} cutting the circle i' in the points L_1' and L_2'. The corresponding projections L_1''', L_2''' on the plane π (Fig. III) are then found by the construction given in the preceding paragraph. Thus the bounding curves of Fig. III corresponding to the toroidal surface of our solid, may be found point by point. Apparently these curves stop abruptly at the points M_1''' and M_2'''. However, these points are not "points d'arrêt", but they are *cusps* of the complete curve of which the remaining portion M_1''' $N'''\ M_2'''$ is the bounding curve of the hidden (back) side of the torus. In fact this curve is a *parallel to the ellipse* into which the axial circle of the torus is projected on the plane π. For, the bounding curves of the (orthographic) projection of any tubular surface (i. e., envelope of a one-parameter family of spheres of constant radius) are parallels to the corresponding projection of the axial curve (i. e., locus of centers of enveloped spheres) of this surface, and the torus is a tubular surface with a circular axial curve.

With the same system of parallel projecting rays which were used to obtain the orthographic projection (Fig. III) of our solid on the plane π, let us also obtain the oblique projection (Fig. IV) of this solid on the plane π_1. In order that Fig. IV may not be superimposed on Fig. I we will drop the former down a little distance on the paper. Hence, now instead of using Q' as the horizontal projection of Q, we will take $\overline{Q}'$ as this projection. Then. to obtain the required oblique projection Q^{IV} of the point Q we merely find the points in which the ray r (r', r'') through the point Q ($\overline{Q}'$, Q'') pierces the plane π.

Since the edges, and other circles of latitude, of our solid are parallel to the plane of (oblique) projection, each of these circles projects into an equal circle and all that is needed is merely to find the (oblique) projections of the centers of such circles. The envelope of the circles thus obtained is the bounding curve of the (oblique) projection (Fig. IV) of our solid. But this may be otherwise obtained by merely finding (as for the point Q) projections of points L of the line of shade.

If in Fig. IV (as in Fig. I) we attach numbers to the circles indicating the altitudes of the planes of the circular sections of which these circles are the (oblique) projection, Fig. IV alone is an adequate plane representation of our solid. This representation has the advantage over that given by Fig. 1 with the corresponding attached numbers, in that it conveys with less effort, on the part of an observer, a notion of the spatial form represented.

In a similar manner, Fig. III with numbered ellipses (not drawn) into which the circles of latitude of our solid project (orthographically), serves as an adequate plane representation of our solid.

However, instead of drawing these ellipses in Fig. III, or the circles in Fig. IV, we might merely give the scale ($O''' Z'''$ in Fig. III, $O^{IV} Z^{IV}$ in Fig. IV) into which the vertical scale OZ (shown in true size in Fig. I) projects. In addition to this scaled axis ($O''' Z'''$ in Fig. III or $O^{IV} Z^{IV}$ in Fig. IV) let us now take two other scaled axes ($O''' X'''$, $O''' Y'''$ in Fig. III or $O^{IV} X^{IV}$, $O^{IV} Y^{IV}$ in Fig. IV), these being the projections of space axes OX and OY, each bearing the scale of OZ and perpendicular to each other and to OZ. In addition to the projection of the general point Q (Q''' in Fig. III or Q^{IV} in Fig. IV), let us find that of the orthographic projection $\tilde{Q}$ of the point Q on the plane π_1 ($\tilde{Q}'''$ in Fig. III or $\tilde{Q}^{IV}$ in Fig. IV). Then this pair of points (Q''', $\tilde{Q}'''$ in Fig. III or Q^{IV}, $\tilde{Q}^{IV}$ in Fig. IV) is sufficient to represent adequately a point of space Q. For we can find from the three axonometric axes ($O''' X'''$, $O''' Y'''$, $O''' Z'''$ in Fig. III or $O^{IV}X^{IV}$, $O^{IV}Y^{IV}$, $O^{IV}Z^{IV}$ in Fig. IV) the coordinates of the point Q in space.

We thus have the *axonometric method* of representation (orthographic in Fig. III, oblique in Fig. IV), in which two points on a line parallel to one of the axonometric axes represent adequately a point of space. Likewise, in general, two lines of the picture plane represent a line of space, and, finally, a plane of space is adequately represented by two lines of the picture plane which intersect on one of the axonometric axes.

While Figs. III and IV were obtained by Mongean processes from the Mongean representation consisting of Figs. I and II, they may be obtained directly by the axonometric method. In the use of this method, one must be able to determine the scales when the directions of the axes are given. The relationship which is involved between these angles and scales is, in the case of orthographic axonometry, expressed by the following theorem.

THEOREM. *Any two conjugate diameters and the minor axis of an ellipse may be regarded, as far as directions are concerned, as the orthographic projection on the plane of the ellipse of three mutually perpendicular axes*; *if these axes be taken as cartesian axes and the plane of the ellipse as the picture plane, the ratios borne by the halves of the chosen conjugate diameters and the focal distance of the ellipse to the half major axis of the ellipse are the foreshortening ratios,* i. e., *the numbers by which the scales on the space axes must be multiplied in order to obtain those on the axonometric axes.* This relationship is indicated in Fig. III, in which the ratios of $O''' A'''$, $O''' B'''$, $O''' C'''$ to $O'''(A)$ are the foreshortening ratios.

In the case of oblique axonometry, however, one may choose at pleasure both the angles and the scales. The truth of this assertion is contained in the following famous theorem, which is due to Pohlke.

POHLKE'S THEOREM. *Three straight-line segments of arbitrary lengths in a plane, drawn from a point and making arbitrary angles with each other, can be regarded as a parallel projection of three equal segments drawn from the origin on three rectangular coordinate axes; however, not more than one of the given segments or one of the given angles can vanish**.

Pictures of the type of Fig. III and Fig. IV make more of an appeal than do those of the type of Figs. I and II. In the case of most of the objects of architecture and technology there are among the bounding surfaces certain mutually perpendicular planes. These planes, or parallel planes, are usually selected as the Mongean planes of representation. Thus certain plane faces appear merely as lines, and hence these projections do not convey as easily a notion of the spatial form of the objects as do projections on planes not parallel to any of the mutually perpendicular planes. Hence, here also, pictures of the type of Fig. III and Fig. IV, which can easily be made directly by the axonometric method, prove very desirable.

All of the methods described in this paper possess the property of unambiguous correspondence. In each is given the plane representative of the space elements: point, line, plane. The first four of the space operations II thus have their counterparts in the picture plane and hence all of the three-dimensional problems of pure geometry of position can be solved in the plane by each of these methods. The fifth of the operations II can, in like manner, be replaced by the third of operations I, and thus also is made possible a graphical solution of all perpendicularity and metrical problems of space by means of constructions in the plane.

WASHINGTON UNIVERSITY, ST. LOUIS.

---

* For a collection of proofs of this theorem see Wendling, *Der Fundamentalsatz der Axonometrie*, Zürich, Speidel, 1912.

THE FIRST CARUS MONOGRAPH

Calculus of Variations. By Gilbert Ames Bliss. Carus Mathematical Monographs, No. 1. Published for the Mathematical Association of America by the Open Court Publishing Company, Chicago, 1925. 13 + 189 pp.

One of the important concepts introduced into mathematical thought by E. H. Moore is that of the "Extensional Attainability" of properties (see New Haven Colloquium Lectures, p. 53). This concept, apart from its technical significance in the theory of properties of classes, admits of another interpretation of perhaps broader applicability, one suggestive of conquest. So, an infant reaching out for its playthings might be said to be experimenting with the extensional attainability of satisfaction for its desires, explorers illustrate the extensional attainability of man's control over the globe, other instances will occur to the reader.

The Carus Mathematical Monographs, of which Professor Bliss's book is the first, are intended "to contribute to the dissemination of mathematical knowledge by making accessible at nominal cost a series of expository presentations of the best thoughts and keenest researches in pure and applied mathematics," "in a manner comprehensible not only to teachers and students specializing in mathematics, but also to scientific workers in other fields, and especially to the wide circle of thoughtful people who, having a moderate acquaintance with elementary mathematics, wish to extend their knowledge without prolonged and critical study of the mathematical journals and treatises." Is this not an exhibition of faith in the extensional attainability of a mathematically informed public? It certainly is most fitting that this series of monographs should have been conceived by a Chicago group and that its first number should come from the pen of one of the members of the Department of Mathematics at the University of Chicago.

To Mrs. Mary Hegeler Carus and to her son Dr. Edward Carus belongs the honor of having recognized the importance of such an undertaking and of having provided the necessary means. Through the publication of these books, the Open Court Publishing Company continues its fine service to mathematical education in this country.

To Professor H. E. Slaught belongs the credit for the inception of the idea which gave rise to the monographs and for having solicitously guided it to successful realization. This series of books will forever be a reminder of his farsighted and intelligent devotion to the cause of mathematical education and to his skill in leading it on into new and significant fields of conquest.

In how far the wider dissemination of knowledge contributes to the enlargement of its domain, not many would venture to say. And, whether mathematics is better served by making results long familiar to the specialists accessible to a wider group than by the publication of new results, is, or should be, a rather futile question. Neither of these tasks should be allowed to be neglected. We must constantly labor to enlarge the foundations if they are to support the ceaselessly expanding superstructure, without, however, using up so much material in the process that the superstructure must suffer.

The simile is manifestly inadequate because it does not allow for the human elements involved. Diffusion of ideas among a larger group is likely to generate curiosity, to lead to the discovery of hitherto unsuspected connections, to suggest new ideas. There is no doubt that a much more healthy development may be expected if broader connecting avenues are laid out between the fields of pure and applied mathematics, if the engineers and the physicists became acquainted with some of the developments and some of the results that have been secured from ideas that perhaps were first brought to light in their fields of knowledge. If the Carus monographs contribute to a removal of the barriers which keep modern mathematical science isolated, they will fully justify the faith of the founders and the hopes of its promoters. May the series prove to be a multiple series stretching out its arms in many directions, so as to attain the widest possible extension.

In accordance with the general aim set for the series, the volume now under review has the purpose of bringing the general methods of the calculus of variations within the reach of a larger public than can be expected to master all the requirements which a systematic study of the subject would require. And to the reviewer it seems that the book succeeds admirably in this purpose. Whether this judgment is correct can only be estimated by the future historian.

The process is inductive. In an introductory chapter of sixteen pages, the calculus of variations is presented to the reader in a semihistorical way, chiefly by means of an analogy with the problem of finding the maxima and minima of functions, and through some of its illustrative problems. Three of these, all belonging to the "simplest problem of the calculus of variations," viz. the problems of the shortest distance, of the brachistochrone, and of the surface of revolution of minimum area, form the subject matter of chapters II, III, and IV, respectively, a total of 111 pages. The fifth chapter treats the minimizing of the integral $\int f(x, y, y')\,dx$. Then follow a list of references, notes, and index.

"The author assumes," so the warning on the jacket reads, "that the reader has an acquaintance with the elementary principles of the Differential and Integral Calculus." Even with this assumption as to the amount of preparation of his readers, the author must have been in doubt many times as to whether or not to presuppose knowledge of a particular fact. In some instances he apparently concluded that a reminder was the only thing necessary. As such we have to consider for instance the clause on page 21 concerning the derivative of an integral with respect to its upper limit, and the brief paragraph on the cycloid on page 52. Would not references to fuller treatment of such questions be useful in these places?

Each of the problems to which separate chapters are devoted receives a more complete treatment than they do when used as examples following the development of the general theory, a treatment moreover which utilizes to the full the more recent work done in connection with the classical theory. In preparation for the derivation of the Euler equation in the shortest distance problem, the lemma, usually attributed to Du Bois Reymond, is proved in a very simple manner, although perhaps a little too cleverly for the unsophiscated reader. The Hilbert independent integral, the concept of a field and related ideas and theorems are introduced in each of chapters II, III, IV, in the special forms which they take for the problems there treated. The usual methods of the sufficiency proofs are exemplified in a similar manner, as are also the cases in which one or two endpoints are variable on arbitrary curves. In the discussion of these latter questions the author is not satisfied to treat merely first order conditions, but gives a complete account of the theory of the focal point, which receives in each of the special cases a more or less simple geometrical interpretation. The proof of the existence of a unique extremal through two given points and the construction of a field in the brachistochrone problem are carried out by drawing in a very skillful way upon well known properties of the cycloid. In connection with the same problem we are introduced to the envelope theorem and to the geometric proof of Jacobi's theorem which depends upon it.

Particularly valuable is the presentation in the fourth chapter of the results obtained by Sinclair and MacNeish in the catenary problem, not heretofore available in a connected form, and the clear discussion of the relation between the catenary and the Goldschmidt straight line solutions of the problem of the surface of revolution of minimum area

The final chapter, more than any of the others, is of the usual textbook character. The reader who has followed the discussions of the three special problems in the preceding chapters should be well prepared to understand now the general theory as here presented. He meets again, in the form of general theorems, statements with whose general character he has had an opportunity to become acquainted. But the chapter contains more than these generalizations of the results previously obtained for special cases. It is here that we learn for the first time of the distinction between weak and strong minima, of the Weierstrass E-function, of Legendre's condition, of the Weierstrass-

Erdmann corner point condition, etc., which for various reasons have not been brought forward before. It is furthermore made clear that the geometrical treatment of the conjugate point condition is not sufficient in all cases. This leads to a discussion of the second variation by the elegant method which the author introduced some years ago and which has since been applied by several of his pupils to a variety of more general problems, but a systematic exposition of which has not hitherto been available outside the journals. Moreover, the reader who merely wants to get the results will find an accurate statement of sets of necessary and of sufficient conditions for the cases both of fixed and of variable endpoints. The chapter closes in the same key in which the first one opened, viz., with historical remarks.

Enough has been said to make clear that in the judgment of the reviewer, Professor Bliss has carried through a difficult task with remarkable success. We have here a new type of mathematical book. It is not a textbook, neither is it addressed to the specialist, actual or *in spe.* It is intended for that very indefinite group, the intelligent general public. This group surely includes teachers of mathematics, even those who may have taken a course in the calculus of variations! Certainly engineers who expect to do more than routine work, physicists and chemists and other workers in natural science can profit largely from familiarizing themselves with the leading ideas set forth in this book. To many it should be of immediate usefulness.

But to write on an advanced mathematical topic for a public of such mixed and uncertain qualifications is much like lecturing to a radio audience: one never can tell whether there is any real catching on. In his selection of material the author had to trust to his judgment; the style of treatment is in a large measure determined by this choice. Now one can readily conceive of other selections which might have been made. A treatment of an isoperimetric problem, mention at least of some problems leading to integrals with more unknown functions, and to extrema of multiple integrals would seem to me to be necessary in order to give an adequate picture of the broad scope of the calculus of variations. This remark however amounts to little else than to asking for "more of the same kind"; for I should not want any of these topics to replace anything that is in the book now. And this indeed is my chief criticism; the book is too short. It is a fine beginning; but it should be continued.

This then is the remark with which I wish to close this review: Let the series of Carus Monographs, for which I anticipate a long and successful career, develop a semi-periodic character which will make it possible to carry forward to other parts of the field the discussion of the calculus of variations which Professor Bliss's volume has so excellently initiated.

ARNOLD DRESDEN

AUERBACH ON MATHEMATICAL PHYSICS

Die Methoden der theoretischen Physik. By Felix Auerbach. Leipzig, Akademische Verlagsgesellschaft, 1925. x + 436 pp.

The book to be reviewed is written by a physicist for students of physics and is meant as an introduction to theoretical, i. e. mathematical physics. The author's program is to present the underlying notions and principles as well as to give a compendium of the mathematical devices which the student is likely to need in a more profound study of the subject. The following extract from the table of contents will show that the ground covered by the author is very extensive:

I. Notions and principles, 27 pp. II. Formal methods, 14 pp.* III. Elementary equations, 26 pp. IV. Difference, differential, and integral calculus, 57 pp. V. Ordinary differential equations, 56 pp. VI. Partial differential equations, 125 pp. VII. Integral and functional equations, 23 pp. VIII. Theory of molecules, Statistical methods, 47 pp. IX. Geometrical, graphical, and vector methods, 51 pp. X. Concluding remarks, 6 pp. Index, 4 pp.

It is rather obvious that the task of treating such a wide range of ideas in an adequate manner is a very trying one. The best parts of the book are those where the author is concerned with the applications of some special theory to particular problems. In fairness it must be said that such problems occupy the greater portion of the book. The examples are usually well chosen and the calculations are carried out in quite an elegant manner. But every now and then we encounter a piece of a general mathematical theory which is presented in an awkward and doubtful manner. The reviewer is perfectly aware of the fact that mathematical rigor can not and perhaps should not be the primary concern of the physicist. But in a book which is meant to introduce a young student into the realm of theoretical physics where he is going to use mathematical tools in all his work, it is only fair to the student to tell him something about the limitations of these tools as instruments of research. This of course applies essentially to questions involving some kind of infinite process or limiting passage where the student needs a timely warning. The author's habit of stating results without mentioning the conditions under which these results hold, is deplorable; it makes the value of the information imparted illusory, and it is scarcely conducive to sound mental habits in the student. Occasionally the statements are wrong or at least misleading.

---

* *Wortsprachliche Methodik.* What is meant is methods based upon reduction to general principles by means of qualitative discussion without the use of calculations.

Examples of such failings are plentiful; the theory of series supplies several. Here are a few samples. On page 95 we are told that in order that the series Σu_n shall be convergent, the limit of u_{n+1}/u_n must be less than unity for sufficiently large values of n. The continuation takes care of the case when the limit is unity but adds other deep mysteries. On page 98 we are told that a series with alternating terms is always convergent if the absolute values (Betrag) of the terms converge to zero with increasing values of n. The treatment of power series on the same page is not very satisfactory: the series $\Sigma a_n x^n$ converges when x lies between $-1/\beta$ and $+1/\beta$ where β is the limit of a_{n+1}/a_n. Apply the rule to the case $a_n = \sin n\alpha$, please! On the next page the author says that for values of x which are greater than one it is better to expand functions according to descending powers of x, because of considerations of convergence. Such considerations do not bother the author on page 278. Here he wants to justify the restriction of the parameter in Legendre's equation to integral values as follows: Wenn nun n keine ganze Zahl ist, gehen die Reihen ohne Ende fort, und fur $\mu = 0$ werden beide Reihen selbst unendlich, was nicht sein darf. These series are power series in $1/\mu$ and only convergent for $|\mu| > 1$.

Only five pages of the seventh chapter deal with integral equations in the technical sense of the word. Evidently our author does not like these equations; he begins by saying that they have not completely come up to expectations, and ends the discussion by remarking that the use of integral equations has not proved to be a great positive gain for physics since in all cases which cannot be solved by other means, the application of integral equations usually encounters difficulties or is wanting in lucidity.

It is hard to estimate the value of the book for the particular class of readers to which it is directed, and it is scarcely fair for a mathematician to judge in this matter. The book does give a great amount of useful information. As to the misinformation, perhaps the habit of slurring over everything abstract that characterizes the average young student will preserve the purity of his mind until he reaches the years of discretion.

EINAR HILLE

SHORTER NOTICES

Curve Sgembe Speciali Algebriche e Transcendenti. Volume primo: Curve algebriche. By Gino Loria. Bologna, N. Zanichelli, 1925. x+374 pp.

The monumental treatise on plane curves* by the same author has now been followed by the first volume of an extensive work of a similar nature on curves of three-dimensional space. The first chapter contains a fairly full derivation of the ordinary differential theory of space curves in general, followed by another on curves derived from a given one, such as radial curves, evolutes, filar and planar, pedals, parallel curves, medians of two given curves, etc. Most of these properties are not made further use of, and for. This reason it is curious to find them in a book with this title.

The subject of the book proper begins with a short chapter on the conic in space, treated parametrically, followed by one hundred and thirty pages on the cubic; first the general projective theory, then a detailed consideration of each metric type. The treatment is very elementary and complete. Then come the elliptic and the rational quartics, and finally a brief outline of quintics, and scarcely more than a mention of certain sextics and a few other isolated curves.

While the text is well supplied with historical and bibliographical notes, it is by no means exhaustive, it does include a number of quite recent papers, including some from our BULLETIN and TRANSACTIONS.

The characteristics of space algebraic curves, analogous to Plucker's numbers for the plane, are not given; in several cases rather primitive methods of deriving some of them are employed, when they could be read out of tables uniquely from others already known.

The book is not provided with an index, either of names or of subject matter. In a work of this kind, such indexes are particularly useful. The matter is so arranged, however, that a topic can be found without undue effort.

The mechanical execution is excellent, and the type large and clear. On the whole but very few typographical errors were noticed, and of these none would be liable to confuse the reader. The citation of German titles suffers most heavily in this regard.

VIRGIL SNYDER

---

* *Spezielle algebraische und transcendente ebene Curven, Theorie und Geschichte, Autorisierte, nach dem italienischen Manuscript bearbeitete deutsche Ausgabe von Fritz Schütte,* 1902; 2nd edition in two volumes, 1910–1911.

L'Analysis Situs et la Géométrie Algébrique. By S. Lefschetz. Paris, Gauthier-Villars, 1924.

As a condensed account of the theory of algebraic manifolds, this little volume is so successful that it would seem beside the point to criticise the somewhat hasty manner in which the author disposes of several rather delicate details. Professor Lefschetz has not only given the theory a new unity, but has clarified and extended it at more than one point by the systematic use of topological methods.

First, we are given a brief résumé dealing with the theory of cycles on an algebraic surface, the connectivity numbers and coefficients of torsion, the intersection numbers associated with cycles intersecting in points. Skilful use is made throughout the book of the simple theorem that if one of two cycles intersecting in points is a bounding cycle, the algebraic sum of the points of intersection is zero. It may be mentioned here, in passing, that there are a number of topological invariants associated with the theory of cycles that meet, not in points, but in curves, surfaces, etc. So far as the reviewer knows, the significance of these invariants in the theory of algebraic manifolds has never been studied.

There follows a discussion of the topological characteristics of an algebraic surface, during the course of which a number of the fundamental constants of algebraic geometry reappear as topological invariants. The surface is thought of as swept out by its intersection with a plane varying in a pencil of planes. A general plane section of the surface is a curve C of genus p on which $2p$ 1-cycles, $\gamma_1, \gamma_2, \cdots, \gamma_{2p}$, independent with respect to C, may be traced. If the plane is made to vary in its pencil so as, ultimately, to return to its original position, the cycles γ_i undergo a linear transformation. It would be interesting to see how far the theory of algebraic manifolds could be carried without making use of anything more than the group of linear transformations engendered on the cycles γ_i. Along these lines, it might be possible to develop a perfectly general theory of surfaces without going into the question of the resolution of singular points and curves.

There is an important theorem due to Lefschetz himself to the effect that a cycle is algebraic if and only if no double integrals of the first kind have periods with respect to it. This and other considerations suggest that, in the theory of algebraic manifolds, it might be advisable to modify the definition of an homology so as to make it read as follows: A cycle C is homologous to zero, $C \sim 0$, if, and only if, it is bounding *or algebraic.* The invariants (Betti numbers, coefficients of torsion, etc.) that would follow from this definition of an homology would have the advantage of being unaltered under birational transformations, whereas the strictly topological invariants that follow from the ordinary definition do not have this property.

Perhaps the most illuminating chapter in the book is the one dealing with systems of curves on a surface. Here we find a fundamental theorem that two curves are algebraically equivalent if, and only if, they are homologous in the ordinary sense of analysis situs. Another interesting fact, as pointed out by Lefschetz, is that the so-called "virtual" curves, which have a purely symbolic existence from the algebraic point of view, may be interpreted in topological terms as non-algebraic cycles of the surface. The book also contains other interesting chapters on algebraic manifolds of higher dimensions and on abelian functions, not to mention two notes in the form of postscripts.

J. W. ALEXANDER

Sophus Lie's Gesammelte Abhandlungen (*Samlede Avhandlinger*). Edited by Friedrich Engel and Poul Heegaard. Volume V: *Abhandlungen uber die Theorie der Transformationsgruppen,* erste Abteilung (*Avhandlinger om Transformationsgruppernes Teori,* første Avdeling), edited by Friedrich Engel. Leipzig, B. G. Teubner, and Kristiania, H. Aschehoug and Co., 1924. xii+776 pages.

The fifth volume of Lie's collected memoirs is the second of the series to be published; it is preceded by the third volume which was published in 1922 (and was reviewed in this BULLETIN, vol. 29 (1923), pp. 367–369). The published memoirs of Lie are to be gathered together in six volumes while a seventh is to be devoted to the principal works found among his literary remains. The memoirs on geometry are to go into volumes I and II; those on differential equations into volumes III and IV; and those on transformation groups into volumes V and VI. Naturally these divisions are not rigorously separated from each other; and each volume will contain memoirs belonging in part to all three divisions. In each of the three divisions the arrangement is chronological; and, generally, the material in the first of any two related volumes is that which was first published at Christiania, while the second of these volumes is devoted principally to Memoirs from the MATHEMATISCHE ANNALEN and the publications of the Leipzig Akademie. In this way it is brought about that no volume will contain two memoirs one of which is mainly a reworking of the other.

As the subtitle indicates, the fifth volume is given to memoirs on transformation groups. It contains the earlier of them, up to February 1889. Of the 560 pages required to print the included memoirs, about 320 pages are given to those in which "Transformationsgruppen" is the main word in the title; a large part of the remaining space is given to memoirs dealing with the applications of the theory of transformation groups to differential equations. The notes from the Norwegian (none of them long) are translated into German, so that the whole body of this volume appears in German. On pages 669–673 of the notes an address of Lie's is published in French, the language in which it was spoken. (In volume III the French articles were published in French, all others appearing in German.) Some verbal changes have again been made in the articles on reprinting; and a list of these is given on pages 561–582.

The notes and supplementary matter in volume III made up more than a third of the whole. The editor stated in that volume that volume V would require much fewer notes; but the event has shown that he was mistaken, the space given to supplementary matter in volume V being only a little less than in volume III. On pages 583–614 we have an account of the beginnings of the theory of transformation groups as set forth by Lie in his letters to Adolph Mayer in 1873 and 1874. Pages 615–755 are given to the editor's notes on the separate memoirs (and include the address already mentioned). These notes contain numerous matters of detail, explanatory remarks, suggestions and cross references, together with a few discussions of considerable length (such as that on pages 643–668). An extensive index to the volume is given on pages 756–774.

The editorial work (here as in volume III) is marked throughout with evidence of that care and patience which belong only to a labor of love. It is done in such way as to render great service to all those who will have occasion to use the memoirs of Lie which are here reproduced. The next volume of the series which is to be printed is the sixth, and in the preface it is indicated that work upon it will begin immediately. We can not hope more for it than that it will be edited and printed with the same care as volumes III and V.

R. D. CARMICHAEL

Johannes Kepler. Mysterium Cosmographicum. Das Weltgeheimnis. Uebersetzt und eingeleitet von Max Caspar. Augsburg, Dr. Benno Filser Verlag, 1923. xxxi+150 pp.

The large increase in the number of students of mathematics and natural science who are unfamiliar with the Latin language makes it more and more useful and desirable to have the greatest works of the pioneers of modern science translated into modern tongues. Among these pioneers Kepler deserves a distinguished place, first because of his work in geometry, in which he prepared the way, through his "infinitesimal method," for the invention of integral calculus; and secondly because in astronomy he laid the foundation for the modern view of the solar system, through his famous three "laws."

Thus there is a real place in the literature of the history of science for such works as the one under review. The *Mysterium Cosmographicum* was Kepler's first work, and it attracted sufficient notice to assure him of a permanent career as an astronomer. The translation of this work is preceded by an interesting and valuable Introduction, in which Dr. Caspar gives a discussion of Kepler's place in the history of science, of his literary style and of the relation of the *Mysterium Cosmographicum* to his other works. It is made clear that in this fantastic-appearing mixture of science, pseudo-science, and mysticism are found the germs, and more than the germs, of the great discoveries of the author's later years.

Kepler's discussion of his theme, in spite of the fact that it is in places labored and tedious, has yet a frankness and naiveté that give it interest and charm. And the light which this work throws upon the beginnings of modern astronomy is alone sufficient to repay the time spent in reading it. Written but fifty years after Copernicus's *De Revolutionibus Orbium* it very naturally begins with a discussion of the relative merits of the Ptolemaic and Copernican systems, in which the greater simplicity of the latter is the chief argument which Kepler stresses. He then proceeds to explain his view of the orbits and motions of the planets. He makes the distances of the planets from the sun equal to the radii of the inscribed and circumscribed spheres of a series of regular polyhedra. Mercury's orbit is placed inside an octahedron, Venus outside it and inside an icosahedron; outside this, and inside a dodecahedron, comes the earth; Mars is then outside this and inside a tetrahedron; Jupiter outside this and inside a cube, while lastly the orbit of Saturn is drawn about the cube. Kepler gives ingenious reasons why the order of the solids should be just this and no other. Thus, the cube must be the outermost, "since the number of sides of the cube, six, is a perfect number." After detailed statements of the numerical relations that exist among the sides and radii of the five solids, he attempts to show that the relative distances of the planets are in agreement with these computations; and while he naturally finds discrepancies of various magnitudes, he has a ready excuse for each, after considering which he is able to convince himself that he has found the true explanation of the relative distances of the planets, as well as of the "fact" that there are exactly six.

The translation has been well done, to judge from numerous passages which the reviewer chose at random and compared. The figures are not well printed, the one on page 89 being actually incorrect; but otherwise there are few misprints.

It would be a valuable contribution to the history of science if we could have a number of translations into English such as this work and the Ostwald "Klassiker" give to the German student. Not to go farther, why can we not have an accessible English translation of Newton's most important works?

R. B. McCLENON

Oeuvres de G. H. Halphen. Publiées par les soins de C. Jordan, H. Poincaré, É. Picard, avec la collaboration de E. Vessiot. Tome IV.* Paris, Gauthier-Villars, 1924. xv + 657 pp.

The works of G. H. Halphen with the exception of the three-volume *Traité des Fonctions Elliptiques,* are contained in four volumes the last of which is now before us. Each volume opens with a "notice" concerning Halphen and his works written either on the occasion of his death or not long afterwards; in fact, the first volume contains two such notices. It has also Halphen's own account (pages 1–47) of his mathematical works written on the occasion of his candidature for membership in the Academy of Sciences. It is natural to find at the beginning of his *Oeuvres* this summary of his own works up to 1885. Except for this paper his memoirs are arranged in the *Oeuvres* in almost chronological order, the deviations from this order being unimportant and not many in number. Owing to Halphen's custom of carrying one investigation to a complete achievement before taking up another and owing to the fact that the order of completion of memoirs is also very nearly their order of publication, the chronological arrangement of Halphen's work is also to a considerable extent a logical arrangement. But the former in some respects departs considerably from the latter so that it is desirable to have added (as on pages 645–653) the "Table Méthodique Générale."

Among the memoirs contained in the first volume are those on the characteristics in the theory of conics, on singular points of plane curves, and on the points where an algebraic curve satisfies a condition expressed by means of a given differential equation. For the second volume we may mention those on the singularities of twisted curves and of algebraic surfaces, on the differential invariants of plane and twisted curves, and on the representation of functions in various types of series. Two of the four memoirs in volume III contain the best work of Halphen. These are the memoir on the reduction of linear differential equations to integrable forms and that on the classification of twisted algebraic curves.

This leaves to the fourth volume the last works of Halphen other than his treatise on elliptic functions, the memoirs and papers appearing in 1883 to 1890, together which such unpublished material as his editors have seen fit to preserve with his collected works. The papers for the most part deal with differential equations, the theory of numbers, and elliptic functions and their applications; they are more fragmentary in character than the comprehensive memoirs in the earlier volumes, doubtless owing to the fact that Halphen's consecutive thought in his last years was being given largely to his treatise on elliptic functions. Here is reproduced also his work on the theory of the singularities of plane curves, originally published as an appendix to the French translation of Salmon's analytical geometry.

---

* The previous volumes have been reviewed in this BULLETIN, vol. 27 1920-21), pp. 466–468 and vol. 28 (1921-1922), pp. 271–272.

In addition to these previously published writings there are two appendices. The "Mémoires et Fragments Inédits" (pages 465–627) contain a considerable variety of material taken from the manuscripts which Halphen left and thus preserve what is most useful in this material. The editors desired also to publish the letters which Halphen had written to several scientists with whom he had been in correspondence; but they were able to bring together only a certain number of extracts from the letters of a single one of his correspondents, namely, Zeuthen. The "Extraits de Lettres à Zeuthen" (pages 628–644) were chosen and transcribed by Zeuthen himself. These letters belong to the period 1876 to 1881 of Halphen's life.

Owing to the fact that the various "notices" in the four volumes contain several brief accounts of Halphen's work the reviewer is relieved of the duty of making an analysis of the separate memoirs.

R. D. CARMICHAEL

The Mathematical Theory of Relativity. 2d edition. By A. S Eddington. Cambridge, University Press, 1924. ix + 270 pages.

The second edition of this important work differs from the first mainly in the addition of thirteen pages (pp. 241–263) of "Supplementary Notes" relating to various sections throughout the whole volume. There is no general revision of the main body of the text. A few corrections have been made, and in such way as not to disturb the paging. An erroneous formula [numbered (59.6)] on page 137 has been suppressed, the paragraph containing it having been replaced by another. The remarks on the problem of the homogeneous sphere on page 170 have been modified, this page having been largely rewritten. The other changes are less important than these two.

R. D. CARMICHAEL

Statistical Methods. By Frederick Cecil Mills. New York, Henry Holt and Co., 1924. xvi+604 pp.

This book has been prepared with particular "reference to the specific needs of quantitative workers in economics and business" and therefore calls for little comment here. It includes, in addition to an extensive elementary treatment of statistical graphs, averages, correlation theory, etc., an extensive treatment of index numbers and time series. It has been carefully prepared and with suitable abbreviations could be used as a textbook in a mathematical course in statistics where no collegiate course in mathematics is a prerequisite.

C. H. FORSYTH

Petit Traité de Perspective. By Raoul Bricard. Paris, Librairie Vuibert, 1924. 87 pp.

This little treatise of perspective contains a partial reproduction of the course given by Bricard at the Conservatoire des Arts et Métiers and at the École Centrale des Arts et Manufactures in Paris.

The method used to present the elements of perspective is that of Cousinery, and is especially emphasized by Bricard. He justly maintains that this method, different from the traditional and later Monge constructions, has, in France, not received the recognition which it deserves.

The method consists in fixing the center of projection C in space by a fixed circle K in the picture plane, so that the end of the perpendicular erected at the center of K, and equal to the radius of K, coincides with C. In this manner a line l in space is determined, when its perspective l', the trace S of l on l' and the vanishing point F' of l on l', are known. The perspective P' of a point P is determined by the intersection of the perspectives l' and g' of two lines l and g through P. By this method also the homology between a figure in a plane Σ after rotating it around its trace down into the picture plane Σ', and its perspective in Σ', is easily obtained.

It may be of interest to state a few historic facts concerning this method of perspective representation. Cousinery published his *Géométrie Perspective ou Principes de Projection Polaire Appliquée à la Description des Corps* in 1828.

Brook Taylor, long before, in 1715 and 1719, made use of the same idea in *New Principles of Linear Perspective.* Joseph Highmoore applied Taylor's method in *The Practise of Perspective on the Principles of Dr. Brook Taylor*, which appeared in 1763.

Also J. H. Lambert, in 1759 and 1774, published a treatise based upon the "distance-circle:" *Die freie Perspective oder Anweisung jeden perspectivischen Aufriss von freien Stücken und ohne Grundriss zu verfertigen.*

Fiedler made systematic use of the "Distanzkreis" in his voluminous *Die Methoden der darstellenden und die Elemente der projectivischen Geometrie*, and his "*Cyclographie,*" after having given an outline of his methods in the Programmabhandlung of the Gewerbeschule of Chemnitz in 1860: *Die Centralprojection als geometrische Wissenschaft.*

Bricard does not presuppose any knowledge of projective geometry and develops as much of the elements, Desargues' theorem, etc., as are needed for the understanding of what follows. The "*Petit Traité*" is an excellent introduction to the essentials of linear perspective, and it is obvious that it has been written by a competent hand.

ARNOLD EMCH

Théorie des Nombres. Tome second: Le Second Degrée Binaire, by E. Cahen, Paris, Hermann, 1924. 736 pp.

This volume is devoted to the development of that part of the theory of numbers which centers around the study of the binary quadratic forms. As the number of pages suggests, the treatment is very comprehensive and besides giving a complete account of all theory necessary in the development, problems in Diophantine analysis, which spring out of the theory of quadratic forms, are taken up for consideration when their interest is of sufficient generality to warrant it. To each chapter is appended a list of exercises in which problems of less general interest are left to the reader.

For any arithmetic theory relative to the field of rational numbers a comprehension of the classical theory of numbers is essential. The author devotes the first four chapters to the development of this theory concluding with the consideration of congruences of the second degree in one unknown. This development is followed by five chapters in which treat the idea of a number in general and various methods of approximation. Special attention is given to the theory of continued fractions and the development of numbers into continued fractions, and this again with special reference to quadratic irrationalities.

Chapter ten is a discussion of Diophantine equations in which one member is a quadratic polynomial in x and y. A very complete study is made of Fermat's equation of the first and second species

$$x^2-Dy^2=\pm 1, \qquad x^2+xy-ky^2=\pm 1.$$

The first of these is what is commonly called Pell's equation, an error of long standing.

The discussion of quadratic Diophantine equations in two unknowns is followed by a general study of binary quadratic forms in which both the algebraic and arithmetic theories are considered. The study includes the theory of the classification of binary quadratic forms and the separation of the classes into genera. The group theory with special reference to finite abelian groups is developed sufficiently for a good comprehension of the theory of composition of forms and the group of classes.

In the study of quadratic forms the author considers the general form $ax^2+bxy+cy^2$ in place of the form $ax^2+2bxy+cy^2$.

Chapter 25 is a study of the equation $x^p+y^p=z^p$ for $p=2, 3, 4$. The study of these equations leads, as is well known, to quadratic irrationalities $i=\sqrt{-1}$ and $j=(-1+\sqrt{-3})/2$. The two fields $C(i)$ and $C(j)$ are studied in detail in chapters 26 and 27.

The remaining chapters contain the development of the general theory of quadratic number fields with the laws governing the factorization of rational primes in such fields and the classification of ideals.

G. E. WAHLIN

Bibliographie de la Relativité, suivie d'un appendice sur les déterminants à deux dimensions, le calcul des variations, les séries trigonométriques, et l'azéotropisme. By Maurice Lecat. Bruxelles, Lamertin, 1924. 292+47 pp.

M. Lecat's bibliography of relativity should find a place in all university libraries and also in the personal libraries of those who are working in this field. In carefully checking the list against my own (limited) store of bibliographical information I found only one omission, viz. the pamphlet by A. N. Whitehead entitled *The Principles of Natural Science*, a lecture delivered at Bryn Mawr on the eighteenth of April, 1922, and published by Bryn Mawr College.. In view of the limited circulation which this pamphlet received I feel justified in saying that M. Lecat's book is a remarkably complete index of the world's literature on relativity as it was on the day of publication, April 10, 1924.

C. N. REYNOLDS, JR.

Höhere Mathematik. Teil I: *Differentialrechnung und Grundformeln der Integralrechnung nebst Anwendungen.* By R. Rothe Leipzig, B. G. Teubner, 1925. vii+185 pp.

This little volume is one of the collection of Teubner's Technische Leitfäden, whose purpose is to give to technical students the theoretical foundations of their work in a brief and handy form.

The volume under discussion, the first of three on "Hohere Mathematik", is devoted mainly to the differential calculus. The chapter headings are: (1) Numbers, variables, and functions, (2) Main theorems of the differential calculus and fundamental integration formulas (3) functions of two or more variables, (4) differential geometry of plane curves, (5) complex numbers, variables and functions. The treatment is brief and concise. but there is no attempt to sacrifice rigor and clearness to brevity. The number of exercises is rather small, especially in comparison with similar books in this country, but on the whole they are well chosen. Of applications (and we assume this to mean applications of a technical nature) there are fewer than one would expect from the title of the book. The chief ones are references to questions of approximations, brief mention of linkages (in connection with curves in polar coordinates) and occasionally a problem in the exercises with an engineering setting.

The book is interesting as indicating perhaps what the mathematica equipment in the direction of the calculus, of an engineer in Germany is supposed to be. The opening chapter, which is really an introduction to the theory of functions of a real variable, would to seem to indicate that a student of calculus is supposed to cope successfully with a rigorous treatment of limits and continuity. American textbooks especially those intended for engineering students, seem to avoid mentioning such concepts rigorously. The closing chapter of the book, which gives the elements of the theory of functions of a complex variable, is another indication of this broadness of scope. Perhaps American engineers would not be so overawed by differentiation and integration processes if the mathematical foundations were broader

T. H. HILDEBRANDT

Les Erreurs Philosophiques de M. Einstein. Étude directe de la Relativité. By G. Joly. Paris, Edition Spes, 1925. 64 pp.

Amidst the mass of temporary literature evoked by the contact between the enthusiastic disciples of Einstein or of his competitors, and the perturbed conservatives of the older schools, this little book finds a natural place. It is an attack vigorously directed against the body of Einstein's conclusions.

The author cordially concedes that all of Einstein's formulas and equations are to be accepted as "guaranteed by several of the most eminent specialists," and a large part of this small book is devoted to the effort of establishing (for the restricted theory) these very formulas upon the basis of classical mechanics. A distinction is drawn between the mathematical calculations and the philosophical conclusions and the latter are rejected *in toto* by this author. The author admits what he calls the *fact* of relativity, which he claims consists in the circumstance that linear measurements reveal nothing concerning actual locations, but only relative distances. In fact he asserts incidentally that "*no point is at rest in the universe*" (italics quoted). However the author insists that the establishment of one of the formulas which he accepts as do the relativists "supposes *implicitly* comparison with a system fixed in absolute space, where lengths and times have an absolute measure". "Relativity has no sense except by comparison with the absolute, it presupposes the latter and may be deduced from it. In denying this reality, M. Einstein is led to a conception which 'is not expressible in words but only in mathematical formulas.'" "The perturbations rightly signalized by M. Einstein are most of the time so slight as to elude experimental verification. ··· There is no need of a special theory to correct aberrations due to these relative movements. ··· It suffices to take account of their causes in laying down the equations."

The modern student of relativity may find interest in some of the simple methods used here in discussing the elementary features of the restricted theory, but the book seems chiefly designed to aid and comfort those who cannot dispute mathematics with Einstein but who would like to feel that the old Newtonian system is as good as ever.

A. A. BENNETT

On the Direct Numerical Calculation of Elliptic Functions and Integrals. By Louis V. King. Cambridge, University Press, 1924. viii+42 pp.

The author undertakes in this little brochure to connect the theory of elliptic functions and integrals with the so-called "arithmetico-geometric mean scales" in order to relate more directly that theory with the computing machine. Using the series of numbers (a_n, b_n) where $a_n = \frac{1}{2}(a_{n-1}+b_{n-1})$, $b_n = \sqrt{a_{n-1}b_{n-1}}$, $a_0 > b_0 > 0$ he derives computation formulas for the various elliptic functions in terms of these. The undertaking appears to be in line with other efforts now being made to use the computing machine to avoid tedious and uncertain interpolations. Large tables will of course always be necessary and for most purposes sufficient, but the use of anything but first differences will always involve danger of errors. This little book contains in convenient form many well known formulas and also many new ones.

D. N. LEHMER

Über Paraboloide, Hyperboloide und Ellipsoide von Archimedes. (Ostwald's *Klassiker der exakten Wissenschaften*, No. 210.) Translated and annotated by A. Czwalina. Leipzig, Akademische Verlagsgesellschaft, 1923. 73 pp.

This little book consists of 66 pages of translation of Archimedes' work on quadric surfaces of revolution, together with 7 pages of notes. It will doubtless prove to be a useful addition to the materials for the study of the history of mathematics in German schools; but for English-speaking students it is without significance, on account of the existence of Sir Thomas Heath's definitive edition of Archimedes in English. The reviewer will merely state that the translation has been carefully done, and that the notes give in modern notation concise explanations of the less evident parts of Archimedes's discussion. The statement of the last theorem on page 64 contains a misprint in that the words "Hälfte der" should be inserted before "Verbindungsstrecke." Some of the figures lack portions of their lettering. Otherwise, no typographical error of consequence was noted.

R. B. MCCLENON

An Introduction to the Theory of Statistics. 7th edition. By G. Udny Yule. London, Charles Griffin and Co., Limited, 1924. xv+415 pp.

The seventh edition of this reliable text differs very little from the preceding edition. Chapter III, on Association, has been practically rewritten; the supplementary list of references has been brought up to date; and some minor alterations and corrections have been made. Additional matter incorporated in all the recent editions has been added in the form of Supplements.

C. H. FORSYTH

NOTES

Beginning with the next issue, this BULLETIN will again be printed in the United States, at the Banta Publishing Company, Menasha, Wisconsin.

The second number of volume 27 of the TRANSACTIONS OF THIS SOCIETY (April, 1925) contains the following papers: *The subgroup composed of the substitutions which omit a letter of a transitive group*, by G. A. Miller; *On the closeness of approach of complex rational fractions to a complex irrational number*, by L. R. Ford; *Solutions of the Einstein equations involving functions of only one variable*, by E. Kasner; *A general theory of linear sets*, by M. H. Ingraham; *On the representation of a certain fundamental law of probability*, by H. L. Rietz; *The group of motions of an Einstein space*, by J. Eiesland; *A generalization of Levi-Civita's parallelism and the Frenet formulas*, by J. H. Taylor. The third number (July, 1925) contains· *The linear complex of conics*, by E. E. Libman; *On the Weddle surface and analogous loci*, by A. Emch; *Algebraic surfaces with reducible bitangent and osculating hyperplanar sections*, by M. Castellani; *Polynomials of several variables and their residue systems*, by A. J. Kempner; *A criterion for the conformal equivalence of a Riemann space to a euclidean space*, by J. Douglas; *The deflection of a rectangular plate fixed at the edges*, by H. W. March; *On irredundant sets of postutates*, by A. Church; *Three-dimensional manifolds of states of motion*, by H. Hotelling; *Relations between the critical points of a real function of n independent variables*, by M. Morse.

The joint September–December, 1924, number of the ANNALS OF MATHEMATICS ((2), vol. 26, Nos. 1–2), contains: *Contact transformations linear in x, y, z; applications to equilong transformations*, by B. H. Brown; *On the trigonometric representation of an ill-defined function*, by D. Jackson; *An irregular boundary value and expansion problem*, by L. E. Ward; *Maximal cuspidal curves*, by T. R. Hollcroft; *Plane cubics associated with the quadrangle-quadrilateral configuration*, by B. M. Turner; *On Pellet's theorem concerning the roots of a polynomial*, by J. L. Walsh; *The setting of a proposition*, by P. J. Daniell; *Theta functions and arithmetic*, by E. T. Bell; *A new type of criteria for the first case of Fermat's last theorem*, by H. S. Vandiver; *Algebraic functions and their divisors*, by G. A. Bliss; *An extension of the definition of the Green's function in one dimension*, by W. M. Whyburn; *Real representations of analytic complex curves*, by W. C. Graustein, *Note on Dirichlet series with complex exponents*, by J. F. Ritt, *On the order of an analytic function at a singular point*, by M. H. Stone;

Representations of integers in certain binary, ternary, quaternary and quinary quadratic forms and allied class number relations, by E. T. Bell. The March, 1925, number ((2), vol. 26, No. 3) contains: *Non-monoidal involutions which contain a web of invariant monoids*, by V. Snyder; *Definite integrals containing a parameter*, by R. L. Jeffery; *Integro-differential expressions invariant under Volterra's group of transformations*, by A. D. Michal; *New proofs of two well known theorems on quadratic forms*, by J. F. Ritt; *Some relations between compound determinants*, by W. H. Metzler; *A contribution to the theory of interpolation*, by N. Wiener; *A property of cyclotomic integers and its relation to Fermat's last theorem (second paper)*, by H. S. Vandiver; *On an infinite system of non-abelian groups of order nm^n*, by W. E. Edington; *Conformal and geodesic mapping*, by A. Bramley.

The second number of volume 47 of the American Journal of Mathematics (April, 1925) contains: *Some properties of the exponential mean*, by J. Dale; *The eliminant of a net of curves*, by F. Morley; *On an equation of planar motion*, by F. Morley; *An extension of the problem of the elastic bar*, by H. T. Davis; *On the Sylow subgroups of the symmetric and alternating groups*, by L. Weisner; *On certain theorems regarding summable series and their application to the double and triple Fourier's series*, by G. M. Merriman; *On the power characters of units in a cyclotomic field*, by H. S. Vandiver

The Sociedad Matemática Argentina has begun publication of a journal entitled Revista Matemática. The first number appeared in October, 1924.

Several numbers have appeared of the Japanese Journal of Mathematics, published under the auspices of the National Research Council of Japan.

The Academy of Mathematical and Physical Sciences of the Royal Society of Naples has been commissioned by the faculty of sciences of the University of Naples to take charge of the award of the quinquennial Gabriele and Ruggero Torelli prize, for the encouragement of research in pure mathematics. Those receiving the doctorate in pure mathematics at an Italian university during the interval from January 1, 1925, to December 1, 1929, are eligible for the next award. Applicants should send copies of their dissertations and other published works to the Academy before January 2, 1930.

The Royal Society of Edinburgh has awarded the Gunning Victoria Jubilee Prize for the period 1920–24 to Sir Joseph Thomson, in recognition of his discoveries in physical science, and the MacDougall-Brisbane Prize for the same period to Professor H. Stanley Allen, for his papers on the quantum and atomic theory.

On the occasion of the celebration of the fiftieth anniversary of his entrance into teaching (see this BULLETIN, vol. 31, p. 185) Professor A. V. Vasilief was elected an honorary member of the Kharkof Mathematical Society.

Professor T. von Kármán, of the Technical School at Aachen, has been elected a corresponding member of the Gottingen Gesellschaft der Wissenschaften.

Professor A. Kneser, of the University of Breslau, has been elected a corresponding member of the Russian Academy of Sciences.

Professors Paul Koebe, of the University of Jena, and Wilhelm Wirtinger, of the University of Vienna, have been elected corresponding members of the physico-mathematical class of the Prussian Academy of Sciences.

The newly founded University of Kovno, Lithuania, conferred an honorary doctorate of mathematics on Professor Aurel Voss, of the University of Munich, on the occasion of his eightieth birthday. Professor Voss had presented his large library to the mathematical seminar of the university.

The faculty of sciences of the University of Padua is collecting a fund for a memorial to Professor Giuseppe Veronese, on the occasion of the tenth anniversary of his death.

The University of Glasgow has conferred an honorary degree on the mathematical physicist Paul Painlevé, premier of France.

Oxford University has conferred an honorary doctorate of science on Professor Niels Bohr, of the University of Copenhagen.

The University of Manchester has conferred the degree of doctor of science on Professor A. S. Eddington, of Cambridge.

Professors Niels Bohr and A. S. Eddington have been elected foreign members of the National Academy of Sciences. Professor S. Lefschetz, of Princeton University, has been elected a member in the section of mathematics.

Professor G. G. Chambers, of the University of Pennsylvania, has received the honorary degree of doctor of science from Dickinson College.

The Franklin Institute has conferred Franklin medals and certificates of honorary membership on Dr. P. Zeeman, professor of physics at the University of Amsterdam, and Dr. Elihu Thomson, of the General Electric Company.

John R. Freeman, consulting engineer, of Providence, has made a gift of securities valued at $25,000 to the Boston Society of Civil Engineers for the establishment of a fund, the income of which is to be used for encouraging research by the younger engineers of the

Society through the award of prizes for papers on hydraulics and allied subjects.

Professor E. V. Huntington, of Harvard University, is acting as Western Exchange Professor during the first half of this academic year; and will be at Beloit College, Carleton College and Knox College.

Columbia University has conferred the honorary degree of doctor of science on Dr. Irving Langmuir, of the General Electric Company.

Yale University has conferred the honorary degree of doctor of science on Professor R. A. Millikan.

The Case School of Applied Science and Union College have conferred honorary degrees on Professor M. I. Pupin, of Columbia University.

Assistant Professor J. R. Kline, of the University of Pennsylvania, has been awarded a Guggenheim Fellowship, to study the analysis situs of three dimensions from a point-set standpoint, principally at the University of Göttingen.

Assistant Professor J. L. Walsh, of Harvard University, has been awarded a National Research Fellowship.

The two Benjamin Peirce Instructorships in Mathematics at Harvard University (see this Bulletin, vol. 21, p. 315) are again open for general competition. Applications for the year 1925-26, accompanied by the necessary papers, should reach Cambridge not later than February 15, 1926.

Mr. Bertrand Russell has been appointed Tarner lecturer in the philosophy of the sciences for 1925-26 at Trinity College, Cambridge.

Mr. E. F. Baxter, of the University of Sheffield, has been appointed lecturer in applied mathematics at the University of Durham.

Mr. W. L. Ferrar, of the University of Edinburgh, has been elected a fellow of Hertford College, Oxford.

Professor J. L. Synge, formerly of the University of Toronto, has been elected a fellow of Trinity College and professor of natural philosophy at the University of Dublin.

Professor C. N. Armstrong, of Ohio Wesleyan University, has been appointed head of the department of mathematics to fill the vacancy caused by the death of Professor C. B. Austin.

At West Virginia University, Assistant Professors Margaret Buchanan, C. N. Reynolds, and B. M. Turner have been promoted to associate professorships of mathematics.

Dr. Julia Dale, recently at the University of Oklahoma, has been appointed professor of mathematics at the Delta State College of Mississippi.

Assistant Professor J. E. Davis, of the Drexel Institute, has been appointed associate professor of engineering extension at Pennsylvania State College.

Dr. Philip Franklin has been promoted to an assistant professorship of mathematics at the Massachusetts Institute of Technology.

At the University of Wyoming, Professor H. C. Gossard has been appointed to a full professorship of mathematics, and Mr. O. H. Rechard has been promoted to an associate professorship.

Dr. C. C. Grove has been appointed assistant professor of mathematics at the Brooklyn Polytechnic Institute.

At the University of Minnesota, Associate Professor W. L. Hart has been promoted to a full professorship of mathematics, and Dr. Gladys Gibbens to an assistant professorship.

Assistant Professor Olive C. Hazlett, of Mount Holyoke College, has been appointed assistant professor of mathematics at the University of Illinois.

Assistant Professor A. J. Kempner, of the University of Illinois, has been appointed full professor of mathematics at the University of Colorado.

Professor Max Mason, of the University of Wisconsin, has been elected president of the University of Chicago.

Professor A. A. Michelson, of the University of Chicago, has been appointed to the first of the distinguished service professorships recently established at that university.

Assistant Professor E. J. Miles, of Yale University, has been promoted to an associate professorship of mathematics.

Mr. Max Morris has been promoted to an assistant professorship of mathematics at the Case School of Applied Science.

The marriage is announced of Professor Anna J. Pell. of Bryn Mawr College and Professor A. L. Wheeler, professor of Latin at Princeton University. They will continue to occupy their present professorships, and they will be at home after this fall in Princeton and in Bryn Mawr.

Dr. J. A. Shohat has been appointed to an assistant professorship at the University of Michigan.

Assistant Professor L. L. Smail, of the University of Oregon, has been appointed to an assistant professorship at the University of Texas.

Mr. Emory P. Starke has been promoted to an assistant professorship of mathematics at Rutgers University.

At the Texas State College at Lubbock, Dr. L. D. Ames, formerly of the University of Missouri, has been appointed associate professor of mathematics, and Miss Elizabeth Stafford has been appointed adjunct professor of mathematics.

Dr. J. H. Taylor has been appointed assistant professor of mathematics at Lehigh University.

Associate Professor H. S. Uhler, of Yale University, has been appointed head of the department of physics at Gettysburg College.

Professor H. A. Wilson, of the University of Glasgow, has accepted reappointment to the professorship of physics at Rice Institute which he held from 1912 to 1924.

Associate Professor R. M. Winger, of the University of Washington, has been promoted to a full professorship.

Professor Frederick Wood, of Lake Forest College, has been appointed professor of mathematics at Wesleyan College, Macon, Ga.

The following appointments to instructorships are announced:
American College for Women, Constantinople, Miss E. Marie Plapp;
Connecticut College for Women, Miss Mildred Carlen;
University of Florida, Mrs. Claire Harkins;
West Virginia University, Mr. H. A. Davis,
University of Wyoming, Miss Greta Neubauer.

The distinguished mathematician Felix Klein, professor of mathematics at the University of Gottingen and editor of the MATHEMATISCHE ANNALEN, died June 22, 1925, at the age of seventy-six.

Dr. Heinrich Müller-Breslau, professor of statics and building construction at the Charlottenburg Technical School, died April 23, 1925, at the age of seventy-three.

Professor Carl Neumann, of Leipzig, died March 27, 1925, in his ninety-third year.

Camille Flammarion, astronomer and author of numerous popular astronomical works, died June 4, 1925, at the age of eighty-three.

Professor Gualtiero Romboli, of the Technical School at Forli, died March 20, 1925, at the age of forty-five.

Dr. W. S. Dennett, of New York City, died March 6, 1925. Dr. Dennett had been a member of the Society since 1892, and served as its treasurer from 1900 to 1907.

Professor S. J. Lockner, of the University of Pittsburgh, died May 10, 1925, at the age of fifty-five.

Dr. Mansfield Merriman, professor of civil engineering at Lehigh University from 1878 to 1907, died June 7, 1925, at the age of seventy-seven. Dr. Merriman had been a member of the American Mathematical Society since 1891.

Professor W. A. Hamilton, of Antioch College, died June 25th, 1925.

Dr. C. E. Stromquist, for many years head of the department of mathematics at the University of Wyoming, died early in April, 1925.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

APPELL (P.). Sur une forme générale des équations de la dynamique. (Mémorial des Sciences Mathématiques, No. 1.) Paris, Gauthier-Villars, 1925. 50 pp.

BRAGG (W.). Concerning the nature of things. Six lectures delivered at the Royal Institution. London, Bell, 1925. 11+232 pp.

BRUNSCHVICG (L.). Le génie de Pascal. Paris, Hachette, 1924. 14+200 pp.

CARR (H. W.). The scientific approach to philosophy. London, Macmillan, 1924. 8+278 pp.

CRENSHAW (B. H.) and KILLEBREW (C. D.). Analytic geometry and calculus. Philadelphia, Blakiston, 1925. 10+222 pp.

FLADT (K.). Unendliche Reihen. (Mathematisch-Physikalische Bibliothek, Nr. 61.) Leipzig, Teubner, 1925. 52 pp

GUT (M.). Ueber die singularen Moduln der Ikosaeder-Modulfunktion. (Diss., Zürich.) Zürich, 1924. 102 pp.

HADAMARD (J.). Cours d'analyse professé à l'Ecole Polytechnique. Tome 1, fascicule 1. Paris, Hermann, 1925. 336 pp.

HILBERT (D). Grundzuge einer allgemeinen Theorie der linearen Integralgleichungen. 2te Auflage Leipzig, Teubner, 1924.

JAMES (R.). See LEWENT (L.).

KILLEBREW (C. D.). See CRENSHAW (B. H).

KLOOSTERMAN (H D.). Over het splitsen van geheele positieve getallen in een som van kwadraten. (Diss., Leiden.) Groningen, Noordhoff, 1924. 79 pp.

LEVI-CIVITA (T.). Lezioni di calcolo differenziale assoluto. Roma, Stock, 1925. 315 pp.

LEWENT (L.). Conformal representation. Translated by R. James and D. H. Williams. London, Methuen, 1925. 8+146 pp.

LOCKE (L.). The ancient quipu or Peruvian knot record. New York, Museum of Natural History, 1923. 84 pp.+59 plates.

LORIA (G.). Pagine di storia della scienza. Torino, Paravia, 1925. 15+151 pp.

MATELL (M.). Asymptotische Eigenschaften gewisser linearer Differentialgleichungen. (Diss., Uppsala) Uppsala, 1924. 67 pp.

NICOD (J.). La géométrie dans le monde sensible. Préface de M. B. Russell. Paris, Alcan, 1924. 16+174 pp.

PÓLYA (G.) und SZEGO (G.). Aufgaben und Lehrsatze der Analysis. Band 2. (Die Grundlehren der mathematischen Wissenschaften, Band 20.) Berlin, Springer, 1925. 10+407 pp.

ROTHE (R.). Höhere Mathematik. Teil 1: Differentialrechnung und Grundformeln der Integralrechnung nebst Anwendungen. Leipzig, Teubner, 1925. 8 + 186 pp.

RUSSELL (B.). See NICOD (J.).

SMAIL (L. L.). History and synopsis of the theory of summable infinite processes. Eugene, University of Oregon Press, 1925. 6 + 175 pp.

SMITH (D. E.). History of mathematics. Volume 2: Special topics of elementary mathematics. Boston, Ginn, 1925.

SZEGÖ (G.). See PÓLYA (G.).

UNGERER (E.). Wissen und Wirken. Karlsruhe, O. Braun, 1924.

VERRIEST (G.). Cours de mathématiques générales. Partie 2. Paris, Gauthier-Villars, 1925. 388 pp.

WILLIAMS (D. H.). See LEWENT (L.).

PART II. APPLIED MATHEMATICS

ACKERET (J.). Das Rotorschiff und seine physikalischen Grundlagen. Gottingen, Vandenhoeck und Rupprecht, 1925. 48 pp.

BELLASIS (E. S.). Hydraulics with working tables. 4th edition, revised. London, E. and F. N. Spon, 1924. 356 pp.

—— River and canal engineering. 2d edition, revised. London, E. and F. N. Spon, 1924.

BERKELEY (L. M.). North star navigation. New York, White Book and Supply Company, 1924. 86 pp.

BOHR (N.). The theory of spectra and atomic constitution. Three essays. 2d edition. Cambridge, University Press, 1924.

TYCHONIS BRAHE Dani opera omnia. Edidit I. L. E. Dreyer. Tomus 11. Hauniae, Libraria Gyldendaliana, 1924. 5 + 422 pp.

CAMPBELL (N. R.). La structure de l'atome. 2e supplément à l'ouvrage La théorie électrique moderne. Traduit de l'anglais par A. Corvisy. Paris, Hermann, 1925. 6 + 166 pp.

CHAUVEAU (B.). Electricité atmosphérique. Fascicule 2: Le champ électrique de l'atmosphère. Paris, Doin, 1925. 10 + 264 pp.

CLARKE (J. R.). See GUYE (C. E.).

CORPS (—.). Le camouflage de la simultanéité, base unique des théories de la relativité. Pontoise, Imprimerie Lucien Paris, 1924. 12 pp.

CORVISY (A.). See CAMPBELL (N. R.).

DE COUSSANGE (J.). See HOFFDING (H.).

CURRIER (E. L.), LENNES (N. J.) and MERRILL (A. S.). Farm accounting. New York, Macmillan, 1924. 9 + 287 pp.

DARRAS (M.). Statique graphique élémentaire et notions préliminaires de résistance des matériaux. Paris, Librairie Centrale des Sciences, 1925. 397 pp.

DE DONDER (T.). La gravifique de Weyl-Eddington-Einstein Paris, Gauthier-Villars, 1924. 32 + 16 pp.

——— Théorie mathématique de l'électricité. Partie 1. Introduction aux equations de Maxwell. Paris, Gauthier-Villars, 1925. 200 pp.

DREYER (I. L. E.). See TYCHO BRAHE.

DUPONT (P.). La mécanique nouvelle démontrée par les principes classiques. Interpretation et transformation des équations de Lorentz et d'Einstein. Paris, Hermann, 1925. 14 + 150 pp.

EXNER (F. M.). Dynamische Meteorologie. Berlin, Springer, 1925 8 + 421 pp.

FABER (O.). Reinforced concrete beams in bending and shear. Theory and tests in support. London, Concrete Publications, Ltd., 1924. 11 + 150 pp.

FOURNIER D'ALBE (E. E.). See NORDMANN (C.).

GEORGII (W.). Wettervorhersage. Dresden, Steinkopff, 1924. 114 pp.

GOULDEN (C.). Refraction of the eye, including elementary physiological optics. London, Churchill, 1925. 12 + 276 pp.

GREINACHER (H.). Ionen und Elektronen. Leipzig, Teubner, 1924. 58 pp.

GUYE (C. E.). Physico-chemical evolution. Translated by J. R. Clarke. London, Methuen, 1925. 12 + 172 pp.

HAHN (K.). Grundriss der Physik. 2te Auflage Teil 2. Leipzig. Teubner, 1924.

HARVEY (F. W.). Examples in mechanics London, Methuen, 1925. 6 + 82 pp.

HAZELTINE (L. A.). Electrical engineering. New York, Macmillan, 1924. 16 + 625 pp.

HEHRE (F. W.). See MORECROFT (J. H.).

HENRY (C.). Essai de généralisation de la théorie du rayonnement. Résonateurs gravifiques et résonateurs biologiques. Exposition intuitive des résultats techniques essentiels de la théorie de la relativité. Paris, Hermann, 1925. 144 pp.

HOBSON (E. W.). The ideal aim of physical science Cambridge, University Press, 1925. 4 + 34 pp.

HOFFDING (H.). La relativité philosophique. Traduit du danois par J. de Coussange. Paris, Alcan, 1924.

HORT (W.). Die Differentialgleichungen des Ingenieurs. 2te umgearbeitete und vermehrte Auflage. Berlin, Springer, 1925. 12 + 700 pp.

JOLLEY (L. B. W.). Alternate current rectification. A mathematical and practical treatment from the engineering view-point. London, Chapman and Hall, 1924. 18 + 352 pp.

JONES (F. W.). The Hodsock ballistic tables for rifles. London, Arnold, 1925. 8 + 82 pp.

KENT (F. C.). Elements of statistics. New York, McGraw-Hill, 1924. 11 + 178 pp.

KOHLHOERSTER (W.). Die durchdringende Strahlung der Atmosphare. Hamburg, H. Grand, 1924. 72 pp.

LAPORTE (M.). Le radium. Paris, Stock, 1924. 128 pp.

LENNES (N. J.). See CURRIER (E. L.).

LENZ (K.). Die Rechenmaschinen und das Maschinenrechnen. 2te Auflage. Leipzig, Teubner, 1924. 6 + 108 pp.

McLAREN (S. B.). Scientific papers mainly on electrodynamics and natural radiation including the substance of an Adams prize essay in the University of Cambridge. Cambridge, University Press, 1925. 8 + 112 pp.

MENGES (C. L. R. E.). Nouvelles vues Faraday-Maxwelliennes. Supplément: Sur la propagation de la lumière. Paris, Gauthier-Villars, 1924. 6 + 44 pp.

MERRILL (A. S.). See CURRIER (E. L.).

MEYERSON (E.). La déduction relativiste. Paris, Payot, 1925. 396 pp.

MORECROFT (J. H.) and HEHRE (F. W.). Electrical circuits and machinery. Volume 2: Alternating currents. New York, Wiley, 1924. 11 + 444 pp.

NIEWENGLOWSKI (G. H.). Les rayons X et le radium. Paris, Hachette, 1925. 184 pp.

NORDMANN (C.). The tyranny of time. Einstein or Bergson? Translated from the French by E. E. Fournier d'Albe. London, T. Fisher Unwin, 1925. 215 pp.

PRZIBRAM (R.). Aufbau mathematischer Biologie. Berlin, Gebruder Borntraeger, 1923.

ROUCH (J.). Les méthodes de prévision du temps Paris, Alcan, 1924. 280 pp.

SALLINGER (F.). Die Gleichstrommaschine. 1 ter und 2 ter Teil. (Sammlung Goschen.) Berlin, de Gruyter, 1923–24.

SCHWENGLER (J.). Der Bau der Starrluftschiffe. Ein Leitfaden fur Konstrukteure und Statiker. Berlin, Springer, 1925. 103 pp.

SHEARCROFT (W. F. F.). The story of the atom. London, Benn, 1925. 79 pp.

DE SPARRE (-.). Calcul du temps et de la dérivation dans les grandes trajectoires des projectiles. Paris, Gauthier-Villars, 1924.

TAYLOR (H. S.). A treatise on physical chemistry. Volumes 1–2. London, Macmillan, 1924. 655 + 709 pp.

THIRTY-FOURTH ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND
SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES
TO THE PLACES OF PUBLICATION

ALEXANDER, J. W. Topological invariants of manifolds Read Oct. 25, 1924. *Proceedings of the National Academy of Sciences*, vol. 10, No. 12, pp. 493–494; Dec., 1924.

BARNETT, I. A. On a class of transformations in function space. Read April 14, 1923. *Annals of Mathematics*, (2), vol. 25, No. 3, pp. 205–220; March, 1924.

—— Integro-differential equations. Read Dec. 29, 1923. *This Bulletin*, vol. 31, Nos. 5–6, pp. 263–265; May–June, 1925.

BELL, E. T. Certain products involving the divisors of numbers. Read (San Francisco Section) April 5, 1924. *Bulletin of the Calcutta Mathematical Society*, vol. 15, No. 1, pp. 21–24; June, 1924.

—— Theta functions and arithmetic. Read (San Francisco Section) April 7, 1923. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 79–87; Sept.–Dec., 1924.

—— Representations of integers in certain binary, ternary, quaternary and quinary quadratic forms and allied class number relations. Read Dec. 27, 1923. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 155–164; Sept.–Dec., 1924.

—— Notes on recurring series of the third order. Read (San Francisco Section) Sept. 18, 1923. *Tôhoku Mathematical Journal*, vol. 24, Nos. 1–2, pp. 168–184; Dec., 1924.

—— On the number of representations of an integer as a sum or difference of two cubes. Read (San Francisco Section) June 19, 1925. *This Bulletin*, vol. 31, No. 7, pp. 309–312, July, 1925.

BENNETT, A. A. Five axioms for point and translation in affine geometry. Read Dec. 30, 1924. *This Bulletin*, vol. 30, Nos. 9–10, pp. 520–526; Nov.–Dec., 1924.

—— On sets of three consecutive integers which are quadratic residues of primes. Read April 10, 1925. *This Bulletin*, vol. 31, No. 8, pp. 411–412; Oct., 1925.

BLISS, G. A. Algebraic functions and their divisors. Read April 19, 1924. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 95–124; Sept.–Dec., 1924.

BRINKMANN, H. W. Einstein spaces which are mapped conformally on each other. Read April 28, 1923. *Mathematische Annalen*, vol. 94, Nos. 1–2, pp. 119–145; April, 1925.

BROWN, B. H. Contact transformations linear in x, y, z; applications to equilong transformations. Read Dec. 27, 1922. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 1–7; Sept.–Dec., 1924.

CAJORI, F. Empirical generalizations on the growth of mathematical notations. Read (San Francisco Section) Sept. 18, 1923. *Isis*, vol. 6, No. 3, pp. 391–394; 1924.

—— Note on our sign of equality. Read (San Francisco Section) April 5, 1924. *Isis*, vol. 6, No. 4, pp. 507–508; 1924.

Cajori, F. Notes on Luca Pacioli's "Summa." Read (San Francisco Section) April 5, 1924. *Archivio di Storia della Scienza*, vol. 5, No. 2, pp. 125–130; June, 1924.

—— Despiau's Select Amusements. Read (San Francisco Section) April 5, 1924. *Mathematical Gazette*, vol. 12, No. 172, p. 216; Oct., 1924.

—— Fanciful hypotheses on the origin of the numeral forms. Read (San Francisco Section) Oct. 25, 1924. *Mathematics Teacher*, vol. 18, No. 3, pp. 129–133; March, 1925.

—— Early printed forms of our dollar mark. Read (San Francisco Section) April 4, 1925. *School and Society*, vol. 21, No. 543, pp. 625–626, May 23, 1925.

Camp, B. H. Probability integrals for a hypergeometric series. Read Oct. 25, 1924. *Biometrica*, vol. 17, Nos. 1–2, pp. 61–67; June, 1925.

Caris, P. A. A solution of the quadratic congruence modulo p, $p = 8n + 1$, n odd. Read April 10, 1925. *American Mathematical Monthly*, vol. 32, No. 6, pp. 294–299; June–July, 1925.

Carpenter, A. F. Projective properties of a ruled surface in the neighborhood of a ruling. Read (San Francisco Section) Sept. 18, 1923, and (San Francisco Section) Dec. 22, 1923. *American Journal of Mathematics*, vol. 46, No. 4, pp. 241–257; Oct., 1924.

—— Cone-cubic configurations of a ruled surface. Read (San Francisco Section) April 5, 1924. *Transactions of this Society*, vol. 27, No. 4, pp. 397–415; Oct., 1925.

Castellani, M. Algebraic surfaces with reducible bitangent and osculating hyperplanar sections. Read May 2, 1925. *Transactions of this Society*, vol. 27, No. 3, pp. 279–286; July, 1925.

Church, A. On irredundant sets of postulates. Read Dec. 30, 1924. *Transactions of this Society*, vol. 27, No. 3, pp. 318–328; July, 1925.

Cummings, L. D. A new type of double sextette closed under a binary (3,3) correspondence. Read Oct. 25, 1924. *This Bulletin*, vol. 31, Nos. 5–6, pp. 266–274; May–June, 1925.

Davis, H. T. An extension of the problem of the elastic bar. Read Dec. 29, 1922, and Dec. 26, 1924. *American Journal of Mathematics*, vol. 47, No. 2, pp. 101–120; April, 1925.

Dickson, L. E. Differential equations from the group standpoint. Read April 15, 1922. *Annals of Mathematics*, (2), vol. 25, No. 4, pp. 287–378; June, 1924.

Dodd, E. L. The frequency law of a function of one variable. Read Dec. 30, 1924. *This Bulletin*, vol. 31, Nos. 1–2, pp. 27–31; Jan.–Feb., 1925.

Douglas, J. A criterion for the conformal equivalence of a Riemann space to a euclidean space. Read Oct. 27, 1923. *Transactions of this Society*, vol. 27, No. 3, pp. 299–306; July, 1925.

Edington, W. E. On an infinite system of non-abelian groups of order nm^n. Read Dec. 29, 1922. *Annals of Mathematics*, (2), vol. 26, No. 3, pp. 233–238; March, 1925.

Eiesland, J. The group of motions of an Einstein space. Read March 26, 1921, Sept. 8, 1921, and Sept. 7, 1923. *Transactions of this Society*, vol. 27, No. 2, pp. 213–245, April, 1925.

Eisenhart, L. P. Fields of parallel vectors in a Riemannian geometry. Read Feb. 28, 1925. *Transactions of this Society*, vol. 27, No. 4, pp. 563–573; Oct., 1925.

EMCH, A. On plane algebraic curves which remain invariant under a quadratic Cremona transformation, their classification, and relation to the Cayley cubic surface. Read Dec. 28, 1921. *Tôhoku Mathematical Journal*, vol. 24, Nos. 1–2, pp. 68–87; Dec., 1924.

—— Determination of plane algebraic curves which are invariant under involutory Cremona transformations. Read April 10, 1925. *Tôhoku Mathematical Journal*, vol. 25, Nos. 1–2, pp. 63–76; May, 1925.

—— On the Weddle surface and analogous loci. Read Dec. 29, 1923. *Transactions of this Society*, vol. 27, No. 3, pp. 270–278; July, 1925.

FORD, L. R. On the closeness of approach of complex rational fractions to a complex irrational number. Read April 24, 1920. *Transactions of this Society*, vol. 27, No. 2, pp. 146–154; April, 1925.

—— The fundamental region of a Fuchsian group. Read Sept. 11, 1925. *This Bulletin*, vol. 31, Nos. 9–10, pp. 531–539; Nov.–Dec., 1925.

FRANKLIN, P. Analytic transformations of everywhere dense point sets. Read May 3, 1924. *Transactions of this Society*, vol. 27, No. 1. pp. 91–100; Jan., 1925.

—— The rotating disc. Read Oct. 25, 1924. *Proceedings of the National Academy of Sciences*, vol. 11, No. 2, pp. 147–149; Feb., 1925.

—— Functions with an essential singularity. Read May 3, 1924. *This Bulletin*, vol. 31, Nos. 3–4, pp. 157–162; March–April, 1925

—— The electric currents in a network. Read Oct. 25, 1924. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 4, No. 2, pp. 97–102; April, 1925.

—— The Weierstrass approximation theorem. Read Dec. 30, 1924. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 4, No. 3, pp. 148–152; May, 1925.

GARABEDIAN, C. A. Quatre méthodes pour résoudre le problème de la poutre rectangulaire. Read Oct. 25, 1924. *Comptes Rendus de l'Académie des Sciences*, vol. 179, No. 7, pp. 381–384; Aug. 18, 1924.

—— Rods of constant or variable cross section. Read April 28, 1923. *American Journal of Mathematics*, vol. 46, No. 4, pp. 273–287; Oct., 1924.

—— Solution du problème de la plaque rectangulaire épaisse encastrée ou posée, portant une charge uniformément répartie ou concentrée en son centre. Read Feb. 28, 1925. *Comptes Rendus de l'Académie des Sciences*, vol. 180, No. 4, pp. 257–259; Jan. 26, 1925.

GLENN, O. E. A note on the abundance of differential combinants in a fundamental system. Read May 3, 1924. *Proceedings of the National Academy of Sciences*, vol. 11, No. 6, pp. 281–284; June, 1925.

GRAUSTEIN, W. C. Real representations of analytic complex curves. Read Sept. 7, 1922. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2. pp. 131–143; Sept.–Dec., 1924.

GRIFFITHS, L. W. Contact curves of the rational plane cubic Read (San Francisco Section) Dec. 22, 1923. *This Bulletin*, vol. 31 No. 7, pp. 312–317; July, 1925.

GRONWALL, T. H. The mutual inductance of two square coils. Read April 28, 1923. *Transactions of this Society*, vol. 27, No. 4, pp. 516–536; Oct., 1925.

HANCOCK, H. The foundations of the theory of algebraic numbers. Read Dec. 31, 1924. *Science*, new ser., vol. 61, No. 1566, pp. 5–10, and No. 1567, pp. 30–35; Jan. 2 and Jan. 9, 1925.

HEDRICK, E. R., and INGOLD, L. Analytic functions in three dimensions. Read Sept. 7, 1923. *Transactions of this Society*, vol. 27, No. 4, pp. 551–555; Oct., 1925.

—— The Beltrami equations in three dimensions. Read (Southwestern Section) Dec. 1, 1923. *Transactions of this Society*, vol. 27, No. 4, pp. 556–562; Oct., 1925.

HENDERSON, A. Is the universe finite? Read Jan. 1, 1925. *American Mathematical Monthly*, vol. 32, No. 5, pp. 213–223; May, 1925.

HENDERSON, R. Life insurance as a social service and as a mathematical problem. Read Dec. 30, 1924. *This Bulletin*, vol. 31, Nos. 5–6, pp. 227–252; May–June, 1925.

HILLE, E. Note on Dirichlet's series with complex exponents. Read April 28, 1923. *Annals of Mathematics*, (2), vol. 25, No. 3, pp. 261–278; March, 1924.

—— A general type of singular points. Read Oct. 25, 1924. *Proceedings of the National Academy of Sciences*, vol. 10, No. 12, pp. 488–493; Dec., 1924.

—— On the zeros of the functions of the parabolic cylinder. Read Dec. 27, 1923. *Arkiv för Matematik, Astronomi och Fysik*, vol. 18, No. 26, pp. 1–56; Jan., 1925.

—— A note on regular singular points. Read Oct. 25, 1924. *Arkiv för Matematik, Astronomi och Fysik*, vol. 19 A, No. 2, pp. 1–21; April, 1925.

HOLLCROFT, T. R. Maximal cuspidal curves. Read Feb. 25, 1922. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 37–46; Sept.–Dec., 1924.

—— Limits for actual double points of space curves. Read Feb. 24, 1923. *This Bulletin*, vol. 31, Nos. 1–2, pp. 42–55; Jan.–Feb., 1925.

HOPKINS, L. A. On periodic orbits for the characteristic planets. Read Sept. 7, 1920. *Astronomical Journal*, vol. 36, Nos. 4–5, pp. 25–33; Nov. 11, 1924.

HOTELLING, H. Three-dimensional manifolds of states of motion. Read (San Francisco Section) Oct. 25, 1924. *Transactions of this Society*, vol. 27, No. 3, pp. 329–344; July, 1925.

HUBER, C. M. On the prime divisors of the cyclotomic functions. Read Oct. 25, 1924. *Transactions of this Society*, vol. 27, No. 1, pp. 43–48; Jan., 1925.

HURWITZ, W. A. Characteristic parameter values for an integral equation. Read Dec. 30, 1924. *This Bulletin*, vol. 31, Nos. 9–10, pp. 503–508; Nov.–Dec., 1925.

HUTCHINSON, J. I. On the roots of the Riemann zeta function. Read Oct. 25, 1924. *Transactions of this Society*, vol. 27, No. 1, pp. 49–60; Jan., 1925.

INGOLD, L. Associated types of linear connection. Read April 11, 1925. *Proceedings of the National Academy of Sciences*, vol. 11, No. 5, pp. 252–256; May, 1925.

—— A symbolic treatment of the geometry of hyperspace. Read (Southwestern Section) Dec. 1, 1923. *Transactions of this Society*, vol. 27, No. 4, pp. 574–599; Oct., 1925.

—— See HEDRICK, E. R.

INGRAHAM, M. H. A general theory of linear sets. Read April 11, 1925. *Transactions of this Society,* vol. 27, No. 2, pp. 163–196; April, 1925.

JACKSON, D. On the method of least *m*th powers for a set of simultaneous equations. Read Dec. 30, 1920. *Annals of Mathematics,* (2), vol. 25, No. 3, pp. 185–192; March, 1924.
—— A general class of problems in approximation. Read Dec. 29, 1922. *American Journal of Mathematics,* vol. 46, No. 4, pp. 215–234; Oct., 1924.
—— On the trigonometric representation of an ill-defined function. Read Dec. 30, 1920. *Annals of Mathematics,* (2), vol. 26, Nos. 1–2, pp. 8–20, Sept.–Dec., 1924.
—— The elementary geometry of function space. Read April 19, 1924. *American Mathematical Monthly,* vol. 31, No. 10, pp. 461–471; Dec., 1924.
—— The geometry of frequency functions. Read Oct. 25, 1924. *This Bulletin,* vol. 31, Nos. 1–2, pp. 63–73, Jan.–Feb., 1925.

JAMES, G. An algebraically reducible solution of the cubic equation, Read Feb. 28, 1925. *American Mathematical Monthly,* vol. 32. No. 4, pp. 162–169; April, 1925.

KASNER, E. An algebraic solution of the Einstein equations. Read Oct. 29, 1921. *Transactions of this Society,* vol. 27, No. 1, pp. 101–105, Jan., 1925.
—— Separable quadratic differential forms and Einstein solutions. Read May 3, 1924. *Proceedings of the National Academy of Sciences,* vol. 11, No. 1, pp. 95–96, Jan, 1925.
—— Solutions of the Einstein equations involving functions of only one variable. Read Sept. 7, 1921. *Transactions of this Society,* vol. 27, No. 2, pp. 155–162, April, 1925.

KEMPNER, A. J. Polynomials of several variables and their residue systems. Read Dec. 26. 1924. *Transactions of this Society,* vol. 27, No. 3, pp. 287–298; July, 1925.

KEYSER, C. J. Concerning groups of dyadic relations of an arbitrary field. Read April 26, 1919. *Fundamenta Mathematicae,* vol. 7, pp. 323–339; 1925.

KLINE, J. R. Concerning the complementary intervals of countable closed sets. Read Oct. 25, 1924. *This Bulletin,* vol. 31, No. 8, pp. 409–410, Oct., 1925.
—— Concerning the sum of two continua each irreducible between the same pair of points. Read May 3, 1924. *Fundamenta Mathematicae,* vol. 7, pp. 314-322; 1925.

LEHMER, D. N. A new method of factorization. Read (San Francisco Section) April 4, 1925. *Proceedings of the National Academy of Sciences,* vol. 11, No. 1, pp. 97–98; Jan., 1925.
—— Note on the construction of tables of linear forms. Read (San Francisco Section) June 19, 1925. *This Bulletin,* vol. 31, Nos. 9–10, pp. 497–498; Nov.–Dec., 1925.

LEVY, H. Normal congruences of curves in Riemann space. Read April 28, 1923. *This Bulletin,* vol. 31, Nos. 1–2, pp. 39–42; Jan.–Feb., 1925
—— Ricci's coefficients of rotation. Read May 3, 1924. *This Bulletin,* vol. 31, Nos 3–4, pp. 142–145; March–April, 1925.

LIBMAN, E. E. The linear complex of conics. Read Feb. 28, 1925. *Transactions of this Society*, vol. 27, No. 3, pp. 265–269; July, 1925.

LOGSDON, M. I. Complete groups of points on a plane cubic curve. Read Dec. 29, 1922. *Transactions of this Society*, vol. 27, No. 4, pp. 474–490; Oct., 1925.

MACDUFFEE, C. C. On the complete independence of the functional equations of involution. Read April 28, 1923. *This Bulletin*, vol. 31, Nos. 1–2, pp. 21–26; Jan.–Feb., 1925.

MACMILLAN, W. D. Some mathematical aspects of cosmology. Read April 10, 1925. *Science*, new ser., vol. 62, No. 1595, pp. 63–72, No. 1596, pp. 96–99, and No. 1597, pp. 121–127; July 24, July 31, and Aug. 7, 1925.

MARCH, H. W. Deflection of a rectangular plate fixed at the edges. Read Dec. 29, 1923. *Transactions of this Society*, vol. 27, No. 3, pp. 307–317; July, 1925.

MERRIMAN, G. M. On certain theorems regarding summable series and their application to the double and triple Fourier's series. Read Dec. 28, 1923, and Dec. 29, 1924. *American Journal of Mathematics*, vol. 47, No. 2, pp. 125–139; April, 1925.

MICHAL, A. D. Functionals of curves admitting one-parameter groups of infinitesimal point transformations. Read Dec. 28, 1923. *Proceedings of the National Academy of Sciences*, vol. 11, No. 1, pp. 98–101; Jan., 1925.

—— Integro-differential expressions invariant under Volterra's group of transformations. Read April 28, 1923. *Annals of Mathematics*, (2), vol. 26, No. 3, pp. 181–201; March, 1925.

—— Functional invariants, with a continuity of order p, of one-parameter Fredholm and Volterra transformation groups. Read Oct. 25, 1924. *This Bulletin*, vol. 31, No. 7, pp. 335–346; July, 1925.

MILLER, G. A. The subgroup composed of the substitutions which omit a letter of a transitive group. Read Dec. 26, 1924. *Transactions of this Society*, vol. 27, No. 2, pp. 137–145; April, 1925.

MOORE, C. L. E. Minimal varieties of two or three dimensions whose element of arc is a perfect square. Read Feb. 28, 1925. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 4, No. 3, pp. 167–178; May, 1925.

MOORE, C. N. On the application of Borel's method to the summation of Fourier's series. Read Dec. 29, 1924. *Proceedings of the National Academy of Sciences*, vol. 11, No. 6, pp. 284–287; June, 1925.

MOORE, R. L. Concerning upper semi-continuous collections of continua which do not separate a given continuum. Read (Southwestern Section) Nov. 29, 1924. *Proceedings of the National Academy of Sciences*, vol. 10, No. 8, pp. 356–360; Aug., 1924.

—— Concerning sets of segments which cover a point set in the Vitali sense. Read (Southwestern Section) Nov. 29, 1924. *Proceedings of the National Academy of Sciences*, vol. 10, No. 11, pp. 464–467; Nov., 1924.

—— Concerning the prime parts of a continuum. Read Sept. 7, 1923. *Mathematische Zeitschrift*, vol. 22, Nos. 3–4, pp. 307–315; Feb., 1925.

—— A characterization of a continuous curve. Read May 3, 1924. *Fundamenta Mathematicae*, vol. 7, pp. 302–307; 1925.

MOORE, R. L. Concerning the separation of point sets by curves. Read Sept. 10, 1925. *Proceedings of the National Academy of Sciences*, vol. 11, No. 8, pp. 469–476; Aug., 1925.
—— Concerning upper semi-continuous collections of continua. Read Dec. 30, 1924. *Transactions of this Society*, vol. 27, No. 4, pp. 416–428; Oct., 1925.

MORITZ, R. E. On the kinematic construction of certain higher plane curves. Read (San Francisco Section) April 4, 1925. *American Mathematical Monthly*, vol. 32, No. 6, pp. 302–305, June–July, 1925.

MORSE, M. Relations between the critical points of a real function of n independent variables. Read Dec. 28, 1923. *Transactions of this Society*, vol. 27, No. 3, pp. 345–396; July, 1925.

MURNAGHAN, F. D. The tensor character of the generalized Kronecker symbol. Read March 1, 1924. *This Bulletin*, vol. 31, No. 7, pp. 323–329; July, 1925.

MURRAY, F. H. Generalization of certain theorems of Bohl. Read March 1, 1924. *American Journal of Mathematics*, vol. 47, No. 1, pp. 25–44; Jan., 1925.

OLSON, H. L. Congruences with constant absolute invariants. Read Feb. 28, 1925. *Transactions of this Society*, vol. 27, No. 1, pp. 15–42; Jan., 1925.

OSGOOD, W. F. On normal forms of differential equations. Read Feb. 28, 1925. *Transactions of this Society*, vol. 27 No. 1, pp. 1–14, Jan., 1925.

PUPIN, M. I. From chaos to cosmos. Read Feb. 29, 1924. *Scribner's Magazine*, vol. 76, No. 1, pp. 3–10; July, 1924.

RAINICH, G. Y. Electrodynamics in the general relativity theory. Read Feb. 24, 1923, Dec. 27, 1923, March 1, 1924, May 3, 1924, and Oct. 25, 1924. *Transactions of this Society*, vol. 27, No. 1, pp. 106–136; Jan., 1925.
—— Second paper on tensor analysis. Read March 1, 1924. *American Journal of Mathematics*, vol. 47, No. 1, pp. 11–24, Jan., 1925.
—— Sur une représentation des surfaces. Read Dec 29, 1924. *Comptes Rendus de l'Académie des Sciences*, vol. 180, No. 11, pp. 801–803; March 16, 1925.

REILLY, J. F. Certain generalizations of osculatory interpolation. Read April 14, 1922. *The Record, American Institute of Actuaries*, vol. 13, No. 1, pp. 4–22, June, 1924.
—— On Lidstone's demonstration of the osculatory interpolation formula. Read (Southwestern Section) Nov. 29, 1924. *The Record, American Institute of Actuaries*, vol. 14, No. 1, pp. 12–20, June, 1925.

REINSCH, B. P. Expansion problems in connection with the hypergeometric differential equation. Read April 19, 1924. *American Journal of Mathematics*, vol. 47, No. 1, pp. 45–70; Jan., 1925.

RIETZ, H. L. On the representation of a certain fundamental law of probability. Read April 13, 1923. *Transactions of this Society*, vol. 27, No. 2, pp. 197–212; April, 1925.

RITT, J. F. Note on Dirichlet's series with complex exponents. Read Dec. 28, 1923. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, p. 144; Sept.–Dec., 1924.

Ritt, J. F. Elementary functions and their inverses. Read Oct. 25, 1924. *Transactions of this Society*, vol. 27, No. 1, pp. 68–90; Jan., 1925.

—— New proofs of two well known theorems on quadratic forms. Read March 1, 1924. *Annals of Mathematics*, (2), vol. 26, No. 3, pp. 202–204, March, 1925.

Roe, E. D. Integral functions as products. Read Sept. 5, 1917. *Publications of the Roe Observatory*, No. 1, pp. 1–6; Jan., 1924.

Roever, W. H. Some phases of descriptive geometry. Read (Southwestern Section) Nov. 29, 1924. *This Bulletin*, vol. 31, Nos. 9–10, pp. 540–550; Nov.–Dec., 1925.

Romig, H. G. Early history of division by zero. Read (San Francisco Section) Sept. 18, 1923. *American Mathematical Monthly*, vol. 31, No. 8, pp. 387–389; Oct., 1924.

Schouten, J. A. On the conditions of integrability of covariant differential equations. Read Sept. 10, 1925. *Transactions of this Society*, vol. 27, No. 4, pp. 441–473; Oct., 1925.

Shohat, J. A. On the development of a continuous function into a series of Tchebycheff polynomials. Read April 19, 1924. *Transactions of this Society*, vol. 27, No. 4, pp. 537–550; Oct., 1925.

—— On the polynomial of the best approximation to a given continuous function. Read Dec. 26, 1924. *This Bulletin*, vol. 31, Nos. 9–10, pp. 509–514; Nov.–Dec., 1925.

Smail, L. L. History and synopsis of the theory of summable infinite processes. Read (San Francisco Section) June 18, 1920, and (San Francisco Section) April 7, 1923. *University of Oregon Publication*, vol. 2, No. 8, 6+175 pp.; Feb., 1925.

Smith, H. L. On the existence of the Stieltjes integral. Read Dec. 29, 1923. *Transactions of this Society*, vol. 27, No. 4, pp. 491–515; Oct., 1925.

Snyder, V. On the types of monoidal involutions. Read Dec. 27, 1923. *Annals of Mathematics*, (2), vol. 25, No. 3, pp. 279–284; March, 1924.

—— Non-monoidal involutions which contain a web of invariant monoids. Read April 18, 1924. *Annals of Mathematics*, (2), vol. 26, No. 3, pp. 165-172; March, 1925.

Stone, M. H. On the order of an analytic function at a singular point. Read March 1, 1924. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 145–154; Sept.–Dec., 1924.

Synge, J. L. A generalization of the Riemannian line-element. Read Dec. 30, 1924. *Transactions of this Society*, vol. 27, No. 1, pp. 61–67; Jan., 1925.

Taylor, J. H. A generalization of Levi-Civita's parallelism and the Frenet formulas. Read April 19, 1924. *Transactions of this Society*, vol. 27, No. 2, pp. 246–264; April, 1925.

—— Reduction of Euler's equations to a canonical form. Read Oct. 25, 1924. *This Bulletin*, vol. 31, Nos. 5–6, pp. 257–262; May–June, 1925.

Thomas, J. M. Congruences of circles studied with reference to the surface of centers. Read Sept. 7, 1923. *Annals of Mathematics*, (2), vol. 25, No. 3, pp. 221–237; March, 1924.

—— Note on the projective geometry of paths. Read Feb. 28, 1925. *Proceedings of the National Academy of Sciences*, vol. 11, No. 4, pp. 207–209; April, 1925.

THOMAS, J. M. Conformal correspondence of Riemann spaces. Read May 2, 1925. *Proceedings of the National Academy of Sciences*, vol. 11, No. 5, pp. 257–259; May, 1925.
—— The number of even and odd absolute permutations of *n* letters. Read Feb. 28, 1925. *This Bulletin*, vol. 31, No. 7, pp. 303–304; July, 1925.
—— See VEBLEN, O.

VANDIVER, H. S. A new type of criteria for the first case of Fermat's last theorem. Read Sept. 7, 1923 *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 88–94; Sept.–Dez., 1924.
—— On sets of three consecutive integers which are quadratic or cubic residues of primes. Read Dec. 29, 1923. *This Bulletin*, vol. 31, Nos. 1–2, pp. 33–38; Jan.–Feb., 1925.
—— A property of cyclotomic integers and its relation to Fermat's last theorem. (Second paper) Read April 24, 1915, and April 27, 1918. *Annals of Mathematics*, (2), vol. 26, No. 3, pp. 217–232; March, 1925.
—— On the power characters of units in a cyclotomic field. Read Dec. 28, 1916. *American Journal of Mathematics*, vol. 47, No. 2, pp. 140–147; April, 1925.
—— Laws of reciprocity and the first case of Fermat's last theorem. Read Sept. 11, 1925. *Proceedings of the National Academy of Sciences*, vol. 11, No. 6, pp. 292–298; June, 1925.
—— On the distribution of quadratic and higher residues. Read Oct. 28, 1916. *This Bulletin*, vol. 31, No. 7, pp. 346–350; July, 1925.

VEBLEN, O. Remarks on the foundations of geometry. Read Dec. 31, 1924. *This Bulletin*, vol. 31, Nos. 3–4, pp. 121–141; March–April, 1925.

VEBLEN, O., and THOMAS, J. M. Projective normal coordinates for the geometry of paths. Read May 2, 1925. *Proceedings of the National Academy of Sciences*, vol 11, No. 4, pp. 204–207, April, 1925.

WAHLIN, G. E. On the solution of diophantine equations by means of ideals. Read Dec. 26, 1924. *This Bulletin*, vol. 31, No. 8, pp. 430–444; Oct., 1925.

WALSH, J. L. Some two-dimensional loci. Read Oct. 25, 1924. *Quarterly Journal of Pure and Applied Mathematics*, vol. 50, No. 2, pp. 154–165, Aug., 1924.
—— On Pellet's theorem concerning the roots of a polynomial. Read Dec. 27, 1923. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 59–64; Sept.–Dec., 1924.

WEDDERBURN, J. H. M. A theorem on simple algebras Read May 3, 1924. *This Bulletin*, vol. 31, Nos. 1–2, pp. 11–13; Jan.–Feb., 1925.
—— The absolute value of the product of two matrices. Read May 2, 1925. *This Bulletin*, vol. 31, No. 7, pp. 304–308; July, 1925.

WEISNER, L. Groups in which the normaliser of every element except identity is abelian. Read Sept. 8, 1925. *This Bulletin*, vol. 31. No. 8, pp. 413–416; Oct., 1925.
—— On the number of elements in a group which have a power in a given conjugate set. Read Feb. 28, 1925. *This Bulletin*, vol. 31, Nos. 9–10, pp. 492–496; Nov.–Dec., 1925.

WIENER, N. The Dirichlet problem. Read May 3, 1924. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 3, No. 3, pp. 127–147; April, 1924.

—— Note on a paper of O. Perron. Read Dec. 30, 1924. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 4, No. 1, pp. 21–32; Jan., 1925.

—— A contribution to the theory of interpolation. Read Dec. 28, 1923. *Annals of Mathematics*, (2), vol. 26, No. 3, pp. 212–216; March, 1925.

—— The solution of a difference equation by trigonometrical integrals. Read Feb. 28, 1925. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 4, No. 3, pp. 153–163; May, 1925.

—— Note on quasi-analytic functions. Read Dec. 30, 1924. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 4, No. 4, pp. 193–199; July, 1925.

WILDER, R. L. A theorem on continua. Read May 3, 1924. *Fundamenta Mathematicae*, vol. 7, pp. 311–313; 1925.

—— Concerning continuous curves. Read April 15, 1922, Dec. 29, 1922, and Febr. 24, 1923. *Fundamenta Mathematicae*, vol. 7, pp. 340–377, 1925.

WILSON, W. A. On the Oscillation of a continuum at a point. Read Sept. 10, 1925. *Transactions of this Society*, vol. 27, No. 4, pp. 429–440; Oct., 1925.

WILSON, W. H. Two general functional equations. Read (Southwestern Section) Nov. 29, 1924. *This Bulletin*, vol. 31, No. 7, pp. 330–334; July, 1925.

WINGER, R. M. On the invariants of the ternary icosahedral group. Read (San Francisco Section) Dec. 22, 1923. *Mathematische Annalen*, vol. 93, Nos. 3–4, pp. 210–216; Jan., 1925.

ZELDIN, S. D. A classification of integral invariants in the calculus of variations. Read Dec. 27, 1923. *Tôhoku Mathematical Journal*, vol. 24, Nos. 1–2, pp. 121–124; Dec., 1924.

INDEX OF VOLUME XXXI

ALEXANDER, J. W. See REVIEWS, under Lefschetz.

BARNETT, I. A. Integro-Differential Equations, 263.

BATEMAN, H. See REVIEWS, under Jeans.

BELL. E. T. On the Number of Representations of an Integer as a Sum or Difference of Two Cubes, 309.

BENNETT, A. A. On Sets of Three Consecutive Integers which are Quadratic Residues of Primes, 411.

—— See REVIEWS, under Joly.

BERNSTEIN, B. A. Reports of Meetings of the San Francisco Section: October Meeting, 110; April Meeting, 385; June Meeting. 488.

BLISS. G. A. See TONELLI, L.

BROWN, B. H. See REVIEWS, under Fouet, Lemoyne.

CAMP, B. H. See REVIEWS, under Rietz.

CARMICHAEL, R. D. See REVIEWS, under Eddington, Halphen, Lamb. Lie, Nörlund.

CARSLAW, H. S. A Historical Note on Gibbs' Phenomenon in Fourier's Series and Integrals, 420.

COWLEY, E. B. See REVIEWS, under Loria.

CRATHORNE, A. R. See REVIEWS, under Borel, Fisher, Knopf, Urban.

CUMMINGS, L. D. A New Type of Double Sextette Closed under a Binary (3, 3) Correspondence, 266.

CURRIER, C. H. See REVIEWS, under Goodwin, Gribble.

DICKSON, L. E. Resolvent Sextics of Quintic Equations, 515.

—— See REVIEWS, under Fricke.

DODD, E. L. The Frequency Law of a Function of One Variable, 27

—— See REVIEWS, under Forsyth.

DRESDEN, A. The International Congress at Toronto, 1.

—— Reports of Meetings of the American Mathematical Society: December Meeting in Chicago, 195, April Meeting in Chicago, 393; May Meeting in New York, 481

—— See REVIEWS, under Bliss.

EMCH, A. See REVIEWS, under Bricard, Freyberger.

ETTLINGER. H. J. See REVIEWS, under Fubini, Kiepert, Knopp.

EVANS, G. C. Note on a Class of Harmonic Functions, 14.

—— See REVIEWS, under Enriques.

FIELD, P. See REVIEWS, under Montel.

FITE, W. B. Report of the February Meeting of the American Mathematical Society in New York, 291

FORD, L. R. The Fundamental Region for a Fuchsian Group, 531.

FORSYTH, C. H. See REVIEWS, under Mills, Yule.

FRANKLIN, P. Functions with an Essential Singularity, 157.

FREEMAN, I. A General Form of the Suspension Bridge Catenary, 424

GRAUSTEIN, W. C. See REVIEWS, under Coolidge.
GRIFFITHS, L. W. Contact Curves of the Rational Plane Cubic, 312.
HENDERSON, R. Life Insurance as a Social Service and as a Mathematical Problem, 227.
HILDEBRANDT, T. H. See REVIEWS, under Rothe.
HILLE, E. See REVIEWS, under Auerbach, Courant.
HOLLCROFT, T. R. Limits for Actual Double Points of Space Curves, 42.
HUNTINGTON, E. V. The Pan-American Scientific Congress, 290.
HURWITZ, W. A. Characteristic Parameter Values for an Integral Equation, 503.
JACKSON, D. The Geometry of Frequency Functions, 63.
KELLOGG, O. D. See REVIEWS, under Kneser.
KLINE, J. R. Concerning the Complementary Intervals of Countable Closed Sets, 409.
KURATOWSKI, C. On the Accessibility of an Arc from its Complement in Space of Three Dimensions, 32.
LEFSCHETZ, S. See REVIEWS, under de Kerékjártó.
LEHMER, D. N. A Correction, 226.
—— Note on the Construction of Tables of Linear Forms, 497.
—— See REVIEWS, under Cunningham, King, Mehmke, Whittaker.
LEVY, H. Normal Congruences of Curves in Riemann Space, 39.
—— Ricci's Coefficients of Rotation, 142.
LONGLEY, W. R. See REVIEWS, under Barton.
MCCLENON, R. B. See REVIEWS, under Caspar, Czwalina, Tropfke.
MACDUFFEE, C. C. On the Complete Independence of the Functional Equations of Involution, 21.
MACMILLAN, W. D. See REVIEWS, under Bieberbach.
MICHAL, A. D. Functional Invariants, with a Continuity of Order p, of One-Parameter Fredholm and Volterra Transformation Groups, 335.
MILLER, G. A. Simplifications relating to a Proof of Sylow's Theorem, 253.
MOORE, C. L. E. See REVIEWS, under Bertini, Schouten.
MOORE, C. N. Note on Gibbs' Phenomenon, 417.
—— See REVIEWS, under Hilb, Lichtenstein, Rosenthal.
MORGENSTERN, G. See REVIEWS, under Sakellarios.
MURNAGHAN, F. D. The Tensor Character of the Generalized Kronecker Symbol, 323.
—— See REVIEWS, under Sheppard.
OUSPENSKY, J. Sur les Valeurs Asymptotiques des Coefficients de Cotes, 145.
PERSONS, W. M. See REVIEWS, under Bowley.
PFEIFFER, G. A. See REVIEWS, under Fraenkel.
PORTER, M. B. See REVIEWS, under de la Vallée-Poussin.
REYNOLDS, C. N. See REVIEWS, under Henderson, Lecat, Lorentz.
RICHARDSON, R. G. D. Reports of Meetings of the American Mathematical Society: October Meeting in New York, 97; Thirty-First Annual Meeting, 203.

RICHARDSON, R. G. D. The Second Award of the Bôcher Memorial Prize, 193.

RITT, J. F. See REVIEWS, under Julia.

ROEVER, W. H. Some Phases of Descriptive Geometry, 540.

SHAW, J. B. See REVIEWS, under Hitchcock, Study.

SHOHAT, J. On the Polynomial of the Best Approximation to a Given Continuous Function, 509.

SHOOK, C. A. Improper Double Integrals, 524.

SNYDER, V. See REVIEWS, under Loria.

STONE, M. H. Three Theorems on Normal Orthogonal Sets, 17.

STOUFFER, E. B. Report of the Seventeenth Regular Meeting of the Southwestern Section, 116.

TAYLOR, J. H. Reduction of Euler's Equations to a Canonical Form, 257.

THOMAS, J. M. The Number of Even and Odd Absolute Permutations of n Letters, 303.

THOMAS, T. Y. Note on the Projective Geometry of Paths, 318.

TONELLI, L. The Calculus of Variations (translated by G. A. BLISS), 163.

UHLER, H. S. Criteria that any number of Real Points in n-Space shall lie in a $(n-k)$-Space, 499.

VANDIVER, H. S. On Sets of Three Consecutive Integers which are Quadratic or Cubic Residues of Primes, 33.

—— On the Distribution of Quadratic and Higher Residues, 346.

VEBLEN, O. Remarks on the Foundations of Geometry, 121.

WAHLIN, G. E. On the Solution of Diophantine Equations by Means of Ideals, 430.

—— See REVIEWS, under Bianchi, Cohen.

WEDDERBURN, J. H. M. A Theorem on Simple Algebras, 11.

—— The Absolute Value of the Product of Two Matrices, 304.

—— An Acknowledgement, 523.

—— See REVIEWS, under Lecat.

WEISNER, L Groups in which the Normaliser of Every Element except Identity is Abelian, 413.

—— On the Number of Elements of a Group which have a Power in a Given Conjugate Set, 492.

WHITE, H. S. The Frank Nelson Cole Prize in Algebra, 289.

WIDDER, D. V. Some Mean-Value Theorems connected with Cotes's Method of Mechanical Quadrature, 56.

WILDER, C. E. See REVIEWS, under Galbrun.

WILSON, W. H. Two General Functional Equations, 330.

WINGER, R. M. Gauss and Non-Euclidean Geometry, 356.

WOODS, F. S. See REVIEWS, under Baker.

YOUNG, J. W. A. See REVIEWS, under Lietzmann.

Reviews

Archimedes. See Czwalina, A.

Auerbach, F. Die Methoden der theoretischen Physik, E. Hille, 555.

Baker, H. F. Principles of Geometry, volume II, F. S. Woods, 370.

Barton, E. H. Analytical Mechanics (second edition), W. R. Longley, 276.

Bertini, E. Einführung in die Projektive Geometrie Mehrdimensionaler Räume. Nach der zweiten italienischen Auflage deutsch herausgegeben von A. Duschek, C. L. E. Moore, 463.

Bianchi, L. Lezioni sulla Teoria dei Numeri Algebrici, G. E. Wahlin, 84.

Bieberbach, L. Theorie der Differentialgleichungen, W. D. MacMillan, 279.

Bliss, G. A. Calculus of Variations. Carus Mathematical Monographs, No. 1, A. Dresden, 551.

Borel, E. Eléments de la Théorie des Probabilités (troisième édition), A. R. Crathorne, 367.

—— Traité du Calcul des Probabilités et de ses Applications, tome III, fascicule 1: Assurances sur la Vie, par H. Galbrun, A. R. Crathorne, 466.

Borel, E., et Deltheil, R. Probabilités, Erreurs, A. R. Crathorne, 367.

Bowley, A. L. The Mathematical Groundwork of Economics, W. M. Persons, 469.

Bricard, R. Petit Traité de Perspective, A. Emch, 564.

Cahen, E. Théorie des Nombres. Tome second: Le Second Degré Binaire, G. E. Wahlin, 565.

Carus Mathematical Monographs. See Bliss, G. A.

Caspar, Max. Johannes Kepler. Mysterium Cosmographicum. Das Weltgeheimniss, R. B. McClenon, 560.

Coolidge, J. L. The Geometry of the Complex Domain, W. C. Graustein, 351.

Courant, R., und Hilbert, D. Methoden der Mathematischen Physik, erster Band, E. Hille, 456.

Cunningham, A. Binomial Factorizations, volumes I and IV, D. N. Lehmer, 180; volumes II, III, and VI, D. N. Lehmer, 462.

Czwalina, A. Ueber Paraboloide, Hyperboloide, und Ellipsoide von Archimedes, R. B. McClenon, 568.

Deltheil, R. See Borel, E.

Duschek, A. See Bertini, E.

Eddington, A. S. The Mathematical Theory of Relativity, (second edition), R. D. Carmichael, 563.

Encyklopadie der Mathematischen Wissenschaften. See Hilb, E., Lichtenstein, L., Rosenthal, A.

Engel, F. See Lie, S.

Enriques, F. Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche, volume I e volume II, G. C. Evans, 449.

Fisher, A. An Elementary Treatise on Frequency Curves and their Application in the Analysis of Death Curves and Life Tables. Translated from the Danish by E. A. Vigfusson, A. R. Crathorne, 275.

Flamant, P. See Julia, G.
Forsyth, C. H. Mathematical Theory of Life Insurance, E. L Dodd, 371.
—— An Introduction to the Mathematical Analysis of Statistics, E. L. Dodd, 374.
Fouet, E. A. Leçons de Géométrie Elémentaire, B. H. Brown, 373.
Fraenkel, A. Einleitung in die Mengenlehre (zweite Auflage), G. A. Pfeiffer, 278.
Fréchet, M. See Rosenthal, A.
Freyberger, H. Zentral-Perspektive (zweite Auflage), herausgegeben von J. Vonderlinn, A. Emch, 88.
Fricke, R. Lehrbuch der Algebra, verfasst mit Benutzung von Heinrich Webers gleichnamigem Buche, erster Band, L. E. Dickson, 372.
Fubini, G. Lezioni di Analisi Matematica, H. J. Ettlinger, 74.
Galbrun, H. Introduction à la Théorie de la Relativité, Calcul Différentiel Absolu, et Géométrie, C. E. Wilder, 178.
—— See Borel, E.
Goodwin, H. B. The Alpha, Beta, Gamma Navigation Tables, C. H. Currier, 88.
Gribble, T. G. Handy Methods of Geodetic Astronomy for Land Surveyors, C. H. Currier, 87.
Halphen, G. H. Oeuvres, tome IV, R. D. Carmichael, 562.
Heegaard, P. See Lie, S.
Henderson, A., Hobbs, A. W., and Lasley, J. W. The Theory of Relativity, Studies and Contributions, C N Reynolds, 277.
Hilb, E., und Riesz, M. Neuere Untersuchungen uber Trigonometrische Reihen. Sonderabdruck aus der Encyklopadie der Mathematischen Wissenschaften, C. N. Moore, 453.
Hilbert, D. See Courant, R.
Hitchcock, F. L., and Robinson, C. S. Differential Equations in Applied Chemistry, J. B. Shaw, 87.
Hobbs, A W. See Henderson, A.
Jeans, J. H. Report on Radiation and the Quantum-Theory, H. Bateman, 85.
Joly, G. Les Erreurs Philosophiques de M. Einstein. Etude Directe de la Relativité, A. A Bennett, 567.
Julia, G. Leçons sur les Fonctions Uniformes à Point Singulier Essentiel Isolé, rédigées par P Flamant, J. F. Ritt, 359.
Kepler, J. See Caspar, M
de Kerékjártó, B. Vorlesungen uber Topologie, erster Band, S. Lefschetz, 176.
Kiepert, L Grundriss der Differential-Rechnung, zweiter Band, H. J. Ettlinger, 74.
King, L. V. On the Direct Numerical Calculation of Elliptic Functions and Integrals, D. N. Lehmer, 568.
Kneser, A. Die Integralgleichungen und ihre Anwendungen in der Mathematischen Physik (zweite Auflage), O. D. Kellogg. 177.
Knopf, O. Wahrscheinlichkeitsrechnung, A. R. Crathorne, 367.

Knopp, K. Aufgabensammlung zur Funktionentheorie, erster Teil, H. J. Ettlinger, 183.

Lamb, H. Hydrodynamics (fifth edition), R. D. Carmichael, 460.

Lasley, J. W. See Henderson, A.

Lecat, M. Bibliographie des Déterminants à Plus de Deux Dimensions, J. H. M. Wedderburn, 86.

—— Bibliographie de la Relativité, C. N. Reynolds, 566.

Lefschetz, S. L'Analysis Situs et la Géométrie Algébrique, J. W. Alexander, 558.

Lemoyne, T. Les Lieux Géométriques en Mathématiques Spéciales, B. H. Brown, 182.

Lichtenstein, L. Neuere Entwicklung der Theorie Partieller Differentialgleichungen zweiter Ordnung vom Elliptischen Typus. Sonderabdruck aus der Encyklopädie der Mathematischen Wissenschaften, C. N. Moore, 453.

Lie, S. Gesammelte Abhandlungen, herausgegeben von F. Engel und P. Heegaard, fünfter Band, erste Abteilung, R. D. Carmichael, 559.

Lietzmann, W. Methodik des Mathematischen Unterrichts, dritter Teil, J. W. A. Young, 374.

Lorentz, H. A. Clerk Maxwell's Electromagnetic Theory, C. N. Reynolds, 277.

Loria, G. Complementi di Geometria Descrittiva, E. B. Cowley, 466.

—— Curve Sgembe Speciali Algebriche e Transcendenti. Volume primo: Curve Algebriche, V. Snyder, 557.

Mehmke, R. Leitfaden zum Graphischen Rechnen (zweite Auflage), D. N. Lehmer, 467.

Montel, P. Statique et Résistance des Matériaux, P. Field, 180.

—— See Rosenthal, A.

Mills, F. C. Statistical Methods, C. H. Forsyth, 563.

National Research Council. See Rietz, H. L.

Norlund, N. E. Vorlesungen uber Differenzenrechnung, R. D. Carmichael, 445.

Riesz, M. See Hilb, M.

Rietz, H. L. (editor). Handbook of Mathematical Statistics, prepared by a committee of the National Research Council, B. H. Camp, 361.

Robinson, C. S. See Hitchcock, F. L.

Robinson, G. See Whittaker, E. T.

Rosenthal, A. Neuere Untersuchungen uber Funktionen Reeller Veranderlichen, nach den Referaten von L. Zoretti, P. Montel, und M. Fréchet. Sonderabdruck aus der Encyklopadie der Mathematischen Wissenschaften, C. N. Moore, 453.

Rothe, R. Höhere Mathematik. Teil I: Differentialrechnung und Grundformeln der Integralrechnung, nebst Anwendungen, T. H. Hildebrandt, 566.

Sakellarios, N. Elements of Analytic Geometry, volume I (in Greek), G. Morgenstern, 465.

Schouten, J A. Der Ricci-Kalkül, C. L. E. Moore, 173.

Sheppard, W. F. From Determinant to Tensor, F. D Murnaghan, 468.

Study, E. Einleitung in die Theorie der Invarianten Linearer Transformationen auf Grund der Vektorenrechnung, erster Teil, J. B. Shaw, 77.

Tropfke, J. Geschichte der Elementarmathematik, funfter und sechster Band, R. B. McClenon, 369; siebenter Band, R. B. McClenon, 461.

Urban, F. M. Grundlagen der Wahrscheinlichkeitsrechnung und der Theorie der Beobachtungsfehler, A. R. Crathorne, 367.

Vigfusson, E. A. See Fisher, A.

de la Vallée Poussin, C. J. Cours d'Analyse Infinitésimale (cinquième édition), volume I, M. B. Porter, 83.

Vonderlinn, J. See Freyberger, H.

Weber, H. See Fricke, R.

Whittaker, E. T., and Robinson, G. The Calculus of Observations, D. N. Lehmer, 181.

Yule, G. U. An Introduction to the Theory of Statistics, (seventh edition), C. H. Forsyth, 568.

Zoretti, L. See Rosenthal, A.

Errata, 226.

Index of Volume XXXI, 589.

New Publications, 93, 189, 285, 379, 476, 575.

Notes, 89, 184, 280, 375, 470, 569.

Papers Read Before the Society and Subsequently Published, Thirty-Fourth Annual List of, 579.

Notes and Other Items

Academies, Associations, Congresses, and Societies:

American Mathematical Society: Annual Meeting, 206; Auditors' Report, 205; Bôcher Prize, 193, 207; Budget, 205; Bulletin, 569, Cole Prize, 102, 289; Colloquium, 375, 396; Committee on Gibbs Lecture, 395; on Membership, 207, on Moore Fund, 101; on Printing, 396; Election of Officers, 206; Endowment Fund, 101, 205, 395, Gibbs Lecture, 101, 207, 375, 395; Librarian's Report, 206; New Members Admitted, 97, 98, 204, 393; Representatives on Council of American Association for the Advancement of Science, 206, 280, at International Mathematical Congress, 101; on National Research Council, 206, 375, at Pan-American Scientific Congress, 101; San Francisco Section, 111, 385, 488; Southwestern Section, 116, 375; Statistics, 205; Summer Meeting, 375, 396; Sustaining Memberships, 97, 101, 204, 393; Transactions, 89, 184, 375, 470, 569; Treasurer's Report, 205, Western Christmas Meeting, 375, 396

Associations for the Advancement of Science, American, 206, 280, 375, 376; British, 89, 185, 376.

Académie des Sciences de l'Oukraine, 281; American Astronomical Society, 375; American Physical Society, 375, Belgian Academy, 376, 471, Bologna Academy, 282; Boston Society of Civil Engineers, 571, Carnegie Institution, 376, Charkow Mathematical Society, 89; Copenhagen Society for the Spread of Natural Philosophy, 376;

Engineering Foundation, Carnot Celebration, 185; Franklin Institute, 185, 571; General Electric Company, 185, Institut de France, 282; International Astronomical Union, 281; International Mathematical Congress, 1, 101, 185, International Mathematical Union, 2, 3, International Physics Union, 281; International Research Council, 281; London Institution of Electrical Engineers, 376; London Mathematical Society, 280; Mathematical Association of America: Carus Monographs, 184; Meeting, 375; Officers, 280; Naples Academy, 570; National Research Council, 90, 206, 375; National Research Council of Japan, 570; Pan-American Scientific Congress, 101, 186, 290, 377: Paris Academy of Sciences, 282, 471; Royal Astronomical Society, 376; Royal Society of Edinburgh, 570; Russian Information Bureau, 281; Sociedad Matemática Argentina, 570; Società Universitaria di Scienze Matematiche, 89, Société Mathématique de France, 281; Unione Matematica Italiana, 282; Westphalian Mathematical Society, Weierstrass Week, 376.

Books, Announcement of New, 184, 281, 376.

Doctorates in Mathematics, American, 472.

Fellowships: American Field Service, 471; Guggenheim, 471, 572; National Research, 572.

Journals:

Acta Literarum ac Scientiarum, Szeged, 281, 291; American Journal of Mathematics, 89, 184, 280, 570; Annales Scientifiques des Institutions Savantes de l'Oukraine, Sect. Math., 89; Annals of Mathematics, 89, 280, 569; Archimede, 89, Astrophysical Journal, 376; Bulletin de l'Académie des Sciences de l'Oukraine, 281; This Bulletin, 569; Communications of the Mathematical Society of Charkow, 89; Japanese Journal of Mathematics, 570; Mémorial des Sciences Mathématiques, 281; Revista Matemática, 570; Transactions of the American Mathematical Society, 89, 184, 375, 470, 569.

Papers and Communications Presented to the Society, Authors:

Adams, C. R., 216, 217; Alexander, J. W., 106, 299, 299, Allen, E. S., 119.

Babcock, R. W., 404; Ballantine, J. P., 487; Barnett, I. A., 201; Barter, J., 385; Bateman, H., 111, 386; Bell, E. T., 111, 111, 112, 112, 113, 113, 386, 488, 488; Bender, H. A., 400, 400; Bennett, A. A., 223, 223, 224, 224, 296, 296, 399, Bernstein, B. A., 113, 386, 488; Betz, H., 200; Birkhoff, G. D., 116; Blichfeldt, H. F., 386; Bliss, G. A., 215.

Cajori, F., 113, 387, 387, 489; Camp, B. H., 102; Camp, C. C., 402; Caris, P. A., 399; Carpenter, A. F., 387, 489; Carson, J. R., 481; Castellani, M., 483; Chittenden, E. W., 117, 223, Church, A., 226, 485; Cole, F. N., 103; Colpitts, J. T., 118; Cramlet, C. M., 489; Crane, R., 397; Cummings, L. D., 102.

Daus, P. H., 387; Davis, H. T., 200; Dickson, L. E., 198, 199; Dodd, E. L., 200, 225, 296.

Edington, W. E., 200; Eisenhart, L. P., 292; Emch, A., 398, Ettlinger, H. J., 201.

Foster, R. M., 222; Franklin, P., 103, 103, 225, 301, 482, 482; Frink, O., 485.

Garabedian, C. A., 104, 298, Gehman, H. M., 103, 220, 301;

Glenn, O. E, 213, 483, Graustein, W. C., 212, 213. Graves, L. M., 211, 211, Green, C. F., 119; Gronwall, T. H, 109, 109, 302, 481, 484, 484, Grove, V. G, 196, 397

Hancock, H., 207; Haskell, M. W., 114, 388. Hazlett, O C., 219, 482. Hedrick, E. R, 114, 218, 388, Henderson, A., 207; Henderson, R., 207, Hille, C. E., 104, 107, 302. Hollcroft, T. R, 482. Hotelling, H, 114, 202, 389, 389, Huber, C. M., 105, Huntington, E. V., 405, Hurwitz, W. A., 215, 215, 216, Hutchinson, J. I., 104

Ingold, L., 114, 119, 201, 294, 407, 407; Ingraham, M. H., 405, 408.

Jackson, D., 107, 109, 403. 404, James, G., 298, 389; Jeffery, R L, 299, Jenkins, W. A., 407.

Kasner, E., 214, 226, 293; Kellogg, O. D., 300; Kempner, A. J., 201; de Kerékjártó, B, 302; Kline, J. R., 110, 300.

Lane, E. P., 196; Langer, R. E., 301, Latimer, C G., 213: Lefschetz, S, 301; Lehmer, D. N., 389, 489; Lehr, A. M M., 209, Levy, H., 294, 294, 485; Libman, E E., 297, 398, Lubben, R G, 225.

McDonald, J. H., 390, 390, 391, 391, 391, MacDuffee, C. C., 400, MacMillan, W. D, 396; MacNeish, H F., 198, 212; March, H. W., 405; Mathews, R M, 397; Merriman, G. M., 210; Michal, A. D, 106, Miller, G. A., 199, 401; Moore, C. L E, 292; Moore, C. N., 210, Moore, R. L., 120. 120, 219, 220, 220; Moritz, R. E., 391, 490. 490, Morley, F, 293; Morris, F. R., 115; Morse, H. M., 216, Mullemeister, H., 392; Murray, F. H., 221; Musselman, J. R., 212,

Olson, H. L, 295; Osgood, W. F., 295

Pell, A. J, 210; Pennell, W. O, 106, Perkins, F. W., 221; Pierce, T. A, 119.

Rainich, G. Y., 108, 109, 209, 209, 292, 293; Reilly, J. F., 120, 399, Reimer, E. H, 115, Reynolds, C. N., 297, Rider, P. R., 197, Ritt, J. F., 107, 108, 218, 300, Robertson, H P, 115, 491, Robinson, G. P., 408, Roever, W. H., 116, Russell, H. N., 207.

Sharpe. F. R., 214; Shaw, J. B., 202, 406; Shohat, J. A., 197, 402; Smith, H. L, 403, Stager, H. W., 491; Stone, M. H, 225, 225; Synge, J. L., 214.

Taylor, J. H., 102, 487, Taylor, J S., 222; Thomas, J. M., 299, 299, 486, 486, 486; Turner, J. S., 118, 118, 408.

de la Vallée Poussin, C. J., 219, Vandiver, H. S., 208, 208, 208; Veblen, O., 207, 486.

Wahlin, G. E., 199 401; Walsh, J. L, 105, Wedderburn. J H. M., 484, 484; Weisner, L., 299, Wiener, N., 106, 221, 221, 221, 302, 481, 482, 484; Wilder, R. L, 224. Williams, K. P., 105, 402, Wilson, W. H., 117, 117, 404; Winger, R. M, 491; Wood F. E., 195.

Zeldin, S. D., 212.

Personal Notes.

Abrich, G. F., 186; Adams, C. R., 474; Agnew, R. P., 475; Albanese, G., 377, Allen, H S., 570, Amaldi, M., 90, Ames, J. S., 90, 375, 375; Ames, L. D., 574, Anderson, A. E., 92; Andoyer, H., 185; Andreoli, G., 377; Appell, P, 471, 471, 471; d'Arcais, F., 283; Archibald, R. C, 280, 280, Armstrong, C N, 572; Arnold, K S., 91; Arnold, S. T., 378, Austin, C. B, 188, 572, Ayres, F., 284.

Babcock, R. W., 472; Bailey, E. A., 474; Baker, B. B., 90; Ball, W. W. R., 475; Balof, C. A., 92; Bateman, H., 282; Bauer, G. N., 186; Baxter, E. F., 91, 572; Beckwith, E. R., 91; Beghin, H., 283; Behnke, H., 378; Belardinelli, G., 283; Bell, E. T., 193, 474; Beloch, M. P., 378; Bennett, A. A., 280, 378; Bennett, J. C., 92; Benson, C. H., 378; Bernstein, S., 89; Berryman, C. H., 188; Berzolari, L., 283; Betz, H., 472; Bianchi, L., 282; Bickerton, A. W., 282; Bieberbach, L., 376; Bill, E. G., 186; Birkhoff, G. D., 375, 375, 474; Bloch, P., 282; Blumberg, A. A., 284; Blumberg, H., 474, Blumenthal, L. M., 378; Bohr, N., 282, 376, 471, 571, 571; Bowie, W., 375; Boyd, E. E., 188; Brackett, F. P., 186; Bradley, F. H., 188; Bragg, W. H., 89, 185; Bray, H. E., 91; Brinkmann, H. W., 475, Brodetsky, S., 90; Bruce, D., 185; Bruton, G. S., 188; Bubb, F. W., 186; Buchanan, M., 572; Burgess, R. W., 474; Byrne, W. E., 471.

Caldonazzo, B., 377; Campbell, A. D., 472; Campbell, G. A., 185; Campbell, J. E., 188, Campbell, M., 186; Carlen, M., 574; Carlson, E., 473; Carnot, S., 185; Carroll, E. T., 91; Cartan, E., 283; Carter, B. E., 284; Carver, W. B., 474; Cassina, U., 283, Chambers, G. G., 571; Chaney, J. C., 188; Chapman, S., 91; Chazy, J., 472; Cherry, T. M., 91; Ciani, E., 283; Coker, E. C., 91; Collignon, G. H., 473; Colpitts, J. T., 473; Comrie, L. J., 187; Conrad, W. A., 284; Coolidge, J. L., 280, Crathorne, A. R., 91; Crozet-Fourneyron, M., 282; Crudeli U., 377.

Dale, J., 473, 572; Daniele, E., 283; Darkow, M. D., 473; Davis, H. A., 574; Davis, J. E., 573; De Long, I. M., 91; Dempster, A. J., 375; Dennett, W. S., 574; De Ronde, L. A., 186; Dixon, A. L., 280; Doetsch, G., 186; de Donder, T., 376; Doshkees, V., 187; Dresden, A., 206, 280; Dyson, F., 376.

Eddington, A. S., 90, 472, 571, 571; Eglin, W. C. L., 185; Einstein, A., 186, 471; Eisenhart, L. P., 280, 283, 375, 375; Elliott, W. W., 473; Eversull, B. M., 473.

Fabry, C., 185, 282; Farnum, F., 475; Fellar, W. L., 472; Fermi, E., 378, Ferrar, W. L, 572; Fichot, E., 472; Filon, L. N. G., 280; Finzi, A., 283; Flammarion, C, 574; Foppl, A., 188; Foley, A. L., 375; Foote, P. D., 375; Fourneyron, B., 282; Fowler, R. H., 471, 472; Franklin, P., 573; Freeman, J. R., 571; Fubini, G., 282; Fulmer, H. K., 187.

Gans, R., 472; Garabedian, C. A., 474; Garabedian, H. L., 475; Garnier, R., 282; Gaylord, L., 91; Gerst, F. J., 473; Gibbens, G., 573; Giddings, H. A., 284; Gigli, D., 90, Gilchrist, L., 91; Gillespie, D. C., 474; Giraud, G., 282, Gleason, R. E., 283; Gossard, H. C., 573, Goursat, E., 282; Gouwens, C, 473; Graves, L. M., 473, 475, Green, W. C., 188; Grove, C. C., 573; Grumann, H. R., 188; Guichard, C., 282, 475; Gummer, C. F., 473; Gutzmer, A., 92.

Hadamard, J., 377; Hadlock, E. H., 188; Hale, G. E., 90, Hall, A. G., 284; Hamilton, W. A., 574; Hammatt, H. B., 475; Hammerstein, A., 90; Hardy, G. H., 280; Harkins, C., 574, Hart, W. L., 573, Haudié, E., 282; Hazlett, O. C., 573; Heaviside, O., 475; Hedley, J. W., 473; Heinke, P. E., 284; Hemke, P. E., 473; Herbert, W. C., 188; Herglotz, G., 377; Hermite, C., 185; Hess, G. W., 187; Highet, H. D., 188; Hilbert, D., 376; Hill, M. J. M., 283; Hill, W. H., 91; Hilton, H., 280; Hinsch, V. B., 378; Hobbs, A. W., 378; Holley, J. L., 473, Hoover, H. A., 92, Hotelling, H., 473; Householder,

F. W., 284; Howie, J. N., 92; Hughes, J. C., 473; Hunt, M., 473; Huntington, E. V., 186, 280, 377, 572, Hussey, L. W., 188.

Ingraham, M. H., 473.

Jackson, D., 280, 375; Janet, B., 377; Jeans, J. H., 92, 472; Jeffreys, H., 472; Jenkins, W. A., 475; Jensen, C. M., 473, Johnson, W. A., 281; Jonah, F. C, 188; Jones, E. M, 185, Jones, J. E., 472, Jones, R. L, 188; Jude, R. H., 92, Julia, G., 472.

von Kármán, T., 572; Kells, L. M., 284; Kempner. A. J., 573, de Kerékjártó, B., 291; Kiernan, A., 284; Klein, F., 90, 376, 574; Kline, J. R., 572, Kneser, A., 571; Knopp, K., 376, von Koch, H., 188, Koebe, P., 376, 571, Koenigs, G., 185; Kohlschutter, A, 376 Koopman, B. O., 475; Kotof, T., 92; Krauss, F, 378, Kryloff, N. M., 281.

Lamb, H., 89; Lambert, P. A., 187, 475; Lane, H. I, 475; Langevin, P., 282; Langmuir, I., 572; Larkin, E. L., 188; Lasley, J. W., 378; Latimer, C. G., 473; von Laue, M., 282, Lazzarino, O., 283; Lecornu, L., 472; Lefschetz, S., 193, 284, 571; Lehmer, D. N., 186, 376; Lehr, A. M. M, 92, Leudesdorf, C., 92; Leuschner, A. O, 377; Levy, H, 473; Linker, J. B, 378, 474, Lockner, S. J., 574; Longbottom, J G., 92; Loring, W. E, 284, Lund, E. H, 188; Lyon, W. H., 187.

McAdam, D. J., 475; McClintick, W S., 378; McFarlan, L. H., 474; McLaughlin, D. B., 188, McLennan, J. C., 90, MacClaren, M., 475, MacPherson, D. H., 92, 284; Maggi, G., 283, Malo, E., 282; Marden, M., 475; Marquis, R. H., 188, Martinson, E. P., 92; Maslow, A. P., 188, Màson. M., 573, Menzel, D. H., 92, Mercer, E., 188, Merrill, A S., 92; Merrill, J. E., 188; Merriman, M, 574; Meyer, F, 378; Michal, A. D., 474; Michelson, A. A., 90, 186, 186, 573; Michie, J. N., 475; Miles, E. J., 573, Miller, D. C., 375, Millikan, R. A., 90, 90, 572; Milne, E. A., 91; Miser, W. L., 475, Mitchell, H. B., 187; Mittag-Leffler, G., 376; Montel, P., 282, 472; Montgomery, R. B., 187; Moorefield, N. L., 188; Morris, I., 187, Morris, M, 573, Morse, H. M., 378, Muller-Breslau, H., 574.

Neder, L., 377, Nekrassoff. P. A, 475; Neubauer, G., 574; Neumann, C., 574; Nicholson, G. W., 188, Nordgaard, M., 186, Nyswander, J. A., 474.

Olson, H. L., 378; Osborn, J. O., 187; Ott, W. P, 187.

Page, L., 375; Painlevé, P., 571, Palatini, A, 283, Palmié, A H, 91; Parsons, C., 185, 185; Patton, L. K., 188; Pell, A. J., 91, 573; Perkins, F. W, 475; Perron, O., 376, Picard, E, 185, 282, 471, Picone, M., 283; Pierpont, J., 375, Pixsley, H. H., 188, Plapp, E. M., 574; Poincaré, H., 471; Pollard, H S., 92, Pólya, G., 377; Porter, C. S., 92, Priulla, G., 89, Proudman, J., 472, Pugh, G T., 187, Pupin, M. I., 90, 280, 572; Putnam, R. G., 474, Putnam, W. L, 284

Rademacher, H., 472; Rainich, G. Y., 470; Ramler, O. J., 187; Rea, P. L, 187; Rechard, O. H, 573; Reeve, W. D., 475, Reinsch, B. P, 474, Reynolds, C. N., 572; Rice, E. W., 185; Richardson, D. P, 188, Richardson, R. G D., 284; Richtmyer, F. K., 375, 375; Riesz, F., 281; Rietz, H. L., 280, Riquier, C., 283; Ritt, J. F, 470, de la Rive, L., 92; Roberts, S. E., 475; Robinson, G. E, 92; Roever, W. H., 280; Romboli, G., 574; Room, T. G, 471; Rowe, J. E., 187; Rowland, S. A., 186; Runge, C, 378; Rupp, C. A., 187; Russell, B., 572; Russell, J., 185; Rutherford, E., 90, 185.

Sannia, G., 283; Sansone, G, 283; Saund, D. S., 474; Schafheitlin, P., 378; Schrutka, L, 472, Scott, C. A., 91; von Seeliger, H., 284, Severi, F., 185, Shaw, E. A., 284, Sherwood, G E. F., 378; Shiefer, H. F, 378; Shohat, J. A., 188, 573, Silla, L., 90, Simpson, C. G., 376; Smail, L. L., 573; Smith, D. E., 475; Smith, P. K., 284; Solum, A., 186, Sparks, F. W., 284; Sperry, C. S., 475; Stabler, H. P., 475; Stafford, E. T., 188, 574; Stager, H. W., 377; Stanton, E. W, 284; Starke, E. P., 574, Stebbins, J., 375, Steinert, E. E, 188, Steinmetz, C. P, 185; Stekloff, V., 185, Stevens, W. P, 284; Stokes, N. C, 92; Stone, M H, 378, Stouffer, E. B., 206, 280; Straneo, P, 377; Stromquist, C E. 574, Symonds, B, 284; Synge, J. L., 572; Synnerdahl, J. H., 187.

Talley, J. L., 188, Taylor, J. H., 474, 574, Taylor, J. S, 187; Terracini, A., 377; Thomson, E., 571, Thomson, J. J., 376, 570, Tienzo, T., 378; Tietze, H., 378; Tinner, J. C., 92, Tisdale, V., 187; Tisdale, W. E, 375; Togliatti, C., 472; Torelli, G, 283, Torrey, M. M., 474, Tricomi, F., 377; Turner, B. M., 572, Tyndall, A. M., 376.

Udinski, W. P., 284; Uhler, H. S., 574; Urysohn, P., 378.

de la Vallée Poussin, C. J., 185; Van Fleet, G. S., 188; Vassilief, A V., 185, 571; Veronese, G. 571; Vessiot, E., 282; Villat, H., 281; Vitali, G., 283, Vivanti, G., 283; Voss, A, 571.

Wagner, E. H., 378; Wallenberg, G. J., 378, Walsh, J. L., 474, 572; Walther, A., 378; Watson, G N., 280, Wavre, R, 90; Webster, A. G., 90; Webster, D. L., 375, Weitzenbock, R, 185, Wells, E. D., 188; Wellstein, J, 378, Western, A. E., 280; Wheeler, A. L., 573; Whitehead, A. N., 91, 186; Whitford, D. E., 475, Whittaker, E. T., 283; Whyburn, W. M, 92; Widder, D. V., 91, 474; Wieghardt, K., 188, Wien, W, 471, Wiggin, E. P., 92, Wilder, R. L., 188; Wiley, F. B., 475, Williams, W. L., 188, Wilson, E. B., 280, Wilson, H. A, 186, 574; Wilson, L. T., 475; Winsor, A. S., 378, Wirtinger, W., 571; Wolfender, J., 475; Wood, F., 474, 574.

Young, W. H., 280.

Zeeman, P., 185, 571; Zeiss, C., 376; Ziwet, A., 475.

Prizes and Medals:

Abbe, 376; Ackermann-Teubner, 376, Arzelà, 282, Barnard, 471, Belgian Academy, 376, 471, Bôcher, 193, 207; Cambridge University: Seely, 281, Adams, 471, Smith's, 471, Cole, 102, 289; Faraday, 376; Franklin Institute, 571; Freeman, 571; Fubini, 282, Lubbock, 185, Oersted, 376, Osiris, 282, Paris Academy, 282, Poincaré, 471, Royal Astronomical Society, 376; Royal Society of Edinburgh, 570; Torelli, 570.

Universities and Technical Schools:

Cambridge, 281, 471,
Columbia, 470, 471,
Harvard, 90, 470;
London, 185,
Oxford, 571,
Rensselaer Polytech Institute, 185,
Szeged, 281,
Union, 185.